An overlapping generations model with just a market for lending

1. Structure of the economy

The model is taken for McCandless and Wallace (1991, ch. 1-2). Time is measured in periods, denoted by t, and indexed by integers: $t \in \{1, 2, 3, \dots\}$. The only agents are consumers. Consumers live for two consecutive periods. In every period t a new generation of $N(t)$ consumers is born. Members of generation t are young in period t and old in period $t+1$; see Fig. 1.

	time period				
generation	1	2	3	4	⋯
0	*old*				
1	*young* →	*old*			
2		*young* →	*old*		
3			*young* →	*old*	
4				*young*	
…					
population	$N(0)+N(1)$	$N(1)+N(2)$	$N(2)+N(3)$	$N(3)+N(4)$	
amount of good	$Y(1)$	$Y(2)$	$Y(3)$	$Y(4)$	

Fig. 1. Demographic structure of the economy

There is only one good in each period. The good is exogenously given: it is a "gift of nature". The amount of good in period t is $Y(t)$. The good available in period t can only be used for consumption in the same period t. The amount of good available in t not consumed in that period is lost: each unit of a good cannot exist at different periods because the good is assumed to completely depreciate in one period. In sum, the good cannot be accumulated nor stored: the amount of good not consumed in the current period is lost.

As endowment, consumer i of generation t has $w_t^i(t)$ units of the good in t and $w_t^i(t+1)$ in $t+1$. The total endowment $Y(t)$ in t is distributed among the people alive in t:

$$\sum_{i \in N(t)} w_t^i(t) + \sum_{i \in N(t-1)} w_{t-1}^i(t) = Y(t).$$

Member i of generation $t \geq 1$ consumes $c_t^i(t)$ units of the good in t and $c_t^i(t+1)$ units in $t+1$. The consumption basket of $i \in N(t)$ is a pair $c_t^i =$

$\left(c_t^i(t),\ c_t^i(t+1)\right)$ that establishes i's consumption $c_t^i(t)$ when young and i's consumption $c_t^i(t+1)$ when old. For $i \in N(0)$, c_0^i is just the number $c_0^i(1)$.

A consumption allocation is a sequence $\{c_t^i\}_{t\geq 0, i\in N(t)}$ of consumption baskets of members of all generations (generation 0 consists only of old people). A consumption allocation is feasible if, for all $t \geq 1$,

$$\sum_{i\in N(t)} c_t^i(t) + \sum_{i\in N(t-1)} c_{t-1}^i(t) \leq Y(t).$$

Consumers have preferences over their consumption baskets. When young, the preference of consumer $i \in N(t)$ is represented by a utility function u_t^i.

The value $u_t^i\left(c_t^i(t), c_t^i(t+1)\right)$ is i's utility when he consumes $c_t^i(t)$ now (as a young individual) and consumes $c_t^i(t+1)$ in the future (as an old individual). When old, i's utility (in period $t+1$) only depends on $c_t^i(t+1)$, which had already been determined when i was young (in period t).

The following recapitulates the main variables of the model and the notation.

- Period of time and generation $\quad t$
- Number of members of generation t $\quad N(t)$
- Amount of good available in period t $\quad Y(t)$
- Consumption in t of individual i of generation t (i young) $\quad c_t^i(t)$
- Consumption in $t+1$ of individual i of generation t (i old) $\quad c_t^i(t+1)$
- Endowment in t of $i \in N(t)$ $\quad w_t^i(t)$
- Endowment in $t+1$ of $i \in N(t)$ $\quad w_t^i(t+1)$
- Utility function of member i of generation t in t (i young) $\quad u_t^i$

The above model could represent a mobile hunter-gatherer economy, with the good representing food or, more precisely, the food's calorie content.

Each young consumer i of each generation $t \geq 1$ has to choose a consumption basket c_t^i that maximizes his utility function u_t^i. Consequently, each i makes all his decisions when young: if i chooses the basket $\left(c_t^i(t), c_t^i(t+1)\right)$ when young, his consumption $c_t^i(t+1)$ when old has been already determined. Hence, old consumers, who also aim at maximizing their utility, do not make any consumption decision: they just follow the plan established when they were young. This means that the analysis can be restricted to what young individuals do.

