Overlapping generations model with endogenous production

1. Representing endogenous production

The main new features of the model are described next; see, for instance, McCandless and Wallace (1991, ch. 9), Acemoglu (2009, ch. 9), Heijdra (2009, ch. 17), and/or Wickens (2008, sec. 6.3).

- People are endowed with labour, not goods.

The lifetime endowment of labour of member i of generation t is denoted by $L_t^i = \left(L_t^i(t), L_t^i(t+1)\right)$, where $L_t^i(t)$ is the amount of labour when i is young and $L_t^i(t+1)$ when old.

In each period t, there is a competitive labour market where people can sell their labour in exchange for a wage $\omega(t)$ paid in good units.

People only care about consumption, not leisure. They (inelastically) supply all their labour in both periods of their life. The total amount of labour $L(t)$ in period t is

$$\sum_{i\in N(t)} L_t^i(t) + \sum_{i\in N(t-1)} L_{t-1}^i(t)\ .$$

- Time t good can be stored from t to $t+1$. The good stored at $t-1$ is called time t capital.

In $t=1$, there is an initial endowment of capital $K(1)$. Every young individual may save a part $K^i(t)$ of his wage $\omega(t)$. $K^i(t)$ is the capital owned in t (when old) by member i of generation $t-1$.

The aggregate savings $\sum_{i\in N(t)} K^i(t)$ in t become the capital stock $K(t+1)$ in $t+1$. All capital available in t depreciates (is completely used up) during t.

- Time t good can be produced by using time t labour and time $t-1$ good. This process is represented by a production function.

A production function takes the form $Y(t) = G\left(A(t), K(t), L(t)\right)$, where $A(t)$ represents the state of technology in t, $L(t)$ is total labour in t, and $K(t)$ is the capital stock in t. For simplicity, for each t, $Y(t) = A(t)\cdot F\left(K(t), L(t)\right)$.

The production function F displays constant returns to scale if, for all $\delta > 0$,

$$F\big(\delta \cdot K(t),\ \delta \cdot L(t)\big) = \delta \cdot F\big(K(t), L(t)\big).$$

Marginal productivities are positive but decreasing: $\frac{\partial F}{\partial K(t)} > 0$, $\frac{\partial F}{\partial L(t)} > 0$, $\frac{\partial^2 F}{\partial K(t)^2} < 0$, and $\frac{\partial^2 F}{\partial L(t)^2} < 0$.

- Production activities are carried out by a new type of agent: firms.

There are many profit-maximizing competitive firms with the same production function. Competitiveness and constant returns imply that firms employ K and L in the same proportion.

This means that all of them can be viewed as larger or smaller copies of a given firm. Hence, given the state of technology, total production $Y(t)$ in t is a function of total capital $K(t)$ and labour $L(t)$ in t. The typical production function is Cobb-Douglas, $Y(t) = A(t) \cdot K(t)^{\alpha} \cdot L(t)^{1-\alpha}$, where $A(t)$ captures the state of technology.

Total production $Y(t)$ in t is: (i) obtained from total labour $L(t)$ and total capital $K(t)$ available in t; and (ii) is either consumed or accumulated for the next period. Formally,

$$\textstyle\sum_{i\in N(t)} c_t^i(t) + \sum_{i\in N(t-1)} c_{t-1}^i(t) + \sum_{i\in N(t)} K^i(t+1) = A(t) \cdot F\big(K(t), L(t)\big)$$

or

$$C(t) + K(t+1) = A(t) \cdot F\big(K(t), L(t)\big).$$

Assumptions: $\frac{\partial F}{\partial K(t)} \to \infty$ if $K(t) \to 0$, $\frac{\partial F}{\partial K(t)} \to 0$ if $K(t) \to \infty$, and the same for $L(t)$.

Since the labour market is competitive, the wage rate equals the marginal productivity of labour: $\omega(t) = \partial F / \partial L(t)$.

