Some more growth models

1. The Mankiw-Romer-Weil model (the Solow-Swan model with human capital)

For any variable, V will be written instead of $V(t)$, whereas V' will be written instead of $V(t+1)$. With $\alpha + \hat{\alpha} < 1$, H designating human capital, and technological progress assumed labour-augmenting, the production function is

$$Y = K^{\alpha} \cdot H^{\hat{\alpha}} \cdot (A \cdot L)^{1-\alpha-\hat{\alpha}}.$$

Notation

N = gross rate of growth of L

$\mathbb{s}$ = propensity to accumulate K

δ = rate at which K depreciates

G = gross rate of growth of A

$\hat{\mathbb{s}}$ = propensity to accumulate human capital

$\hat{\delta}$ = rate at which human capital depreciates

Per capita variables are defined in effective labour units: $k = \frac{K}{A \cdot L}$, $h = \frac{H}{A \cdot L}$, and $y = \frac{Y}{A \cdot L}$.

$$y = \frac{Y}{A \cdot L} = \frac{K^{\alpha} \cdot H^{\hat{\alpha}} \cdot (A \cdot L)^{1-\alpha-\hat{\alpha}}}{A \cdot L} = \frac{K^{\alpha} \cdot H^{\hat{\alpha}}}{(A \cdot L)^{\alpha+\hat{\alpha}}} = \frac{K^{\alpha}}{(A \cdot L)^{\alpha}} \cdot \frac{H^{\hat{\alpha}}}{(A \cdot L)^{\hat{\alpha}}} = \left(\frac{K}{A \cdot L}\right)^{\alpha} \left(\frac{H}{A \cdot L}\right)^{\hat{\alpha}} = k^{\alpha} \cdot h^{\hat{\alpha}}.$$

As in the SS model, $K' = \mathbb{s} \cdot Y + (1-\delta)K$. After dividing both sides by $A' \cdot L'$,

$$k' = \frac{K'}{A' \cdot L'} = \frac{\mathbb{s} \cdot Y}{A' \cdot L'} + \frac{(1-\delta) \cdot K}{A' \cdot L'} = \mathbb{s} \cdot \frac{K^{\alpha} \cdot H^{\hat{\alpha}} \cdot (A \cdot L)^{1-\alpha-\hat{\alpha}}}{(G \cdot A) \cdot (N \cdot L)} + \frac{(1-\delta) \cdot K}{(G \cdot A) \cdot (N \cdot L)} =$$

$$= \frac{\mathbb{s}}{G \cdot N} \cdot \frac{K^{\alpha} \cdot H^{\hat{\alpha}} \cdot (A \cdot L)^{1-\alpha-\hat{\alpha}}}{A \cdot L} + \frac{1-\delta}{G \cdot N} \cdot \frac{K}{A \cdot L} = \frac{\mathbb{s}}{G \cdot N} \cdot \frac{K^{\alpha} \cdot H^{\hat{\alpha}}}{(A \cdot L)^{\alpha+\hat{\alpha}}} + \left(\frac{1-\delta}{G \cdot N}\right) \cdot k =$$

$$= \frac{\mathbb{s}}{G \cdot N} \cdot \frac{K^{\alpha}}{(A \cdot L)^{\alpha}} \frac{H^{\hat{\alpha}}}{(A \cdot L)^{\hat{\alpha}}} + \frac{1-\delta}{G \cdot N} \cdot k = \frac{\mathbb{s}}{G \cdot N} \cdot k^{\alpha} \cdot h^{\hat{\alpha}} + \frac{1-\delta}{G \cdot N} \cdot k = \frac{\mathbb{s}}{G \cdot N} \cdot y + \frac{1-\delta}{G \cdot N} \cdot k.$$

By substracting k from both sides,

$$\Delta k = \frac{\mathbb{s}}{G \cdot N} \cdot y - \frac{\delta + G \cdot N - 1}{G \cdot N} \cdot k.$$

This equation coincides with the one from the SS model (with $G = 1 + a$ and $N = 1 + n$). For $\Delta k = 0$ it must be that $\mathbb{s} \cdot y = (\delta + G \cdot N - 1) \cdot k$. That is,

$$\mathbb{s} \cdot k^{\alpha} \cdot h^{\hat{\alpha}} = (\delta + G \cdot N - 1) \cdot k$$

Solving for h,

$$h(t) = \left(\frac{\delta + GN - 1}{\mathbb{s}}\right)^{1/\hat{\alpha}} \cdot k(t)^{(1-\alpha)/\hat{\alpha}}.$$

The above equation represents the condition $\Delta k(t) = 0$, represented in Fig. 1.

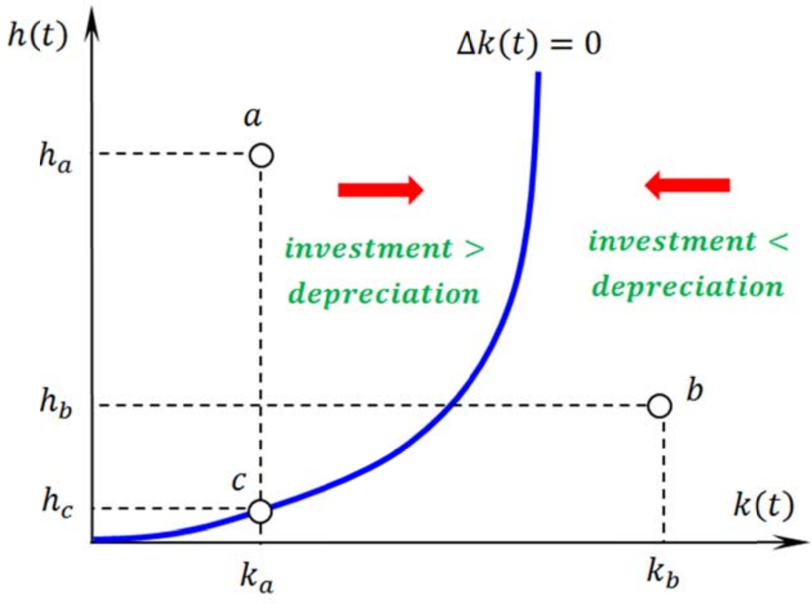

Fig. 1. Graphical representation of $\Delta k(t) = 0$

At points like a in Fig. 1, investment in k is higher than depreciation of k, so k increases.

Investment is higher than depreciation at a because, given k_a, it is enough to have per capita human capital equal to h_c for investment to equal depreciacion (that is, for $\Delta k(t) = 0$ to hold).

As $h_b > h_c$, there is a human capital excess creating too much output, which generates too much investment (in comparison with the depreciation corresponding to k_a).

Similarly, k decreases at points to the right of the curve $\Delta k(t) = 0$ (like point b in Fig. 1).

