Axiomatic approaches to preference aggregation (strict preferences)

0. Definitions

N	finite set of n individuals (N can be defined as $\{1, \dots, n\}$)
A	finite set m of alternatives
a preference p over A	is interpreted as a strict preference (no indifference)
$x\, p\, y$	means that $x \in A$ is preferred to $y \in A \setminus \{x\}$ in preference p
L	set of (strict) preferences over the set A
preference profile P	a vector $P = (P_1, \dots, P_n)$ in which P_i is individual i's preference over A
L^n	set of all preference profiles
social welfare function f	a mapping $f: L^n \to L$ assigning a collective preference $f(P)$ to each preference profile $P \in L^n$
dictatorial SWF	a social welfare function f satisfying the following: there is $i \in N$ such that, for all $p \in L^n$, $f(P) = P_i$.

1. "Vertical" procedures

In vertical procedures, axioms are stated in terms of the preference of one alternative over another alternative, so they establish whether an alternative is collectively preferred to another one. As a result, the axioms contribute to define the collective preference in relative, rather than absolute, terms: they tell if alternative x is preferred to alternative y given certain conditions, not if x is the most (or second most, or third most…) preferred alternative. Accordingly, the axioms do not allow a direct construction of the collective preference: they just provide pieces that, when all collected, can be assembled to generate the collective preference. The axioms in Arrow's theorem illustrate vertical procedures.

PAR. *Pareto principle*
For all $P \in L^n$, $x \in A$ and $y \in A \setminus \{x\}$, if, for all $i \in N$, $x\, P_i\, y$, then $x\, f(P)\, y$.

IIA. *Independence of irrelevant alternatives*
For all $P \in L^n$, $Q \in L^n$, $x \in A$ and $y \in A \setminus \{x\}$, if, for all $i \in N$, $x\, P_i\, y \Leftrightarrow x\, Q_i\, y$, then $x\, f(P)\, y \Leftrightarrow x\, f(Q)\, y$.

Theorem 1 (K. Arrow). *If $n \geq 2$ and $m \geq 3$, then a SWF f satisfies PAR and IIA if and only if f is dictatorial.*

2. "Horizontal" procedures based on positions

Now, axioms are positional axioms: they indicate the position that an alternative occupies in the collective preference on the basis of the position that the alternative occupies in the individual preferences. Axioms of this type allow one to construct the collective preference directly, position by position. A result by Nicolas Houy[1] illustrates this kind of procedure.

For preference p and $k \in \{1, \dots, m\}$, ${}^{k}p$ designates the alternative in A that occupies the kth position in p, with the first position associated with the most preferred alternative.

POS. *Position-wise aggregation*

For every $k \in \{1, \dots, m\}$ there is a choice function $g_k : A^n \to A$ such that:

(i) if $a_1 = \dots = a_n = a$ then $g_k(a_1, \dots, a_n) = a$; and

(ii) for all $P \in L^n$ and $k \in \{1, \dots, m\}$, ${}^{k}f(P) = g_k({}^{k}P_1, \dots, {}^{k}P_n)$.

POS asserts that, for every position k in the collective preference, there is a function g_k mapping vectors of alternatives into alternatives such that:

(i) if all the alternatives in the vector $(a_1, \dots, a_n)$ are the same, then g_k picks that alternative; and

(ii) for every preference profile P, the alternative ${}^{k}f(P)$ occupying position k in the collective preference is obtained by letting g_k select one alternative among those occupying position k in the individuals' preferences.

Theorem 2 (N. Houy). *If $n \geq 2$ and $m \geq 3$, then a SWF f satisfies POS if and only if f is dictatorial.*

3. "Horizontal" procedures based on alternatives

This procedure is symmetric with respect to the second one, as each alternative x is assigned a mapping that determines the position that x occupies in the collective preference by taking into account only the positions that x occupies in the individuals' preferences.

[1] Nicolas Houy (2006): "Positional independence in preference aggregation: A remark", Social Choice and Welfare 27, 341–345.

For preference p and alternative x, $\pi(x, p)$ designates the position that x occupies in p, with the first position corresponding to the most preferred alternative. Therefore, $\pi(x, p) = k$ if and only if ${}^{k}p = x$.

PUN. *Positional unanimity*
For all $P \in L^n$ and $x \in A$, if there is $k \in \{1, \dots, m\}$ such that, for all $i \in N$, $\pi(x, P_i) = k$ then $\pi(x, f(P)) = k$.