The capital market is also assumed to be competitive, so the price $\sigma(t)$ of capital equals the marginal productivity of capital: $\sigma(t) = \partial F / \partial K(t)$.

Constant returns guarantee that ω and σ depend on the relative, not the absolute, amounts of K and L.

2. Cobb-Douglas example

Let $Y(t) = A(t) \cdot K(t)^{\alpha} \cdot L(t)^{1-\alpha}$. Then:

$$\omega(t) = \frac{\partial F}{\partial L(t)} = (1-\alpha) \cdot A(t) \cdot \left(\frac{K(t)}{L(t)}\right)^{\alpha}$$

$$\sigma(t) = \frac{\partial F}{\partial K(t)} = \alpha \cdot A(t) \cdot \left(\frac{L(t)}{K(t)}\right)^{1-\alpha}$$

By the uniqueness of the input prices, all firms use K and L in the same proportion: firms using more K will be using more L.

Since all labour is hired, the total wage bill is $\omega(t) \cdot L(t) = (1-\alpha) \cdot A(t) \cdot \left(\frac{K(t)}{L(t)}\right)^{\alpha} \cdot L(t) = (1-\alpha) \cdot Y(t)$. Similarly, $\sigma(t) \cdot K(t) = \alpha \cdot Y(t)$.

This says that the total payment to labour is the fraction $1-\alpha$ of output, whereas the total payment to capital is the fraction α. As a result,

$$\omega(t) \cdot L(t) + \sigma(t) \cdot K(t) = Y(t).$$

Production is distributed between labour and capital in fixed proportions. This holds for production functions with constant returns. Another implication of this result is that firms earn no profit.

3. General competitive equilibrium and steady (stationary) state

Every individual i aims at maximizing his or her utility subject to his or her lifetime budget constraint. When young and old, i's budget constraints are

$$c_t^i(t) + l^i(t) + K^i(t+1) = \omega(t) \cdot L_t^i(t)$$

$$c_t^i(t+1) = R(t) \cdot l^i(t) + \sigma(t+1) \cdot K^i(t+1) + \omega(t+1) \cdot L_t^i(t+1).$$

By combining the two constraints,

$$c_t^i(t) + \frac{c_t^i(t+1)}{R(t)} = \omega(t) \cdot L_t^i(t) + \frac{\omega(t+1) \cdot L_t^i(t+1)}{R(t)} + K^i(t+1) \cdot \left(\frac{\sigma(t+1)}{R(t)} - 1\right).$$

If $\sigma(t+1) > R(t)$, then everyone would like to borrow as much of the good to invest in capital. This cannot be in equilibrium, because no one would lend.

If $\sigma(t+1) < R(t)$, nobody would like to hold capital, so $K(t+1) = 0$. This makes the marginal productivity of K, arbitrarily large. Hence, $\sigma(t+1)$ is also arbitrarily large, contradicting the assumption that $\sigma(t+1) < R(t)$.

Therefore, in equilibrium, only $\sigma(t+1) = R(t)$ is possible, for which reason $K^i(t+1)\left(\frac{\sigma(t+1)}{R(t)} - 1\right) = 0$.

The decision problem of every $i \in N(t)$ is the same as with exogenous production (see (2) in the Lecture 1 on the OLG model with just private lending) because the lifetime budget constraints in the two cases are analogous: endowments $w_t^i(s)$ are now the wage incomes $\omega(s) \cdot L_t^i(s)$.

The only qualification to be made is that $\omega(t+1)$ is not known in t (and neither is $\sigma(t+1)$ known). Accordingly, for both problems to be the same, it is necessary to postulate perfect foresight: individuals know in each period t the market prices prevailing in $t+1$.