On the other hand, starting with

$$H' = \hat{\mathbb{s}} \cdot Y + \left(1 - \hat{\delta}\right)H$$

a similar procedure leads to

$$\Delta h = \frac{\hat{\mathbb{s}}}{G \cdot N} \cdot y - \frac{\hat{\delta} + G \cdot N - 1}{G \cdot N} \cdot h\,.$$

For $\Delta h = 0$ it must be that

$$\hat{\mathbb{s}} \cdot k^{\alpha} \cdot h^{\hat{\alpha}} = \left(\hat{\delta} + G \cdot N - 1\right) \cdot h\,.$$

Solving for h, the condition representing $\Delta h(t) = 0$, represented in Fig. 2, is

$$h(t) = \left(\frac{\hat{\mathbb{s}}}{\hat{\delta} + GN - 1}\right)^{1/(1-\hat{\alpha})} \cdot k(t)^{\alpha/(1-\hat{\alpha})}\,.$$

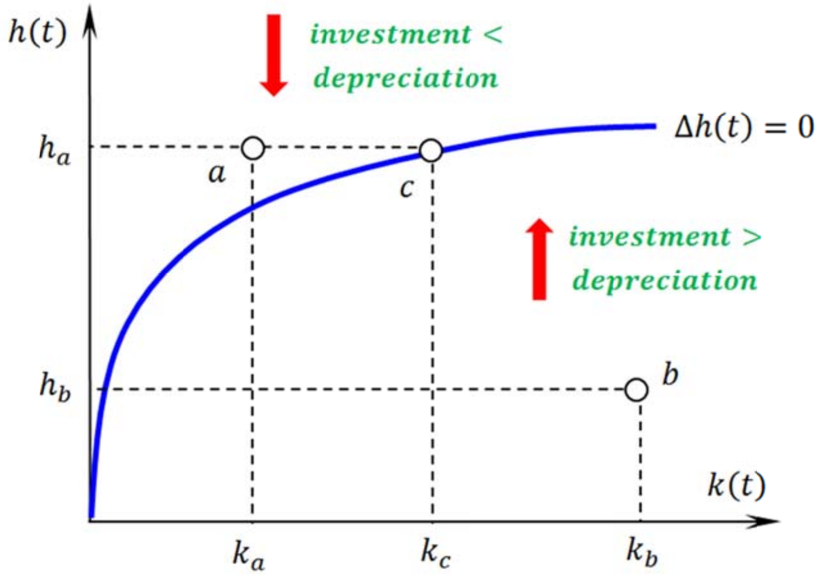

Fig. 2. Graphical representation of $\Delta h(t) = 0$

At points like a in Fig. 2, investment in h is smaller than depreciation of h, so h decreases.

Investment is smaller than depreciation at a because, given h_a, it is necessary to have capital stock per capita equal to k_c for investment to equal depreciacion (for $\Delta h(t) = 0$ to hold).

Since $k_a < k_c$, there is a capital shortage creating an output gap that generates insufficient investment in h (in comparison with the depreciation corresponding to h_a).

Analogously, h increases at points below the curve $\Delta h(t) = 0$ (like point b in Fig. 2).

Graphically, the solution of the model is obtained by combining Fig. 1 with Fig. 2; see Fig. 3. The dynamics of k and h ensure convergence to the point e in Fig. 3 where both curves intersect.

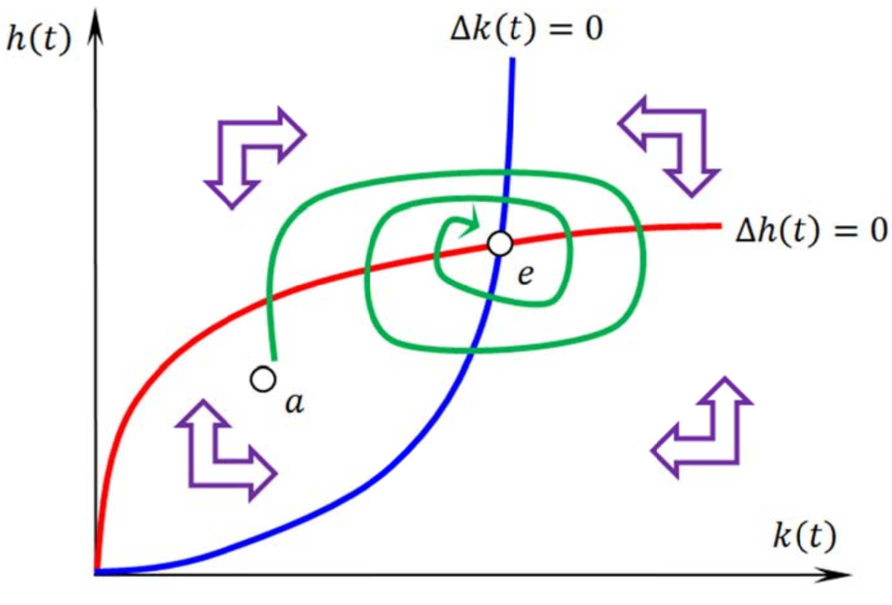

Fig. 3. Solution of the model: $\Delta k(t) = \Delta k(t) = 0$

2. The Harrod-Domar (HD) model

The production function in the HD model is

$$Y = F(K, L) = min\{A \cdot K, B \cdot L\},$$

where A, $B > 0$ are fixed coefficients. If K is the limiting factor (that is, $A \cdot K < B \cdot L$), then $Y = A \cdot K$. Hence, $y = A \cdot k = f(k)$. In this case,

$$\frac{1}{v} = \frac{1}{\frac{K}{Y}} = \frac{Y}{K} = A\,.$$

Define the effective growth rate g (of output per capita) as the one resulting from the equilibrium condition $S = I$. Therefore,

$$\mathbb{s} \cdot Y = S = I = \Delta K + \delta \cdot K$$

Since $Y = A \cdot K$,

$$\mathbb{s} \cdot A \cdot K = \Delta K + \delta \cdot K$$

Dividing by K,

$$\mathbb{s} \cdot A = \frac{\Delta K}{K} + \delta$$

or

$$\mathbb{s} \cdot A = g_K + \delta$$

In sum,

$$g_K = \mathbb{s} \cdot A - \delta$$

Ginve that $Y = A \cdot K$, it follows that $g_Y = g_K = \mathbb{s} \cdot A - \delta$. All in all, the effective growth rate g coincides with the growth rate g_Y ensuring the full employment of capital.

The growth rate $\boldsymbol{g}_Y$ that could be considered socially optimal is the one ensuring the full employment of labour. It can be defined as the growth rate $g_{B \cdot L}$ of effective labour $B \cdot L$ (assuming that technology progress is labour augmenting). Letting n designate the population (labour) growth rate and b represent the rate of technological progress,

$$g_{B \cdot L} = \frac{B' \cdot L'}{B \cdot L} - 1 = \frac{B'}{B} \cdot \frac{L'}{L} - 1 = (g_B + 1) \cdot (g_L + 1) - 1 = (b + 1) \cdot (n + 1) - 1 = n + b \cdot (1 + n)$$

Hence, set $\boldsymbol{g}_Y = g_{B \cdot L} = n + b \cdot (1 + n)$.

If $g_Y < \boldsymbol{g}_Y$, then the effective growth rate of the economy is insufficient to ensure the full employment of labour, so structural unemployment arises. This occurs when

$$\mathbb{s} \cdot A - \delta < n + b \cdot (1 + n)$$

To recap, if

$$\mathbb{s} < \frac{n + b \cdot (1 + n) + \delta}{A}$$

then the economy generates structural unemployment because of an insufficient saving rate. Even if the effective labour remains constant ($n = b = 0$), the saving rate guaranteeing full employment of capital does not ensure the full employment of labour (structural unemployment would occur if $\mathbb{s} < \delta/A$).

When $\mathbb{s} < \delta/A$, equality between g_Y and $\boldsymbol{g}_Y$ could be obtained by lowering δ/A. This could be achieved by increasing A, which amounts to improving the productivity of capital (the output to capital ratio Y/K). But this approach has a limit, because it has been assumed that K is the limiting factor, in the sense that $A \cdot K < B \cdot L$.