PIN. *Positional independence*
For all $P \in L^n$, $Q \in L^n \setminus \{P\}$ and $x \in A$, if, for all $i \in N$, $\pi(x, P_i) = \pi(x, Q_i)$ then $\pi(x, f(P)) = \pi(x, f(Q))$.

Theorem 3. *If $n \geq 2$ and $m \geq 3$, then a SWF f satisfies PUN and PIN if and only if f is dictatorial.*

4. A mixed approach

To describe the new approach, it is convenient to summarize the approaches so far presented. To this end, consider the aggregation problem represented next.

$$\begin{matrix} x & y & z \\ y & z & x \\ z & x & y \end{matrix} \tag{1}$$

For $n \in \{1, 2, 3\}$, column n shows the preference of individual n over the set of alternatives $\{x, y, z\}$. For instance, column 2 means that individual 2 prefers y to both z and x, and prefers z to x. Call Procedure 1 the type of aggregation mechanism formalized by the IIA axiom. Procedure 1 solves (1) by solving first the following three simpler problems, obtained by restricting the preferences to just two alternatives.

preference of x against y			preference of y against z			preference of x against z		
x	y	x	y	y	z	x	z	z
y	x	y	z	z	y	z	x	x

Consequently, Procedure 1 specifies three mappings f_{xy}, f_{yz} and f_{xz} that aggregate preferences over sets with two alternatives and applies transitivity to the partial aggregation outcomes in order to finally generate the aggregation of (1).

Specifically, under Procedure 1, every two alternatives α and β are given a function $f_{\alpha\beta}$ that establishes the ranking between α and β in the collective preference $f(P)$ taking only into account the preference between α and β held by individuals.

Problem (1) is defined in terms of two inputs: alternatives and preference positions. Procedure 1 emphasizes alternatives. By symmetry, one could consider a variation on Procedure 1 in which independence applies to positions instead of alternatives: when determining the alternative occupying position k the information concerning the rest of positions is irrelevant. This leads to what may be called Procedure 2. Procedure 2 solves problem (1) by solving first the following three simpler problems, obtained by restricting the preferences to positions.

alternatives in position 1			alternatives in position 2			alternatives in position 3		
x	y	z	y	z	x	z	x	y

Procedure 2 is based on three mappings f_1, f_2 and f_3 that aggregate preferences position-wise. In particular, for $k \in \{1, 2, 3\}$, f_k determines the alternative that occupies position k in the summarizing preference using as inputs the alternatives occupying position k in the individual preferences. Procedure 2 is simpler than Procedure 1 in the sense that it never requires more mappings than Procedure 1. For instance, when 5 alternatives are involved, Procedure 1 has to specify 10 mappings of the type $f_{\alpha\beta}$, whereas Procedure 2 needs to specify only 5. In addition, whereas the mappings in Procedure 2 construct the summarizing preference directly, those in Procedure 1 construct the preference indirectly: f_{xy} states whether x occupies a higher or lower position than y but not the position to which x is actually assigned.

Procedure 3 is symmetrical with respect to Procedure 2. The mappings in Procedure 2 are assigned to positions and treat alternatives as inputs. The mappings in Procedure 3 are assigned to alternatives and treat positions as inputs. Procedure 3 solves (1) by solving first the following three simpler problems, obtained by listing the position that each alternative occupies in the preference of every individual.

positions of x			positions of y			positions of z		
1	3	2	2	1	3	3	2	1

To be more precise, Procedure 3 specifies three mappings f_x, f_y and f_z that associate with each alternative the position that the alternative occupies in the final aggregation.

The final procedure considered, Procedure 4, is like Procedure 2 with the difference that the input to which the mappings apply is not fixed but depends on the previous solutions. Procedure 4 solves (1) sequentially by solving first the same initial problem as Procedure 2, given by the profile of alternatives occupying position 1:

$$x \qquad y \qquad z.$$

But the next aggregation problem to be solved depends on the solution of the first problem. If, for instance, the solution is x then Procedure 4 solves the problem given by the profile of alternatives occupying position 1 when x is deleted in (1). That is, at the second stage, Procedure 4 solves

$$y \qquad y \qquad z.$$

When the solution to that problem is determined the next problem is obtained from (1) by deleting all the solutions already found. As Procedure 2, Procedure 4 can be associated with three mappings f_1, f_2 and f_3 that are sequentially applied to a variable input, which consists of what is left in (1) after the removal of alternatives selected by the preceding mappings.