Since the endowment i has as a young individual cannot be stored to be used when he becomes old, in the absence of a market for lending and borrowing the good, all individuals will consume their endowments. This would the autarkic solution: for all generation t and $i \in N(t)$, $\left(c_t^i(t), c_t^i(t+1)\right) = \left(w_t^i(t), w_t^i(t+1)\right)$.

2. The market for lending and borrowing

Assume now that a competitive market for lending and borrowing the good is created. Let $r(t) \geq 0$ designate the (real) interest rate in t: lending 1 unit of the good in t implies receiving $1 + r(t)$ units of the good in $t+1$. Define the gross interest rate as $R(t) = 1 + r(t)$. Given that the market is competitive, each consumer i takes $R(t)$ as given.

Intergenerational lending is not possible: old persons in t cannot pay/collect debts in $t+1$, because such persons are not alive in $t+1$. Thus, lending/borrowing can only take place among members of the same generation.

Let $l^i(t)$ designate the lending of consumer i of generation t. It is written $l^i(t)$ instead of $l_t^i(t)$ because i does not lend when old, so $l_t^i(t+1) = 0$. The presumptions that old consumers try to maximize their utility and that their utility coincides with consumption imply that old consumers consume all the units of the good available to them. It therefore does not make sense for an old consumer to lend units of good because he will not be alive to consume the repayment of the loan in the next period. Obviously, old consumers would like to borrow good, since they will not be alive when the loan has to be paid back. The problem is that lenders will know that and, consequently, no lender is willing to lend to an old consumer. The final conclusion of this analysis is that only young consumers participate in the market for lending and borrowing.

3. Defining budget constraints

Consumer i's budget constraint when young is

$$c_t^i(t) + l^i(t) \leq w_t^i(t).$$

Since i is supposed to maximize his utility from consumption, the budget constraint can be assumed to hold with equality: $c_t^i(t) + l^i(t) = w_t^i(t)$. This says that the amount $c_t^i(t)$ of good that young consumer i of generation t consumes plus the amount of good $l^i(t)$ that he lends (if $l^i(t) > 0$) or borrows (if $l^i(t) < 0$) must equal his own endowment $w_t^i(t)$. When old, i's budget constraint is

$$c_t^i(t+1) \leq w_t^i(t+1) + R(t)l^i(t).$$

It follows from the assumption that an old consumer maximizes his utility (his consumption) that the constraint holds as an equality: $c_t^i(t+1) = w_t^i(t+1) + R(t)l^i(t)$.

If $l^i(t) > 0$, then i lends when young and receives $R(t)l^i(t)$ when old. If $l^i(t) < 0$, then i borrows when young and pays back $R(t)l^i(t)$ when old.

By combining the two constraints, $c_t^i(t) + l^i(t) = w_t^i(t)$ and $c_t^i(t+1) = w_t^i(t+1) + R(t)l^i(t)$, the <u>lifetime budget constraint</u> of consumer i obtains.

$$\underbrace{c_t^i(t) + \frac{c_t^i(t+1)}{R(t)}}_{\substack{\textit{present value of} \\ \textit{lifetime consumption}}} = \underbrace{w_t^i(t) + \frac{w_t^i(t+1)}{R(t)}}_{\substack{\textit{present value of} \\ \textit{lifetime endowments}}} \qquad (1)$$

Condition (1) determines the set of consumption baskets $\left(c_t^i(t), c_t^i(t+1)\right)$ that are feasible for consumer i of generation t given endowments $w_t^i = \left(w_t^i(t), w_t^i(t+1)\right)$ and the gross interest rate $R(t)$.

4. The consumer's decision problem

Each consumer $i \in N(t)$, for $t \geq 1$, is assumed to choose a consumption basket c_t^i that <u>maximizes</u> u_t^i given w_t^i and $R(t)$. Formally, i's aim is to

$$\begin{aligned} &\textit{maximize}_{\{c_t^i(t),\, c_t^i(t+1)\}} \ u_t^i\left(c_t^i(t), c_t^i(t+1)\right) \\ &\textit{subject to} \ \ c_t^i(t) + \frac{c_t^i(t+1)}{R(t)} = w_t^i(t) + \frac{w_t^i(t+1)}{R(t)}. \end{aligned} \qquad (2)$$

The utility function u_t^i is typically assumed to satisfy the properties ensuring that indifference curves are differentiable, decreasing, and (strictly) convex (u_t^i is increasing, strictly quasi-concave, and continuously differentiable).