Developed economies can be characterized by the condition $g_Y > \boldsymbol{g}_Y$. One explanation for this fact is having a small population growth rate n. An implication of this condition is that savings are excessive or that investment is not enough to meet savings plans.

In non-developed economies the opposite occurs: $g_Y < \boldsymbol{g}_Y$. This is consistent with a small saving rate $\mathbb{s}$, which manifests itself in an imbalance between the population growth and the rate at capital accumulation. In fact, as $g_Y = g_K$, it follows from $g_Y < \boldsymbol{g}_Y$ that $g_K < \boldsymbol{g}_Y$. Thus, $g_Y < \boldsymbol{g}_Y$ implies

$$g_K < n + b \cdot (1 + n)$$

In particular, if the productivity B of labour remains constant, so $b = 0$, having $g_Y < \boldsymbol{g}_Y$ amounts to having $g_K < n$.

In any case, in the HD model growth can be permanent, whereas in the SS model growth is temporary (it is a by-product of convergence to a steady state and, to be sustained, must be exogenously induced).

Specifically, since $Y = A \cdot K$, it is plain that $y = A \cdot k = f(k)$ and $g_y = g_k$. As in the SS model,

$$\Delta k = \mathbb{s} \cdot f(k) - \delta \cdot k.$$

Therefore,

$$g_y = g_k = \frac{\Delta k}{k} = \mathbb{s} \cdot \frac{f(k)}{k} - \delta = \mathbb{s} \cdot A - \delta$$

With population growing at rate n,

$$g_y = g_k = \frac{\mathbb{s} \cdot A}{1 + n} - \frac{\delta + n}{1 + n}.$$

The main lesson of these results is that a sustained increase in output per capita ($g_y > 0$ permanently) is possible with a sufficiently high saving rate:

$$g_y > 0 \quad \Leftrightarrow \quad \mathbb{s} \cdot A > \delta + n$$

That is, sustained growth can be achieved when

$$\mathbb{s} > \frac{\delta + n}{A}.$$

The corresponding policy recommendation for prosperity is simple: “save more”.

Define the golden age as the state in which $g_Y = \boldsymbol{g}_Y$. This is equivalent to

$$\mathbb{s} = \frac{n + b \cdot (1 + n) + \delta}{A}$$

But there is no mechanism in the model to ensure this equality. First, $\mathbb{s}$ is an exogenous decision (by savers, the families). Second, n is also exogenous and determined by an unmodelled demographic dynamics. Third, the capital to output ratio $\frac{K}{Y} = \frac{1}{A}$ arises from the expectations of investors (assumed to be correct). Finally, b is also a technological exogenous parameter.

The basic conclusions obtained from the model are:

(i) that the economy may be considered unstable in the sense that there is no guaranteed convergence to an equilibrium (the existence of a stationary state is not ensured); and

(ii) that it is highly unlikely that the effective rate of growth of output is exactly the rate that generates the full employment of labour.

The preceding results should be qualified, because they rely on the presumption that K is the limiting factor. But if $g_k > 0$, then, eventually, K will no longer be the limiting factor.

That K is the limiting factor means that $A \cdot K < B \cdot L$; equivalently, $k = K/L < B/A$. Hence, if k grows, the condition $k < B/A$ expressing the fact that K is the limiting factor will eventually no longer hold: at some point in time, K/L will exceed B/A. In this case, the model should be solved again with production function $Y = B \cdot L$.

When $Y = B \cdot L$, $g_Y = n$ and, if B does not grow, Y/L cannot grow. If L is the limiting factor,

$$g_k = \mathbb{s} \cdot B \cdot \frac{1}{k} - \delta.$$

In sum:

(i) with K being the limiting factor, the HD model could account for a (limited) sustained growth of output per capita (long-run growth), whereas the SS model cannot;

(ii) conversely, the HD model fails to account for convergence among economies, which is a phenomenon the SS model can explain.

3. The AK model

It is like the HD (or the SS) model with $Y = A \cdot K$ always. This model can explain sustained long-run growth. In fact,

$$k(t+1) = (\mathbb{s} \cdot A + 1 - \delta) \cdot k(t).$$

Hence, if $\mathbb{s} \cdot A + 1 - \delta > 1$ (that is, $\mathbb{s} \cdot A > \delta$), then k grows forever.

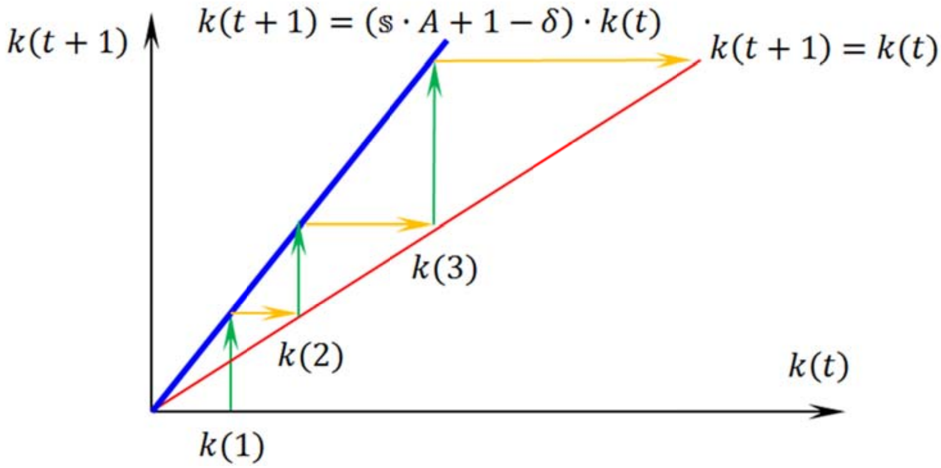

Fig. 4. Sustained growth in the AK model

Justification of the AK model: the accumulation of capital generates, through learning by doing, technical progress that prevents the productivity of capital from falling (see the Arrow model). Knowledge externalities can also explain sustained long-run growth.

The AK model fails to account for convergence, as it suggests different growth rates and does not distinguish between capital accumulation and technological progress.

The HD model suggests that the processes of accumulation and growth very likely generate instability: the discrepancy between effective capacity and desired capacity, between actual growth and expectations by investors, leads to growth with persistent unemployment (in the model, the constant $v = K/Y$ captures the investors' expectations).

The SS model claims the opposite: accumulation and growth do not generate instability. Long-run growth is only sustained with correct expectations and full employment is achieved thanks to a variable capital to output ratio v.

As a summary of the SS, HD, and AK models, the common framework adopted in the three models is the following (all three models rely on the conceptual sequence **profitability (rate of return on capital) $\rightarrow$ capital accumulation $\rightarrow$ economic growth**).

- Equilibrium condition $S = I$
- Saving function $S = \mathbb{s} \cdot Y$ constant saving rate $0 < \mathbb{s} < 1$
- Gross investment $I = \Delta K + \delta \cdot K$
- Net investment $I - \delta \cdot K = \Delta K$

Inserting the equilibrium condition and the saving function into net investment,

$$\Delta K = \mathbb{s} \cdot Y - \delta \cdot K$$

or, dividing both sides by K,

$$\frac{\Delta K}{K} = \mathbb{s} \cdot \frac{Y}{K} - \delta \,. \tag{1}$$

Define the capital to output ratio as

$$v = \frac{K}{Y}$$

and the growh rate of the stock of capital as

$$g_K = \frac{\Delta K}{K} \,.$$

Then (1) can be equivalently expressed as

$$g_K = \frac{\mathbb{s}}{v} - \delta \,.$$

For the stock of capital K to grow it must be that $g_K > 0$; that is, $\mathbb{s} > \delta \cdot v$.