A general competitive equilibrium (with initial capital $K(1) > 0$, production function F, labour endowments, and perfect foresight) is a sequence $\{\hat{R}(t),\ \hat{\sigma}(t),\ \hat{\omega}(t),\ \hat{K}(t)\}_{t \geq 1}$ such that, for all $t \geq 1$:

(i) $S_t\left(\hat{R}(t)\right) = \hat{K}(t+1)$, where S_t is the total savings function obtained by maximizing each individual's utility;

(ii) $\hat{\sigma}(t+1) = \hat{R}(t)$;

(iii) $\hat{\sigma}(t) = \partial F / \partial K(t)$; and

(iv) $\hat{\omega}(t) = \partial F / \partial L(t)$.

A steady state of the economy is characterized by the condition $K(t+1) = K(t)$.

4. Example on computing the general competitive equilibrium

Let $u_t^i = c_t^i(t) \cdot c_t^i(t+1)$ and $Y(t) = A(t) \cdot K(t)^{\alpha} \cdot L(t)^{1-\alpha}$. Then, defining $L_t(s) = \sum_{i \in N(s)} L_t^i(s)$ and $L(t) = L_t(t) + L_{t-1}(t)$,

$$S_t = \frac{\omega(t) \cdot L_t(t)}{2} - \frac{\omega(t+1) \cdot L_t(t+1)}{2 \cdot R(t)}$$

$$\omega(t) = (1-\alpha) \cdot A(t) \cdot \left(\frac{K(t)}{L(t)}\right)^{\alpha}$$

$$\sigma(t) = \alpha \cdot A(t) \cdot \left(\frac{L(t)}{K(t)}\right)^{1-\alpha}$$

Substituting all three equations into the equilibrium condition $S_t = K(t+1)$ and solving for $K(t+1)$,

$$K(t+1) = \left(\frac{\frac{(1-\alpha)\cdot A(t)}{2} \cdot \frac{L_t(t)}{L(t)^{\alpha}}}{1 + \frac{1-\alpha}{2\alpha} \cdot \frac{L_t(t+1)}{L(t+1)}} \right) \cdot K(t)^{\alpha}\,. \qquad (1)$$

• **Case 1**: population and technology constant.

If A, L, and L_t all remain constant, then the term within the parenthesis in (1) is a positive constant. Denote this constant by a. The equation describing the dynamics of capital accumulation in equilibrium is then

$$K(t+1) = a \cdot K(t)^{\alpha}.$$

The steady state capital stock $\bar{K}$ is obtained when $K(t+1) = K(t) = \bar{K}$. That is, $\bar{K} = a \cdot \bar{K}^{\alpha}$. Accordingly,

$$\bar{K} = a^{1/(1-\alpha)}\,.$$

Fig. 1 next represents $\bar{K}$ and the equation $K(t+1) = a \cdot K(t)^{\alpha}$. No matter the initial stock $K(1) > 0$, the economy's capital stock converges to $\bar{K}$.

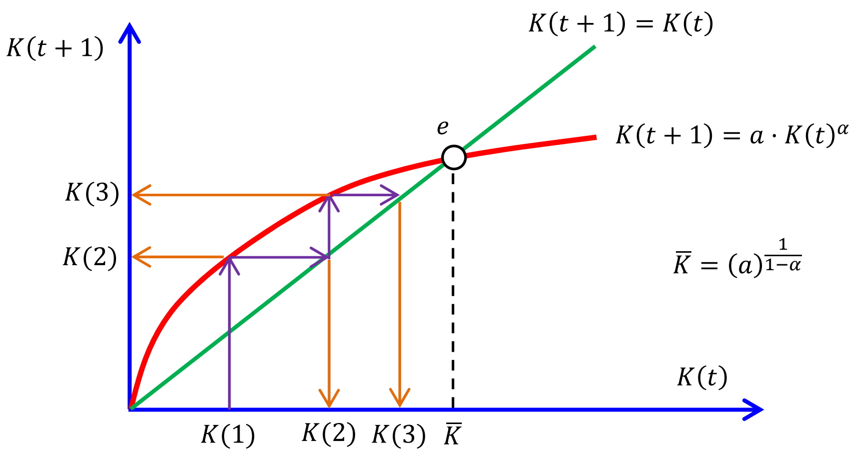

Fig. 1. Capital stock dynamics and convergence to the steady state (with technology fixed and population constant)