Procedure 4 has been defined so that the synthesizing preference is generated from the most preferred alternative to the least preferred. An alternative procedure, Procedure 4*, could be defined to operate the other way round: the least preferred alternative is identified first, next the second least preferred alternative and so on. In this context, it seems natural to invoke a symmetry axiom: an aggregation mechanism should be the result of applying both a Procedure 4 and a Procedure 4*, indicating that the aggregation proceeding downwards has to produce the same outcome as the aggregation proceeding upwards. Theorem 4 below states that the only social welfare function satisfying this requirement are dictatorial (a voter always impose his preference) or constant (a given preference over the candidates is always imposed).

DOWN$_r$. For every $t \in \{1, \dots, r\}$, there is a mapping $d_t : A^n \to A$ such that, for all $P \in L^n$: (i) ${}^1f(P) = d_1({}^1P_1, \dots, {}^1P_n)$; and (ii) if $r \geq 2$ then, for all $t \in \{2, \dots, r\}$, ${}^tf(P) = d_t({}^1P_1|_{A_t}, \dots, {}^1P_n|_{A_t})$, where $A_t = A \setminus \{{}^1f(P), \dots, {}^{t-1}f(P)\}$.

DOWN$_r$ postulates the existence of r choice functions $d_1, \dots, d_r$, each one of them mapping A^n into A, such that: (i) d_1 determines the most preferred alternative in the collective preference using as input the profile $({}^1P_1, \dots, {}^1P_n)$ of most preferred alternatives; and (ii) for $t \geq 2$, d_t determines the alternative occupying position t in the collective preference using as input the profile of most preferred alternatives when the alternatives already selected by the $t-1$ first functions $d_1, \ldots, d_{t-1}$ have been removed from the set A of alternatives. DOWN$_r$ sequentially generates the r first alternatives in the collective preference: the alternative in position 1 is selected first, next the alternative in position 2, and so on.

UP$_r$. For every $t \in \{1, \ldots, r\}$, there is a mapping $u_t : A^n \to A$ such that, for all $P \in L^n$: (i) ${}^1f(P)^* = u_1({}^1P_1^*, \ldots, {}^1P_n^*)$; and (ii) if $r \geq 2$ then, for all $t \in \{2, \ldots, r\}$, ${}^tf(P)^* = u_t({}^1P_1^* |_{A_t}, \ldots, {}^1P_n^* |_{A_t})$, where $A_t = A \backslash \{{}^1f(P)^*, \ldots, {}^{t-1}f(P)^*\}$.

UP$_r$ is DOWN$_r$ in reverse. UP$_r$ sequentially generates the last r alternatives in the collective preference: the alternative in the last position (position m) is selected first, next the alternative in position $m-1$, …, and lastly the alternative in position $m-r+1$.

Theorem 4. *With $n \geq 2 < m$, let $f : L^n \to L$ be a social welfare function. Then f satisfies DOWN$_m$ and UP$_m$ if and only if f is either constant or dictatorial.*

5. A non-negative interpretation of dictatorial social welfare functions

When a dictatorial SWF declares x to be socially preferred to y and some voter i different from the dictator d also prefers x to y, it could be argued that the reason why, in the specific preference profile P under consideration, x was declared socially preferred to y was not d's preference for x against y but i's. In particular, if there are k voters in P preferring x to y, one could at least provide k different justifications for having x socially preferred to y when the voters have preferences as in P. Hence, by considering justifications related to individuals and not groups, and by accepting only justifications relying on individuals that share the social preference to be justified, one could associate with each of those k voters, and for profile P, an average power of $1/k$. By defining a voter's average power in a dictatorial SWF as the sum, over each preference profile P, of his average power in P, it turns out that the dictator's average power is never greater than three times the average power of any other voter.

Definition 5.1. For SWF f and different alternatives x and y, a justification function for x against y under f is a mapping $J_{xy} : L^n \to N$ such that, for every preference profile P and voter i, $J_{xy}(P) = i$ implies $x \, P_i \, y$ and $x \, f(P) \, y$.