Now consider the capital to labour ratio $k = \frac{K}{L}$. Then the growth rate g_k of k is

$$g_k = \frac{\Delta k}{k} = \frac{k' - k}{k} = \frac{k'}{k} - 1 = \frac{\frac{K'}{L'}}{\frac{K}{L}} - 1 = \frac{\frac{K'}{K}}{\frac{L'}{L}} - 1 = \frac{g_K + 1}{g_L + 1} - 1 = \frac{g_K - g_L}{g_L + 1} \,.$$

Accordingly, if population grows at a constant rate $n > 0$,

$$g_k = \frac{g_K - g_L}{g_L + 1} = \frac{\frac{\mathbb{s}}{v} - \delta - n}{n + 1} \,.$$

The capital to labour ratio k grows if $g_k > 0$; that is, if

$$\frac{\mathbb{s}}{v} - \delta - n > 0$$

or

$$\frac{\mathbb{s}}{v} > \delta + n \,.$$

If technological progress A is labour augmenting, labour is defined as effective labour $A \cdot L$, and the capital to (effective) labour ratio $k = \frac{K}{A \cdot L}$, then the growth rate g_k of k is

$$g_k = \frac{\Delta k}{k} = \frac{k'}{k} - 1 = \frac{\frac{K'}{A' \cdot L'}}{\frac{K}{A \cdot L}} - 1 = \frac{\frac{K'}{K}}{\frac{L'}{L} \cdot \frac{A'}{A}} - 1 = \frac{g_K + 1}{(g_L + 1) \cdot (g_A + 1)} - 1 = \frac{g_K - g_L - g_A \cdot (g_L + 1)}{(g_L + 1) \cdot (g_A + 1)}.$$

In view of this, if population grows at a constant rate $n > 0$ and technology progresses at a constant rate $a > 0$,

$$g_k = \frac{g_K - g_L - g_A \cdot (g_L + 1)}{(g_L + 1) \cdot (g_A + 1)} = \frac{\frac{\mathbb{s}}{v} - \delta - n - a \cdot (1 + n)}{(1 + a) \cdot (1 + n)}.$$

As a result,

$$g_k > 0 \quad \Leftrightarrow \quad \frac{\mathbb{s}}{v} > \delta + n + a \cdot (1 + n).$$

The growth models so far considered (except the neoclassical growth model, the last one to be presented) can be classified according to two basic choices, concerning the saving rate $\mathbb{s}$ and the capital to output ratio v.

$\mathbb{s}$ exogenous / endogenous

S1. $\mathbb{s}$ exogenous Solow-Swan model, Harrod-Domar model, AK model

S2. $\mathbb{s}$ endogenous overlapping generations model, neoclassical growth model

v constant / variable

V1. v constant Harrod-Domar model, AK model

V2. v variable Solow-Swan model, overlapping generations, neoclassical growth model

Ejercicio 1. Harrod-Domar. (i) En el modelo de Harrod-Domar cuando K es el factor limitante, calcula la tasa de crecimiento del capital per cápita si la población crece a la tasa (neta) $n > 0$. (ii) En el modelo de Harrod-Domar cuando L es el factor limitante y no crece, calcula la tasa de crecimiento del stock de capital.

Ejercicio 2. Modelo de Mankiw-Romer-Weil. Verifica que la expresión

$$h(t) = \left(\frac{\hat{\mathbb{s}}}{\hat{\delta} + GN - 1}\right)^{1/(1-\hat{\alpha})} \cdot k(t)^{\alpha/(1-\hat{\alpha})}$$

que representa la condición $\Delta h(t) = 0$ es correcta.

4. The Barro (1990) model

There is a government that provides a public good, which is interpreted as a positive externality for production. The amount of public good expenditure is denoted by E.

- **Aggregate production function** $Y = A \cdot K^{\alpha} \cdot E^{1-\alpha}$, where $0 < \alpha < 1$ and $A > 0$

Population is assumed to grow at rate $n > 0$. Dividing both sides by population L yields the per capita production function (where $k = K/L$ and $e = E/L$).

$$y = \frac{Y}{L} = \frac{A \cdot K^{\alpha} \cdot E^{1-\alpha}}{L} = A \cdot \frac{K^{\alpha}}{L^{\alpha}} \cdot \frac{E^{1-\alpha}}{L^{1-\alpha}} = A \cdot \left(\frac{K}{L}\right)^{\alpha} \cdot \left(\frac{G}{L}\right)^{1-\alpha} = A \cdot k^{\alpha} \cdot e^{1-\alpha}$$

- **Per capita production function** $y = A \cdot k^{\alpha} \cdot e^{1-\alpha}$

The public good is financed by taxes. The tax rate is $0 < \tau < 1$. Total disposable income Y_d is then $Y_d = (1-\tau) \cdot Y$.

- **Aggregate savings function** $S = \mathbb{s} \cdot Y_d = \mathbb{s} \cdot (1-\tau) \cdot Y$, where $0 < \mathbb{s} < 1$

- **Macroeconomic equilibrium** $S = I$

The stock of capital accumulates as in the Solow-Swan model:

$$K(t+1) = I(t) + (1-\delta) \cdot K(t).$$

Dividing both sides by $L(t+1)$ and knowing that $L(t+1) = (1+n) \cdot L(t)$,

$$k(t+1) = \frac{K(t+1)}{L(t+1)} = \frac{I(t)}{(1+n) \cdot L(t)} + \frac{(1-\delta) \cdot K(t)}{(1+n) \cdot L(t)}.$$

Using the macroeconomic equilibrium condition,

$$k(t+1) = \frac{\mathbb{s} \cdot (1-\tau) \cdot Y(t)}{(1+n) \cdot L(t)} + \frac{(1-\delta) \cdot K(t)}{(1+n) \cdot L(t)} = \frac{\mathbb{s} \cdot (1-\tau)}{1+n} \cdot y(t) + \frac{1-\delta}{1+n} \cdot k(t)\,.$$

In $k(t)$ is subtracted from both sides,

$$k(t+1) - k(t) = \frac{\mathbb{s} \cdot (1-\tau)}{1+n} \cdot y(t) + \left(\frac{1-\delta}{1+n} - 1\right) \cdot k(t) = \frac{\mathbb{s} \cdot (1-\tau)}{1+n} \cdot y(t) - \left(\frac{\delta+n}{1+n}\right) \cdot k(t)$$

or, in a more compact notation:

$$\Delta k = \frac{\mathbb{s} \cdot (1 - \tau)}{1 + n} \cdot y - \left(\frac{\delta + n}{1 + n}\right) \cdot k \, .$$

- **Dynamics of capital per capita** $g_k = \frac{\Delta k}{k} = \frac{\mathbb{s} \cdot (1-\tau)}{1+n} \cdot \frac{y}{k} - \frac{\delta + n}{1+n}$

Introducing the value of $y = A \cdot k^{\alpha} \cdot e^{1-\alpha}$ leads to

$$g_k = \frac{\mathbb{s} \cdot (1 - \tau)}{1 + n} \cdot A \cdot \left(\frac{e}{k}\right)^{1-\alpha} - \frac{\delta + n}{1 + n}.$$

The government budget is assumed to be always balanced: taxes are equal to the public good expenditure.