Once found a steady state value $\bar{K}$, then, assuming L and A constant, the value $\bar{Y}$ of output in the steady state can also be found: $\bar{Y} = A \cdot \bar{K}^{\alpha} \cdot L^{1-\alpha}$. Knowing this, both $\bar{\omega}$ and $\bar{\sigma}$ can be subsequently determined.

From the equilibrium condition $S_t = K(t+1)$, it follows that $\bar{S} = \bar{K}$. Given this, as S_t is a function of $R(t)$, $\bar{R}$ can also be ascertained (in equilibrium, $\bar{R} = \bar{\sigma}$).

- **Case 2**: population grows and technology constant.

With everything else the same as in Case 1, let $N(t+1) = N \cdot N(t)$, for some $N > 1$, and all generations be endowed with the same amount of labour.

Let $L_0(0)$ be the labour endowment of the young in $t = 0$ and $L_0(1)$ the labour of the old in $t = 1$. Define $L(0) = L_0(0) + L_0(1)/N$.

The total labour endowment (supply) of the young in t is

$$L_t(t) = N^t \cdot L_0(0)$$

and the labour endowment of the old in t is

$$L_{t-1}(t) = N^{t-1} \cdot L_0(1).$$

Therefore, total labour supply in t is

$$L(t) = L_t(t) + L_{t-1}(t) = N^t \cdot L_0(0) + N^{t-1} \cdot L_0(1) = N^t \left(L_0(0) + \frac{L_0(1)}{N} \right) = N^t \cdot L(0).$$

The savings function of each individual i in t is

$$s^i(t) = \frac{1}{2}\left(\omega(t) \cdot L_t^i(t) - \frac{\omega(t+1) \cdot L_t^i(t+1)}{R(t)} \right).$$

Aggregate savings in t are

$$S_t = N(t) \cdot s^i(t) = N^t \cdot N(0) \cdot s^i(t) = \frac{1}{2}\left(\omega(t) \cdot N^t \cdot L_0(0) - \frac{\omega(t+1) \cdot N^t \cdot L_0(1)}{R(t)} \right).$$

The wage in t is

$$\omega(t) = \frac{\partial F}{\partial L(t)} = (1-\alpha) \cdot A(t) \cdot \left(\frac{K(t)}{N^t \cdot L(0)} \right)^{\alpha}.$$

The price of capital in $t+1$ (which equals $R(t)$ in equilibrium) is

$$\sigma(t+1) = \frac{\partial F}{\partial K(t+1)} = \alpha \cdot A(t) \cdot \left(\frac{K(t+1)}{N^{t+1} \cdot L(0)}\right)^{\alpha-1}.$$

Using these equations and the equilibrium condition $S_t = K(t+1)$, or simply recalling that,

$$K(t+1) = \left(\frac{\frac{(1-\alpha)\cdot A(t)}{2} \cdot \frac{L_t(t)}{L(t)^\alpha}}{1 + \frac{1-\alpha}{2\alpha} \cdot \frac{L_t(t+1)}{L(t+1)}}\right) \cdot K(t)^\alpha$$

which is the equation (1) describing the equilibrium path of capital,

$$K(t+1) = \left(\frac{\frac{(1-\alpha)\cdot A(0)}{2} \cdot \frac{L_0(0)}{L(0)^\alpha}}{1 + \frac{1-\alpha}{2\alpha} \cdot \frac{L_0(1)}{N \cdot L(0)}}\right) \cdot N^{t(1-\alpha)} \cdot K(t)^\alpha.$$