• Method 1 for solving the consumer's problem: Lagrange multiplier approach.

The solution to problem (2) can be obtained by solving problem (3).

$$maximize_{\{c_t^i(t),\, c_t^i(t+1)\}} \; L\left(c_t^i(t), c_t^i(t+1), \lambda\right) = u_t^i\left(c_t^i(t), c_t^i(t+1)\right) + \lambda\left(w_t^i(t) + \frac{w_t^i(t+1)}{R(t)} - c_t^i(t) - \frac{c_t^i(t+1)}{R(t)}\right) \quad (3)$$

The first-order conditions to solve (3) are given by

$$0 = \frac{\partial L}{\partial c_t^i(t)} = \frac{\partial u_t^i}{\partial c_t^i(t)} - \lambda \quad (4)$$

$$0 = \frac{\partial L}{\partial c_t^i(t+1)} = \frac{\partial u_t^i}{\partial c_t^i(t+1)} - \frac{\lambda}{R(t)} \quad (5)$$

$$0 = \frac{\partial L}{\partial \lambda} = w_t^i(t) + \frac{w_t^i(t+1)}{R(t)} - c_t^i(t) - \frac{c_t^i(t+1)}{R(t)}. \quad (6)$$

Condition (6) is the budget constraint (1). Solving for λ in (4), $\lambda = \frac{\partial u_t^i}{\partial c_t^i(t)}$. Solving for λ in (5), $\lambda = \frac{\partial u_t^i}{\partial c_t^i(t+1)} R(t)$. Therefore, $\frac{\partial u_t^i}{\partial c_t^i(t)} = \frac{\partial u_t^i}{\partial c_t^i(t+1)} R(t)$. Equivalently,

$$R(t) = \frac{\partial u_t^i / \partial c_t^i(t)}{\partial u_t^i / \partial c_t^i(t+1)}. \quad (7)$$

The term $\frac{\partial u_t^i}{\partial c_t^i(t)}$ is the marginal utility that the (young) consumer i obtains from the current consumption $c_t^i(t)$. The term $\frac{\partial u_t^i}{\partial c_t^i(t+1)}$ is the marginal utility that the (young) consumer i obtains from the future consumption $c_t^i(t+1)$. The ratio between the two is i's marginal rate of substitution MRS_t^i.

$$MRS_t^i = \frac{\partial u_t^i / \partial c_t^i(t)}{\partial u_t^i / \partial c_t^i(t+1)}$$

Geometrically, the MRS_t^i evaluated at $c_t^i = \left(c_t^i(t), c_t^i(t+1)\right)$ is the slope (in absolute value) of the indifference curve containing the basket c_t^i. The economic interpretation is that MRS_t^i represents the increase in $c_t^i(t+1)$ necessary to keep utility constant given a decrease of $c_t^i(t)$.

To sum up, the solution $\left(c_t^i(t), c_t^i(t+1)\right)$ of consumer i's problem solves equations (1) and (7).

• Method 2 for solving the consumer's problem: direct substitution of (1) into the utility function.

Solving for $c_t^i(t+1)$ in (1) yields $c_t^i(t+1) = R(t)[w_t^i(t) - c_t^i(t)] + w_t^i(t+1)$. After inserting this into the utility function u_t^i, (2) becomes

$$max_{\{c_t^i(t)\}} \; u_t^i\left(c_t^i(t), R(t)[w_t^i(t) - c_t^i(t)] + w_t^i(t+1)\right).$$