- **Government budget** $E = \tau \cdot Y = \tau \cdot A \cdot K^{\alpha} \cdot E^{1-\alpha}$

Expressed in per capita terms, $e = \tau \cdot A \cdot k^{\alpha} \cdot e^{1-\alpha}$. That is,

$$e = k \cdot (\tau \cdot A)^{\frac{1}{\alpha}} \, . \qquad (2)$$

If (2) is inserted into the equation describing the dynamics of capital per capita,

$$g_k = \frac{\mathbb{s} \cdot (1 - \tau)}{1 + n} \cdot A \cdot \left(\frac{e}{k}\right)^{1-\alpha} - \frac{\delta + n}{1 + n} = \frac{\mathbb{s} \cdot (1 - \tau)}{1 + n} \cdot A \cdot (\tau \cdot A)^{\frac{1-\alpha}{\alpha}} - \frac{\delta + n}{1 + n} =$$

$$= \frac{\mathbb{s} \cdot (1 - \tau)}{1 + n} \cdot A^{\frac{1}{\alpha}} \cdot \tau^{\frac{1-\alpha}{\alpha}} - \frac{\delta + n}{1 + n}.$$

If (1) is inserted into the equation describing output per capital,

$$y = A \cdot k^{\alpha} \cdot e^{1-\alpha} = A \cdot k^{\alpha} \cdot \left(k \cdot (\tau \cdot A)^{\frac{1}{\alpha}}\right)^{1-\alpha} = A \cdot k \cdot (\tau \cdot A)^{\frac{1-\alpha}{\alpha}} = A^{\frac{1}{\alpha}} \cdot \tau^{\frac{1-\alpha}{\alpha}} \cdot k \, .$$

This expression is equivalent to the per capita production function of an AK model, with the only difference that the constant A in an AK model now takes the form of the constant $A^{\frac{1}{\alpha}} \cdot \tau^{\frac{1-\alpha}{\alpha}}$.

Since y is proportional to k, as in an AK model, it follows that the rate of growth g_y of y is equal to the rate of growth g_k of k. Accordingly,

$$g_y = g_k = \frac{\mathbb{s} \cdot (1 - \tau)}{1 + n} \cdot A^{\frac{1}{\alpha}} \cdot \tau^{\frac{1-\alpha}{\alpha}} - \frac{\delta + n}{1 + n}.$$

An immediate implication of this equation is that $g_y > 0$ if and only if

$$\mathbb{s} \cdot (1 - \tau) \cdot A^{\frac{1}{\alpha}} \cdot \tau^{\frac{1-\alpha}{\alpha}} > \delta + n$$

that is, if and only if

$$\mathbb{s} > \frac{\delta + n}{(1-\tau) \cdot A^{\frac{1}{\alpha}} \cdot \tau^{\frac{1-\alpha}{\alpha}}}.$$

The lesson of this result is that a sufficiently high saving rate $\mathbb{s}$ could ensure a sustained growth of output per capita y, in contrast to the eventual convergence of g_y to zero in the Solow-Swan model.

An interesting question concerns the value of τ that maximizes the rate of growth g_y of y. But that is another story that you can try to tell yourself.

Ejercicio 3. Calcula la derivada parcial de $(\delta + n)/\left((1-\tau) \cdot A^{\frac{1}{\alpha}} \cdot \tau^{\frac{1-\alpha}{\alpha}}\right)$ respecto de τ e interpreta el resultado en el contexto del problema de garantizar que $g_y > 0$.

5. The Frankel (1962) model

The Frankel model tries to get the best of the Solow-Swan and the Harrod-Domar model: the first operates at the micro level (to determine allocation) and the second at the macro level (to generate growth).

The economy has m firms. Each firm i is endowed with a Cobb-Douglas technology

$$Y_i = A \cdot H \cdot {K_i}^{\alpha} \cdot {L_i}^{1-\alpha}$$

that includes a term H interpreted as a development modifier. The value of H represents the level of development of the economy. This level embodies a positive externality for the production Y_i made by firm i when the firm makes uses of K_i units of capital and L_i units of labour. The term A captures the common technology available to all the firms. From the standpoint of a firm H is treated as a parameter because, just by itself, the firm is not capable to alter the level of development of the economy.

Let firm i use the proportion π_i of all the factors: $K_i = \pi_i \cdot K$ and $L_i = \pi_i \cdot L$, where K is the total stock of capital in the economy and L is the total amount of labour. Therefore, total ouput Y is

$$Y = \sum_{i=1}^{m} Y_i = \sum_{i=1}^{m} A \cdot H \cdot {K_i}^{\alpha} \cdot {L_i}^{1-\alpha} = A \cdot H \cdot \sum_{i=1}^{m} {K_i}^{\alpha} \cdot {L_i}^{1-\alpha} =$$

$$= A \cdot H \cdot \sum_{i=1}^{m} {\pi_i}^{\alpha} \cdot K^{\alpha} \cdot {\pi_i}^{1-\alpha} \cdot L^{1-\alpha} = A \cdot H \cdot \sum_{i=1}^{m} \pi_i \cdot K^{\alpha} \cdot L^{1-\alpha} =$$

$$A \cdot H \cdot K^{\alpha} \cdot L^{1-\alpha} \cdot \sum_{i=1}^{m} \pi_i = A \cdot H \cdot K^{\alpha} \cdot L^{1-\alpha}.$$

- **Aggregate production function** $Y = A \cdot H \cdot K^{\alpha} \cdot L^{1-\alpha}$, with $0 < \alpha < 1$, $A > 0$ and $H > 0$

The development modifier H is assumed to depend on the level of capital per capita, which can be viewed as a proxy for development: the higher the amount of capital per worker in an economy, the more developed the economy.

- **Level of development** $H = \left(\frac{K}{L}\right)^{\gamma}$

Inserting this definition of the modifier into the production function,

$$Y = A \cdot H \cdot K^{\alpha} \cdot L^{1-\alpha} = A \cdot \left(\frac{K}{L}\right)^{\gamma} \cdot K^{\alpha} \cdot L^{1-\alpha} = A \cdot K^{\alpha+\gamma} \cdot L^{1-\alpha-\gamma}$$

When $\alpha + \gamma = 1$, the resulting production function is the one from the model: $Y = A \cdot K$. In this case, each firms is endowed with a Cobb-Douglas production function, as the one in the Solow-Swan model, but the economy as a whole is endowed with a fixed coeficient function, like the one in the Harrod-Domar model.

The rest of the Frankel model is conventional. For simplicity, let $\delta = 0$.