The interpretation is that $J_{xy}(P)$ is the voter whose preference for x against y is presumed to determine the social preference for x against y in preferene profile P: it is because $J_{xy}(P)$ prefers x to y that $x\,f(P)\,y$. Measures of a voter's power could be defined in terms of justification functions J_{xy}. Two such measures are suggested next.

Definition 5.2. For different alternatives x and y, SWF f and justification function J_{xy}, the power of voter i to impose x over y in f under J_{xy} is defined as the proportion

$$W_{xy}^{i} = \frac{|\{P \in L^n : J_{xy}(P) = i\}|}{|\{P \in L^n : x\, f(P)\, y\}|}.$$

The proportion W_{xy}^{i} is the number of profiles in which the justification function makes i decisive for x against y divided by the number of profiles in which x is socially preferred to y. The traditional negative interpretation of Arrow's theorem seems to rely on the presumption that, if a SWF f is dictatorial with dictator d, then, for alternatives x and y, the justification function J_{xy} must be such that, for every preference profile P, $J_{xy}(P) = d$. In that case, for all different alternatives x and y, $W_{xy}^{d} = 1$ and, for every voter $i \neq d$, $W_{xy}^{i} = 0$: only d has power to impose an alternative against another one. Nonetheless, there are alternative justification functions that can be associated with a dictatorial SWF.

Since the measure of the power of a voter, even in a dictatorial SWF, depends on the justification functions chosen, it may be worth considering, as a yardstick, some form of average power.

Definition 5.3. For SWF f and different alternatives x and y:

(i) the average power of voter i to impose x over y in preference profile P such that $x\,P_i\,y$ is $w_{xy}^{i}(P) = \dfrac{1}{|\{j \in N : x P_j y\}|}$, with $w_{xy}^{i}(P) = 0$ if $y\,P_i\,x$.

(ii) letting $T_{xy} = \{P \in L^n : x\,f(P)\,y\}$, the average power of voter i to impose x over y in f is $w_{xy}^{i} = \dfrac{\sum_{P \in T_{xy}} w_{xy}^{i}(P)}{|T_{xy}|}$.

The value $w_{xy}^{i}(P)$ can be viewed as i's expected power in profile P when a justification function J_{xy} is chosen at random, with every such function having the same probability of being chosen. It also represents the proportion of justification functions J_{xy} such that $J_{xy}(P) = i$, that is, the proportion of justification functions under which i can be seen as imposing his preference for x over y. The value w^i_{xy} is just a general average over the set of preference profiles T_{xy} in which the SWF makes x socially preferred to y.

It is worth remarking that, since a dictatorial SWF is symmetric with respect to alternatives, for every voter i and three different alternatives x, y and z, it holds $w^i_{xy} = w^i_{xz} = w^i_{zy}$. Similarly, since a dictatorial SWF is symmetric with respect to voters different from the dictator d, for every two such individuals i and j, and different alternatives x and y, it holds $w^i_{xy} = w^j_{xy}$. These remarks motivate the following definition.

Definition 5.4. For a dictatorial SWF with dictator d, define d^* to be the average power of the dictator d and v^* the average power of any other voter different from the dictator (that is, for each voter $i \neq d$ and different alternatives x and y, $d^* = w^d_{xy}$ and $v^* = w^i_{xy}$).

Theorem 5 below asserts that the dictator d of a dictatorial SWF with $n \geq 2$ voters: (a) has an average power d^* greater than the egalitarian power $1/n$; but (b) for sufficiently large number n of voters, d^* converges from above to $2v^*$: the dictator's average power d^* is approximately equal to twice the average power v^* of any other voter.

As the number n of voters grows, d^* approaches $\frac{2}{n+1}$ and v^* approaches $\frac{1}{n+1}$.

Theorem 5. *Let f be a dictatorial SWF with n ≥ 2 voters and dictator d. Then:*

(i) the average power d of the dictator and the average power v* of a voter different from the dictator satisfy*

$$d^* = \frac{2}{n}\left(1 - \frac{1}{2^n}\right) > \frac{1}{n} \tag{2}$$

$$v^* = \frac{1 - d^*}{n - 1}; \tag{3}$$

(ii) as the number n of voters increases, the ratio $\frac{d^*}{v^*}$ *approaches 2 from above; and*

(iii) for sufficiently large n, $d^* \approx \frac{2}{n+1}$ *and* $v^* \approx \frac{1}{n+1}$ *(the dictator counts as just two voters).*