Since $c_t^i(t+1) = R(t)[w_t^i(t) - c_t^i(t)] + w_t^i(t+1)$, take the total derivative of u_t^i

$$du_t^i = \frac{\partial u_t^i}{\partial c_t^i(t)} dc_t^i + \frac{\partial u_t^i}{\partial c_t^i(t+1)} \frac{\partial c_t^i(t+1)}{\partial c_t^i(t)} dc_t^i,$$

that is,

$$\frac{du_t^i}{dc_t^i(t)} = \frac{\partial u_t^i}{\partial c_t^i(t)} - \frac{\partial u_t^i}{\partial c_t^i(t+1)} R(t).$$

To maximize u_t^i, it must be that $\frac{du_t^i}{dc_t^i(t)} = 0$. As a result, (7) is obtained. This and $c_t^i(t+1) = R(t)[w_t^i(t) - c_t^i(t)] + w_t^i(t+1)$, which is the budget constraint (1), solve the problem. Hence, both methods involve the same two conditions to solve the consumer's problem: (1) and (7).

5. The savings function

Define the savings $s^i(t)$ of consumer i of generation t as $s^i(t) = w_t^i(t) - c_t^i(t)$: the difference between endowment and consumption. Using (1) and (7), a demand function for consumption when the consumer is young is obtained: $c_t^i(t) = C_t^i\left(w_t^i(t), w_t^i(t+1), R(t)\right)$. This expresses i's consumption (when he is young) as a function of both his liftime endowments $w_t^i(t)$ and $w_t^i(t+1)$, and the gross interest rate $R(t)$. Given C_t^i and $s^i(t) = w_t^i(t) - c_t^i(t)$, it is easy to determine the corresponding savings function $S_t^i\big(w_t^i(t), w_t^i(t+1), R(t)\big)$.

6. A Cobb-Douglas example

Example 1. Each consumer i (young in period t) has utility function $u_t^i = c_t^i(t) \cdot c_t^i(t+1)$. Consumer i's endowments are left unspecified as $w_t^i(t)$ and $w_t^i(t+1)$.

For Example 1, (7) becomes $R(t) = c_t^i(t+1)/c_t^i(t)$. By (1), $c_t^i(t+1) = R(t)[w_t^i(t) - c_t^i(t)] + w_t^i(t+1)$. Accordingly, the demand function for consumption (when young) is $c_t^i(t) = \frac{1}{2}\left(w_t^i(t) + \frac{w_t^i(t+1)}{R(t)}\right)$. Consumption depends positively on wealth and negatively on the interest rate. The corresponding savings function is $s^i(t) = w_t^i(t) - \frac{1}{2}\left(w_t^i(t) + \frac{w_t^i(t+1)}{R(t)}\right) = \frac{1}{2}\left(w_t^i(t) - \frac{w_t^i(t+1)}{R(t)}\right)$. In view of this, a rise in the interest rate stimulates savings (verify that $\frac{\partial s^i(t)}{\partial R(t)} > 0$).

7. General competitive equilibrium

Definition 1. A general competitive equilibrium (GCE) is a sequence $\{\hat{R}(t)\}_{t\geq 1}$ of (gross real) <u>interest rates</u> and a <u>consumption allocation</u> $\{\hat{c}_t^i\}_{t\geq 0, i\in N(t)}$ such that:

E1. for every period $t \geq 1$ and $i \in N(t)$, $\hat{c}_t^i$ <u>maximizes</u> u_t^i given $\hat{R}(t)$ and i's endowments w_t^i (for $t = 0$, $\hat{c}_t^i$ is just the available wealth of the old); and

E2. for every period $t \geq 1$, the <u>goods market clearing condition is satisfied</u>, $\sum_{i\in N(t)} \hat{c}_t^i(t) + \sum_{i\in N(t-1)} \hat{c}_{t-1}^i(t) = Y(t) = \sum_{i\in N(t)} w_t^i(t) + \sum_{i\in N(t-1)} w_{t-1}^i(t)$.

Condition E1 asserts that, in every period $t \geq 1$ and for each young consumer i in t, $\hat{c}_t^i$ is the value of i's <u>demand function for consumption</u> given $\hat{R}(t)$ and i's endowment vector w_t^i. In other words, E1 simply says that $\hat{c}_t^i$ is obtained using (1) and (7).