- **Aggregate savings function** $S = \mathbb{s} \cdot Y$, where $0 < \mathbb{s} < 1$

- **Macroeconomic equilibrium** $S = I$

- **Capital accumulation** $K(t+1) = I(t) + K(t)$ or $I = \Delta K$

Combining the preceding equations, with " ' " designating a variable referring to the next period,

$$I = \Delta K \Rightarrow S = \Delta K \Rightarrow \mathbb{s} \cdot Y = \Delta K \Rightarrow \frac{\mathbb{s} \cdot Y}{L'} = \frac{K' - K}{L'} \Rightarrow \frac{\mathbb{s} \cdot Y}{(1+n) \cdot L} = \frac{K'}{L'} - \frac{K}{(1+n) \cdot L}.$$

In sum,

$$\frac{\mathbb{s}}{1+n} \cdot y = k' - \frac{1}{1+n} \cdot k\,.$$

Therefore,

$$\frac{\mathbb{s}}{1+n} \cdot y = k' - k + k - \frac{1}{1+n} \cdot k = \Delta k + \frac{n}{1+n} \cdot k.$$

As a result,

$$\Delta k = \frac{\mathbb{s}}{1+n} \cdot y - \frac{n}{1+n} \cdot k$$

and

$$g_k = \frac{\Delta k}{k} = \frac{\mathbb{s}}{1+n} \cdot \frac{y}{k} - \frac{n}{1+n}.$$

On the other hand,

$$y = \frac{Y}{L} = \frac{A \cdot K^{\alpha+\gamma} \cdot L^{1-\alpha-\gamma}}{L} = A \cdot \frac{K^{\alpha+\gamma}}{L^{\alpha+\gamma}} \cdot \frac{L^{1-\alpha-\gamma}}{L^{1-\alpha-\gamma}} = A \cdot \left(\frac{K}{L}\right)^{\alpha+\gamma} = A \cdot k^{\alpha+\gamma}.$$

In view of this,

$$g_k = \frac{\mathbb{s}}{1+n} \cdot \frac{y}{k} - \frac{n}{1+n} = \frac{\mathbb{s} \cdot A}{1+n} \cdot k^{\alpha+\gamma-1} - \frac{n}{1+n}.$$

When $\delta = 0$, the first part of the above expression holds in the Solow-Swan model:

$$\boldsymbol{g}_k = \frac{\mathbb{s}}{1+n} \cdot \frac{y}{k} - \frac{n}{1+n} = \frac{\mathbb{s}}{1+n} \cdot \frac{f(k)}{k} - \frac{n}{1+n}.$$

In the latter case,

$$\frac{\partial \boldsymbol{g}_k}{\partial k} = \frac{\mathbb{s}}{1+n} \cdot \frac{\partial (f(k)/k)}{\partial k} < 0$$

because $\frac{\partial (f(k)/k)}{\partial k} < 0$ (remember that $f(k)/k$ is a decreasing function). The fact that $\frac{\partial \boldsymbol{g}_k}{\partial k} < 0$ means that, as k grows, the rate at which k is each time smaller and converges to zero (the steady state).

As distinguished from this result, in the Frankel model,

$$\frac{\partial g_k}{\partial k} = (\alpha + \gamma - 1) \cdot \frac{\mathbb{s} \cdot A}{1+n} \cdot k^{\alpha+\gamma-2}.$$

Only if $\alpha + \gamma < 1$, the dynamics of k is the same as in the Solow-Swan model, because in this case $\frac{\partial g_k}{\partial k} < 0$. If $\alpha + \gamma = 1$, then k accumulates at a constant rate (not necessarily zero). Finally, if $\alpha + \gamma > 1$, the more capital per capita is accumulaed, the higher the rate at which it accumulates.

Since $y = A \cdot k^{\alpha+\gamma}$, and A, α, and γ are constants,

$$g_y \approx (\alpha + \gamma) \cdot g_k.$$

Consequently, $g_k > 0$ implies $g_y > 0$. Hence, sustained growth in output per capita is possible, as in the Harrod-Domar model but unlike the Solow-Swan model.

Ejercicio 4 (Modelo de Arrow, 1962). Calcula la tasa de variación g_y del producto per cápita en el modelo que se diferencia del modelo de Frankel en que la función de producción agregada es $Y = A \cdot K^{\alpha} \cdot (H \cdot L)^{1-\alpha}$, donde $H = K^{\gamma}$. (Intuición: la productividad del trabajo se incrementa mediante la experiencia en el uso del capital, *learning by doing*).

6. The Romer (1990) model

There are four inputs. There is labour, L. There is also a set of (physical) capital goods. The amount of capital good i is K_i. The third input is technology, which is interpreted as a non-rival component in production: when someone makes use of technology everybody else can also make use of it. It can be viewed as freely available knowledge. Technology is represented by the number A of capital goods available. Lastly, there is human capital, H, which is assumed to be a rival component in production. A part H_Y of the human capital H is used to produce the good, whereas the rest $H_A = H - H_Y$ is employed to improve technology in the research sector of the economy.

Each capital good is supposed to have the same production cost and the same productivity. For this reason, it is assumed that the same amount Ꝁ is produced of each capital good. Therefore, the total amount K of capital goods in the economy is

$$K = A \cdot Ꝁ.$$

The model consists of the equations listed next (it is assumed that $\delta = 0$).

- **Aggregate production function** $Y = {H_Y}^{\alpha} \cdot L^{\beta} \cdot (A \cdot Ꝁ^{1-\alpha-\beta}),\ 0 < \alpha < 1$ and $0 < \beta < 1$
- **Aggregate savings function** $S = \mathbb{s} \cdot Y,\ 0 < \mathbb{s} < 1$
- **Macroeconomic equilibrium** $S = I$
- **Capital accumulation** $I = \Delta K$
- **Technological change** $\Delta A = \phi \cdot H_A \cdot A\ ,\ \phi > 0$

The last equation describes the process of creation of generic (non-rival) knowledge. It asserts that the change in technology depends on the productivity ϕ of researchers, the human capital H_A spent in research activities, and the existing techonology A.

It is assumed that the human capital H, the human capital used in production H_Y, labour L, and the amount Ꝁ produced of each capital good remain all constant. In view of this, it follows from $Y = A \cdot {H_Y}^{\alpha} \cdot L^{\beta} \cdot Ꝁ^{1-\alpha-\beta}$ that the rate of change in output equals the rate of change in technology. That is,

$$g_Y = g_A.$$

The technological change equation $\Delta A = \phi \cdot H_A \cdot A$ implies $g_A = \frac{\Delta A}{A} = \phi \cdot H_A$. The final conclusion is then

$$\boxed{g_Y = \phi \cdot H_A\,.}$$

Interpretation: the output growth rate is proportional to both the human capital H_A devoted to research (to increase the stock of knowledge) and the productivity ϕ of researchers (knowledge generated per researcher).

Given that the population has been assumed constant, $g_y = g_Y$: output and output per capita both grow at the same rate. As a result, $g_y = \phi \cdot H_A$: prosperity can be sustained by just investing in accumulating knowledge.

7. Representative-agent (Ramsey, 1928) model

Time is discrete. There is only one good in each period. Expressing variables in per capita terms, at each t, production at t equals consumption at t plus investment at t.

$$y_t = c_t + i_t \tag{3}$$

Output can only be consumed or saved: $y_t = c_t + s_t$. Therefore, $i_t = s_t$.

Each period a fraction $0 < \delta < 1$ of capital depreciates. Capital at $t+1$ is investment at t plus the remaining capital from period t.

$$k_{t+1} = i_t + (1-\delta) \cdot k_t \tag{4}$$

The production function f makes output per capita depend on capital per capita.

$$y_t = f(k_t) \tag{5}$$

f satisfies the typical properties: $f \geq 0$, $f' > 0$, $f'' < 0$, $\lim_{k_t \to 0} f'(k_t) = \infty$, and $\lim_{k_t \to \infty} f'(k_t) = 0$.