Denoting by B the term in parenthesis, the final conclusion is

$$K(t+1) = B \cdot N^{t(1-\alpha)} \cdot K(t)^\alpha.$$

The gross growth rate of capital is

$$G_K(t+1) = \frac{K(t+1)}{K(t)} = \frac{B \cdot N^{t(1-\alpha)} \cdot K(t)^\alpha}{B \cdot N^{(t-1)(1-\alpha)} \cdot K(t-1)^\alpha} = \frac{1}{N^{\alpha-1}} \cdot G_K(t)^\alpha = N^{1-\alpha} \cdot G_K(t)^\alpha.$$

Let G_K designate the limit of the gross growth rate of capital. As a result,

$$G_K = N^{1-\alpha} \cdot {G_K}^\alpha .$$

Solving for G_K, ${G_K}^{1-\alpha} = N^{1-\alpha}$. In sum,

$$G_K = N .$$

To recap, in the equilibrium steady state, capital accumulates at the same rate as population grows: $K(t+1) = N \cdot K(t)$.

The growth rate of the capital stock K and the growth rate of ouput Y eventually equal the growth rate of the population.

• **Case 3**: population constant and technology grows.

Suppose now that technology improves at gross rate $G > 1$: $A(t+1) = G \cdot A(t)$. Since $Y = A \cdot K^{\alpha} \cdot L^{1-\alpha}$, technological growth is called neutral, due to the fact that changes in A affect the productivity of both capital and labour.

Given $A(t) = G^t \cdot A(0)$ and constant population, the equilibrium path of capital (1) becomes

$$K(t+1) = \left(\frac{\frac{(1-\alpha)\cdot A(0)}{2} \cdot \frac{L_0(0)}{L(0)^{\alpha}}}{1 + \frac{1-\alpha}{2\alpha} \cdot \frac{L_0(1)}{N \cdot L(0)}} \right) \cdot G^t \cdot K(t)^{\alpha}.$$

Denoting by B the term in parenthesis, $K(t+1) = B \cdot G^t \cdot K(t)^{\alpha}$.

The gross growth rate of capital is

$$G_K(t+1) = \frac{K(t+1)}{K(t)} = \frac{B \cdot G^t \cdot K(t)^{\alpha}}{B \cdot G^{t-1} \cdot K(t-1)^{\alpha}} == G \cdot G_K(t)^{\alpha}.$$

If G_K is the limit of $G_K(t)$, $G_K = G \cdot {G_K}^{\alpha}$and

$$G_K = G^{\frac{1}{1-\alpha}}.$$

As $\frac{1}{1-\alpha} > 1$, $G_K > G$: the capital stock growth rate (which equals the ouput growth rate) is greater than the technology growth rate.

5. Exercises

Ejercicio 1. Equilibrio con producción endógena. La función de utilidad de cada individuo joven i es $u_t^i = ln\, c_t^i(t) + \beta \cdot ln\, c_t^i(t+1)$, donde $0 < \beta < 1$. Cada generación está formada por 100 individuos, 50 con dotación $(0,1)$ y 50 con dotación $(2,0)$. La función de producción es $Y(t) = K(t)^{\alpha} L(t)^{1-\alpha}$ y $K(1) > 0$.

(i) Determina la ecuación en diferencias que establece la trayectoria del stock de capital.

(ii) Calcula un estado estacionario con stock de capital positivo y el equilibrio general correspondiente.

(iii) Responde a los apartados (i) y (ii) si, para todo t, la generación $t+1$ tiene un 50 % más de miembros que la generación t.

(iv) Responde a los apartados (i) y (ii) si, para todo t, si en el período 2 fallece la mitad de los jóvenes de cada tipo.

(v) Responde a los apartados (i) y (ii) si, para todo t, si en el período 2 se destruye la mitad del stock de capital.