Condition E2 holds that <u>the market for the good is in equilibrium</u> in every period $t \geq 1$. That is, for each $t \geq 1$, the aggregate (or total) demand for the good $\sum_{i\in N(t)} \hat{c}_t^i(t) + \sum_{i\in N(t-1)} \hat{c}_{t-1}^i(t)$ equals the aggregate (or total) supply of the good $\sum_{i\in N(t)} w_t^i(t) + \sum_{i\in N(t-1)} w_{t-1}^i(t)$. Since there is no production of the good, total supply coincides with the total endowment $Y(t)$.

There are only two markets in this economy: the market for the good and the market for loans of the good. As only those who are young in t lend or borrow in t, the loan market is in equilibrium when $\sum_{i\in N(t)} l^i(t) = 0$. By Walras' law, condition $\sum_{i\in N(t)} l^i(t) = 0$ could replace condition E2 in Definition 1.

It follows from the budget constraint $c_t^i(t) + l^i(t) = w_t^i(t)$ when i is young that $l^i(t) = w_t^i(t) - c_t^i(t)$. On the other hand, by definition, $s^i(t) = w_t^i(t) - c_t^i(t)$. As a consequence, $l^i(t) = s^i(t)$: what i lends or borrows is exactly what i saves. When i lends ($l^i(t) > 0$), i is saving; when i borrows ($l^i(t) < 0$), i is dissaving. The point is that $\sum_{i\in N(t)} l^i(t) = 0$ amounts to $\sum_{i\in N(t)} s^i(t) = 0$: in a GCE, <u>aggregate savings must be zero</u> (total lending should equal total borrowing).

As a result of the previous analysis, if $\{\hat{R}(t)\}$ and $\{\hat{c}_t^i\}$ constitute a GCE, then, for each $\hat{R}(t)$, $\sum_{i\in N(t)} S^i\left(w_t^i(t), w_t^i(t+1), \hat{R}(t)\right) = 0$.

Conversely, if $\{\hat{R}(t)\}$ is such that, for all $\hat{R}(t)$, $\sum_{i\in N(t)} S^i\left(w_t^i(t), w_t^i(t+1), \hat{R}(t)\right) = 0$, then, for some $\{\hat{c}_t^i\}$, $\{\hat{R}(t)\}$ and $\{\hat{c}_t^i\}$ constitute a GCE.

8. An example on computing a GCE

Example 2. For all t: $\left(w_t^i(t), w_t^i(t+1)\right) = (4, 1)$; $u_t^i = c_t^i(t) \cdot c_t^i(t+1)$; and $N(t) = 200$.

Example 2 has the same utility functions as Example 1. As shown in Section 6, $s^i(t) = \frac{1}{2}\left(w_t^i(t) - \frac{w_t^i(t+1)}{R(t)}\right)$. Thus, the equilibrium condition (given by E2) $0 = \sum_{i\in N(t)} s^i(t) = 100\left(4 - \frac{1}{R(t)}\right)$ leads to $R(t) = 1/4$.

With $R(t) = 1/4$, for all i, $s^i(t) = \frac{4-1\cdot 4}{2} = 0$. This means that <u>no individual saves</u>: there is no lending nor borrowing and, accordingly, consumption in each period coincides with the endowment in that period.

9. Failure of Pareto efficiency

Definition 2. A consumption allocation $C = \{c_t^i\}_{t\geq 0, i\in N(t)}$ is Pareto efficient if there does not exist another consumption allocation $\tilde{C} = \{\tilde{c}_t^i\}_{t\geq 0, i\in N(t)}$ such that:

P1. for some $t \geq 1$ and some $i \in N(t)$, $u_t^i(\tilde{c}_t^i) > u_t^i(c_t^i)$; and

P2. for all $t \geq 1$ and all $i \in N(t)$, $u_t^i(\tilde{c}_t^i) \geq u_t^i(c_t^i)$.

C Pareto efficient means that for no other $\tilde{C}$ some consumer i has more utility and no consumer i has less utility.

The First Fundamental Welfare Theorem states that, for the standard atemporal Walrasian economies, the allocation of goods in a Walrasian (or general competitive) equilibrium is Pareto efficient. For economies represented by overlapping generation models this result fails: the consumption allocation of a GCE need not be Pareto efficient.