Combining (3), (4), and (5),

$$f(k_t) = c_t + k_{t+1} - (1-\delta) \cdot k_t \tag{6}$$

or, by defining $\Delta k_{t+1} = k_{t+1} - k_t$,

$$f(k_t) = c_t + \Delta k_{t+1} + \delta \cdot k_t\,. \tag{7}$$

Equation (7) defines the dynamic constraint the economy faces, which is represented in Fig. 5. **Interpretation 1**: given k_t, c_t and k_{t+1} are determined; given k_{t+1}, c_{t+1} and k_{t+2} are determined… **Interpretation 2**: given k_t, decision is over c_t, c_{t+1}, c_{t+2}…

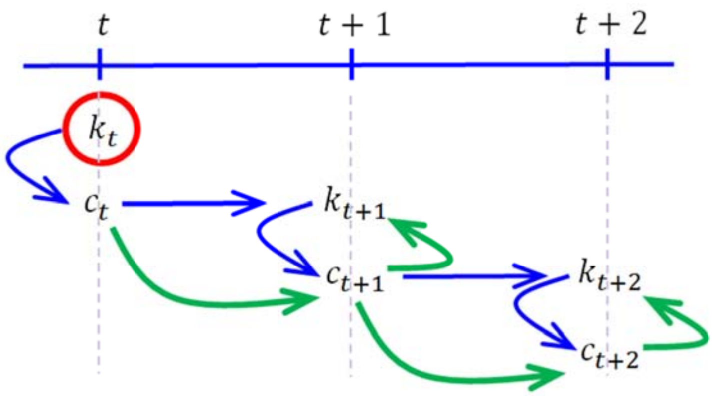

Fig. 5. Dynamic constraint on the economy

Golden rule

There is a representative agent. If population is constant, then variables can be seen as per capita variables (c_t would be what the agent consumes in period t). Suppose the aim of the agent is to maximize consumption each period (no discounting).

The problem can be solved by considering the steady state (the long run of the economy). Let c and k be the steady state values.

From (7), $f(k) = c + \delta \cdot k$; that is,

$$c = f(k) - \delta \cdot k.$$

This is the familiar idea that steady-state consumption is the output that remains once taken the output necessary to replace the lost capital, so that capital remains constant.

The first-order condition to maximize c is $\frac{\partial c}{\partial k} = 0$; that is, $f'(k) = \delta$. Since $f'' < 0$, the second-order condition $\left(\frac{\partial^2 c}{\partial k^2} < 0\right)$ holds.

The condition $f'(k) = \delta$ says that the marginal product of capital equals its depreciation rate. This solution is known as "the golden rule". If $f'(k) < \delta$, c can be increased by rising k. If $f'(k) > \delta$, c can be increased by lowering k.

Let (c_G, k_G) be the golden rule solution. Suppose capital is exogenously reduced to $k < k_G$ but the agent tries to maintain c_G.

Then $c_G = f(k_G) - \delta \cdot k_G$ and $c = f(k) - \delta \cdot k - \Delta k$. If $c_G = c$, then $f(k_G) - \delta \cdot k_G = f(k) - \delta \cdot k - \Delta k$. Solving for Δk,

$$\Delta k = (f(k) - \delta \cdot k) - (f(k_G) - \delta \cdot k_G).$$

As (c_G, k_G) is the golden rule solution $f(k_G) - \delta \cdot k_G > f(k) - \delta \cdot k$. In sum, $\Delta k < 0$.

With less capital, future output would be smaller. The attempt to keep c_G will further decrease the stock of capital, making the consumption level c_G eventually untenable.

Lesson: "too much" consumption sooner or later exhausts the capital stock, so the economy will be unable to sustain that consumption level.

Solution to the negative shock on k: divert consumption temporarily to rebuild the capital stock. Once k_G is restored, c can be increased to reach level c_G.

If consumption in different periods is valued differently, the agent may choose to maximize the present value of the infinite sequence of consumption $(c_0, c_1, c_2, \ldots)$ or, given a utility function u common for each t, the present value of $(u(c_0), u(c_1), u(c_2), \ldots)$.

$$max_{\{c_t,\, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{subject to } \; c_t + k_{t+1} = f(k_t) + (1-\delta) \cdot k_t$$

Assumptions on u: $u \geq 0$, $u' > 0$, and $u'' < 0$. Parameter $\beta \in (0,1)$ is the discount factor.

Using the method of Lagrange multipliers, define the Lagrangian as

$$\mathcal{L}_t = \sum_{t=0}^{\infty} [\beta^t \cdot u(c_t) + \lambda_t (f(k_t) + (1-\delta) \cdot k_t - c_t - k_{t+1})]$$

which is maximized with respect to c_t, k_{t+1}, and λ_t ($\mathcal{L}_t$ is not maximized with respect to k_t because k_t is known at t).

- First-order conditions (FOC)

$$0 = \frac{\partial \mathcal{L}_t}{\partial c_t} = \beta^t \cdot u'(c_t) - \lambda_t$$

$$0 = \frac{\partial \mathcal{L}_t}{\partial k_{t+1}} = \lambda_{t+1}(f'(k_{t+1}) + 1 - \delta) - \lambda_t$$

$$0 = \frac{\partial \mathcal{L}_t}{\partial \lambda_t} = f(k_t) + (1-\delta) \cdot k_t - c_t - k_{t+1}$$

- Transversality condition

$$lim_{t \to \infty} \; \beta^t \cdot u'(c_t) \cdot k_{t+1} = 0$$

To interpret the transversality condition, suppose t is the last period.

If $k_{t+1} > 0$ (some capital is left at the last period), then $u'(c_t) = 0$: consuming that capital should have no impact on utility.

If $u'(c_t) > 0$, then it cannot be that some capital is saved for the next (non-existent) period, because utility would be increased by consuming that capital now. Therefore, it must be that $k_{t+1} = 0$.

From the first FOC, $\lambda_t = \beta^t \cdot u'(c_t)$ and $\lambda_{t+1} = \beta^{t+1} \cdot u'(c_{t+1})$. Substituting for λ_t and λ_{t-1} in the second FOC,

$$\beta^{t+1} \cdot u'(c_{t+1}) \cdot [f'(k_{t+1}) + 1 - \delta] = \beta^t \cdot u'(c_t)\,.$$

The result is the so-called <u>Euler equation</u>:

$$\beta \cdot u'(c_{t+1}) \cdot [f'(k_{t+1}) + 1 - \delta] = u'(c_t)\,. \tag{8}$$

<u>Interpretation</u>. How much additional c_{t+1} can be obtained by just reducing c_t while leaving total utility (and everything beyond period $t+1$) constant?

Since periods after $t+1$ are unaffected, attention can be restricted to $u(c_t) + \beta \cdot u(c_{t+1})$, which must remain constant. Taking the total differential,

$$0 = du(c_t) + d[\beta \cdot u(c_{t+1})] = du(c_t) + \beta \cdot du(c_{t+1}) =$$

$$= u'(c_t) \cdot dc_t + \beta \cdot u'(c_{t+1}) \cdot dc_{t+1}\,.$$

In sum,

$$-\frac{dc_{t+1}}{dc_t} = \frac{u'(c_t)}{\beta \cdot u'(c_{t+1})}\,. \tag{9}$$

This is nothing else but the MRS. The resour-ce constraints at t and $t+1$ must hold, so

$$dc_t + dk_{t+1} = df(k_t) + (1-\delta) \cdot dk_t$$

$$dc_{t+1} + dk_{t+2} = df(k_{t+1}) + (1-\delta) \cdot dk_{t+1}.$$

That is,

$$dc_t + dk_{t+1} = f'(k_t) \cdot dk_t + (1-\delta) \cdot dk_t$$

$$dc_{t+1} + dk_{t+2} = f'(k_{t+1}) \cdot dk_{t+1} + (1-\delta) \cdot dk_{t+1}$$

Since k_t is given at t, $dk_t = 0$. The first equation then becomes $dk_{t+1} = -dc_t$: the additional capital at $t+1$ comes from the consumption cut at t.