Ejercicio 2. Evasión fiscal. Cada generación consta de 100 miembros: 50 ("los pobres") con dotación de trabajo $(1,0)$ y los otros 50 ("los ricos") con dotación de trabajo $(4,0)$. Todos los jóvenes de todas las generaciones tienen la misma función de utilidad $u_t^i = c_t^i(t) \cdot c_t^i(t+1)$. No hay capital: la producción sólo depende del trabajo: $Y(t) = L(t)^{1/2}$. El salario es $\omega(t) = L(t)^{-1/2}$.

Hay un gobierno que establece un impost τ a pagar por los ricos jóvenes. Para cada t, la recaudación tributaria en t se distribuye entre todos los mayores en t (sistema de pensiones de reparto). Cada individuo mayor recibirà $\tilde{\tau}$.

Los ricos jóvenes pueden dedicar una parte x de su dotación de trabajo a evadir el pago del impuesto. Cuando un rico emplea x para defraudar el pago, acaba pagando $\tau \cdot g(x)$ en lugar de τ, donde $g(x) = \left(1 - \frac{x}{4}\right)^2$.

En cada período t, el presupuesto del gobierno está equilibrado: los ingresos tributarios obtenidos de los ricos son iguales a las transferencias realizadas a los mayores $(100 \ \cdot \tilde{\tau})$. Los ingresos provenientes de los ricos no son necesariamente $50 \cdot \tau$ dado que que hay que determinar el nivel de evasión fiscal que deciden los ricos. Halla la ecuación que determina $\tilde{\tau}$ en función de τ y calcula $\tilde{\tau}$ si $\tau = 1$.

Ejercicio 3. Independencia. En la economía hay un único bien, que puede acumular-se sólo un período en forma de capital (sin depreciación) y que puede producirse combinando los factores capital y trabajo. El bien puede ser prestado en un mercado de préstamos. Se asume que el arbitraje entre préstamos y capital iguala la rentabilidad de prestar el bien y de acumular capital.

Cada generación está formada per dos grupos, 1 y 2. El grupo 1 integra $2 \cdot n$ individuos idénticos, cada uno con una unidad de trabajo de joven y dos unidades de trabajo de mayor. La función de utilidad de cada joven del grupo 1 en el período t es $u_t = c_t \cdot (c_{t+1})^{\beta}$, donde c_t es el consumo que el individuo realiza de joven, c_{t+1} el consumo que el mismo individuo tendrá de mayor y $0 < \beta < 1$. La función d'utilidad de cada mayor coincide con su consumo.

El grupo 2 está formado por n individuos idénticos, cada uno con cuatro unidades de trabajo de joven y dos unidades de mayor. La función de utilidad de cada joven del grupo 2 en el período t es $u_t = c_t \cdot c_{t+1}$, donde c_t es el consumo que el individuo realiza de joven y c_{t+1} el consum que el individuo tendrá de mayor. La función d'utilidad de cada mayor coincide con su consumo.

La función de producción de la economía en cada período t es $Y_t = K_t \cdot L_t$, donde K_t es el capital total en t y L_t es la cantidad total de trabajo ofrecida en t. Todos los individuos ofrecen su dotación de trabajo. La remuneración del capital es la mitad de la productividad marginal del capital. La remuneración del trabajo es la mitad de la productividad marginal del treball. Se asumeix que, por arbitraje, el tipo de interès bruto de un préstamo en el período t coincide con la remuneración del capital en el período $t + 1$.

(i) Determina la ecuación que describe la trayectoria de acumulación del capital y representala gráficamente.

(ii) Imagina que los miembros del grupo 2 se independizan y constituyen una economía pròpia, separada de la economía que formarían los miembros del grupo 1. En cada economía se mantienen las dotaciones de los miembros de los grupos respectivos, la función de producción de la economía original y las reglas que dictan la remuneración de cada factor de producción. Determina la ecuación que representa la trayectoria de acumulación del capital de cada economia y compárala con la obtnieda en el apartado (i) para juzgar si a alguno de los grupos le conviene la secesión [supón que conviene si se puede acumular más capital].

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