For instance, in Example 2, the GCE is such that $\left(c_t^i(t), c_t^i(t+1)\right) = (4,1)$ for all $t \geq 1$ (the old in period 1 consume $w_0^i(1) = 1$). It will be shown that this consumption allocation is not Pareto efficient. To this end, for each period $t \geq 1$, define a bijection $\beta_t : N(t) \to N(t-1)$. This bijection exists because the sets $N(t)$ and $N(t-1)$ have the same number of members: 200. Each mapping β_t matches every young consumer $i \in N(t)$ in period t with some old consumer $\beta_t(i) \in N(t-1)$ from the same period t.

Let a new consumption allocation $\tilde{C}$ be defined from the GCE consumption allocation by letting each young consumer i transfer ε units of the good to the old consumer $\beta_t(i)$. The old are all clearly better off in $\tilde{C}$: whereas in the GCE consumption allocation each old consumer of each generation consumes 1 unit of the good, in $\tilde{C}$ each such individual consumes $1 + ε$.

As regards the young consumers, the utility of each young consumer in the GCE consumption allocation is $u_t^i(4,1) = 4$. After the transfer, in $\tilde{C}$, the utility is $u_t^i(4-ε, 1+ε) = (4-ε)(1+ε) = 4 + ε(3-ε)$. For sufficiently small ε (specifically, for $ε < 3$), $u_t^i(4-ε, 1+ε) > 4$. Consequently, all the young consumers are better off in $\tilde{C}$. This proves that the GCE allocation is not Pareto efficient.

10. Exercises

Ejercicio 1. Consumo y ahorro. Obtén las funciones de consumo y ahorro si:

(i) $u_t^i\left(c_t^i(t), c_t^i(t+1)\right) = c_t^i(t)^{\alpha} \cdot c_t^i(t+1)^{\beta}$, donde α y β son constantes positivas;

(ii) $u_t^i\left(c_t^i(t), c_t^i(t+1)\right) = c_t^i(t)^{\alpha} + c_t^i(t+1)^{\beta}$.

Ejercicio 2. Equilibrio. Determina el equilibri general competitivo en el caso (i) del Ejercicio 1 si los consumidores mayores no tienen dotación, los jóvenes tienen una unidad y hay 100 consumidores en cada generación.

Ejercicio 3. Equilibrio y Paretoeficiencia. Verifica que el equilibrio general competitivo de la siguiente economía no es Paretoeficiente:

(i) cada generación está formada por 100 consumidores;

(ii) la dotación de cada consumidor joven es 2;

(iii) la dotación de cada consumidor mayor es 1; y

(iv) para todo i y t, $u_t^i\left(c_t^i(t), c_t^i(t+1)\right) = c_t^i(t) \cdot c_t^i(t+1)$.

Ejercicio 4. Equilibrio con ladrones. Cada generación consta de 100 miembros: 80 ("los pobres") con dotación $(1, 0)$ y los otros 20 ("los ricos") con dotación $(4, 2)$. Todos los consumidores jóvenes de todas las generaciones tienen la función de utilidad $u_t^i = c_t^i(t) \cdot c_t^i(t+1)$.

(i) Halla el vector de consumo de equilibrio de cada consumidor joven.

(ii) Para cada período, determina el vector de consumo agregado de equilibri del total de ricos (y del total de pobres) del período.

Para todo t, cada joven pobre en t roba b unidades del grupo de ricos en t. El robo agregado asciende a $80 \cdot b$, donde $3 \cdot b$ unidades se toman de cada joven rico y b unidades de cada rico mayor. Por consiguiente, el total apropiado por los jóvenes ($80 \cdot b$) coincide con lo que pierden los ricos ($20 \cdot 3 \cdot b + 20 \cdot b$).

(iii) Vuelve a responder a (i) y a (ii) si $b = 1$. ¿Causa el robo un aumento o una reducción de la desigualdad?

11. References

McCandless, G. T. and N. Wallace (1991): *Introduction to Dynamic Macroeconomic Theory: An Overlapping Generations Approach*, Harvard University Press, Harvard (MA).