By assumption, $dk_{t+2} = 0$. Given $dk_{t+1} = -dc_t$, the second equation is equivalent to

$$dc_{t+1} = -f'(k_{t+1}) \cdot dc_t - (1-\delta) \cdot dc_t$$

or

$$-\frac{dc_{t+1}}{dc_t} = f'(k_{t+1}) + (1-\delta).$$

From this and (9) the Euler equation (8) follows.

Interpretation. The output dc_t not consumed at t yields a utility loss at t of $|u'(c_t) \cdot dc_t|$. This output is invested at $t+1$, as dk_{t+1}, to increase output at $t+1$.

The additional output $|f'(k_{t+1}) \cdot dc_t|$ and the undepreciated part $(1-\delta) \cdot dk_{t+1} = |(1-\delta) \cdot dc_t|$ of the extra capital are consumed at $t+1$. All in all,

$$dc_{t+1} = [f'(k_{t+1}) + (1-\delta)] \cdot |dc_t|.$$

The discounted utility of dc_{t+1} is

$$\beta \cdot u'(c_{t+1}) \cdot dc_{t+1} = \beta \cdot u'(c_{t+1}) \cdot [f'(k_{t+1}) + (1-\delta)] \cdot |dc_t|.$$

But to keep utility constant, the utility $\beta \cdot u'(c_{t+1}) \cdot dc_{t+1}$ gained at $t+1$ must equal the utility $u'(c_t) \cdot |dc_t|$ lost at t. As a result,

$$u'(c_t) \cdot |dc_t| = \beta \cdot u'(c_{t+1}) \cdot [f'(k_{t+1}) + (1-\delta)] \cdot |dc_t|$$

which is the Euler equation once the common term $|dc_t|$ is cancelled out.

Steady state solution and dynamics

For steady-state values c and k, the Euler equation can be written as

$$\beta \cdot u'(c) \cdot [f'(k) + 1 - \delta] = u'(c)$$

so

$$f'(k) = \delta + \frac{1}{\beta} - 1\,.$$

The golden rule solution is $f'(k_G) = \delta$. Since $\frac{1}{\beta} - 1 > 0$, $f'(k) > f'(k_G)$. As $f'' < 0$, $k < k_G$. There is less capital than under the golden rule because now future utility is discounted at a rate $\frac{1}{\beta} - 1$. Moreover, $k < k_G$ yields $c < c_G$: discounting lowers consumption.

The dynamic analysis relies on the two equations giving the solution at each t: Euler equation (8) and the resource constraint (7).

$$\beta \cdot \frac{u'(c_{t+1})}{u'(c_t)} \cdot [f'(k_{t+1}) + 1 - \delta] = 1$$

$$\Delta k_{t+1} = f(k_t) - c_t - \delta \cdot k_t \tag{10}$$

Linearizing the Euler equation by taking a Taylor series expansion of $u'(c_{t+1})$ around c_t,

$$u'(c_{t+1}) \approx u'(c_t) + \Delta c_{t+1} \cdot u''(c_t)$$

or

$$\frac{u'(c_{t+1})}{u'(c_t)} \approx 1 + \Delta c_{t+1} \cdot \frac{u''(c_t)}{u'(c_t)}.$$

Inserting the previous approximation into the Euler equation yields (11), where $\frac{u''}{u'} < 0$.

$$\Delta c_{t+1} = \frac{u''(c_t)}{u'(c_t)} \cdot \left(\frac{1}{\beta \cdot [f'(k_{t+1}) + 1 - \delta]} - 1\right) \tag{11}$$

Equations (10) and (11) establish the changes in the capital stock and consumption.

Let c and k be the steady-state values (the solutions of (10) and (11) if $\Delta k_{t+1} = \Delta c_{t+1} = 0$).

If $k_{t+1} < k$, then $f'(k_{t+1}) > f'(k)$. Hence,

$$\beta \cdot [f'(k_{t+1}) + 1 - \delta] > \beta \cdot [f'(k) + 1 - \delta]\,.$$

As shown previously, $f'(k) = \delta + \frac{1}{\beta} - 1$. Thus, $\beta \cdot [f'(k) + 1 - \delta] = 1$.

Consequently, $\beta \cdot [f'(k_{t+1}) + 1 - \delta] > 1$ and, in (11), $\frac{1}{\beta \cdot [f'(k_{t+1}) + 1 - \delta]} < 1$. Since $\frac{u''}{u'} < 0$, the final conclusion is that

$$k_{t+1} < k \Rightarrow \Delta c_{t+1} > 0\,.$$

A similar reasoning proves that

$$k_{t+1} = k \Rightarrow \Delta c_{t+1} < 0.$$

This consumption dynamics is represented in Fig. 6: for capital stock to the left of the steady-state value k, consumption increases; for stock to the right of k, consumption decreases.

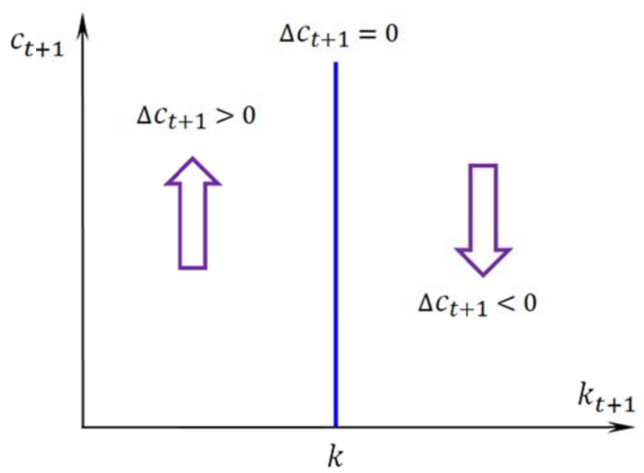

Fig. 6. Consumption dynamics

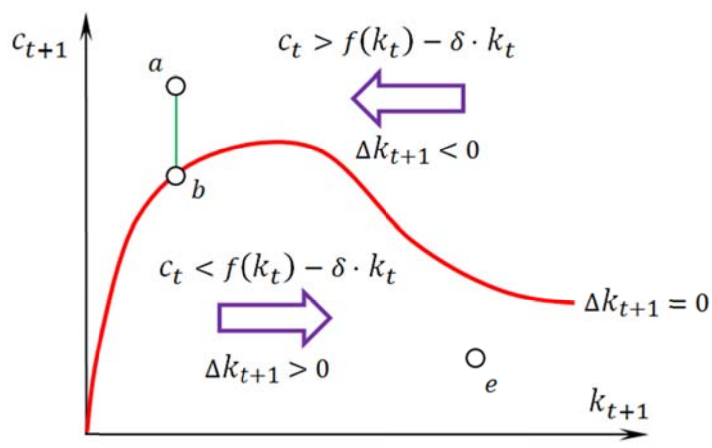

Fig. 7. Capital dynamics

Fig. 7 displays the capital dynamics following (10). Clearly,

$$\Delta k_{t+1} > 0 \quad \Leftrightarrow f(k_t) - \delta \cdot k_t > -c_t \, .$$

Above the curve $\Delta k_{t+1} = 0$, consumption is higher that the steady-state consumption, so capital must decumulate.

At point a, consumption exceeds the level (given by b) compatible with the steady state (with $\Delta k_{t+1} = 0$). Capital has to decrease to compensate excessive consumption.

Below the curve $\Delta k_{t+1} = 0$, consumption allows capital to acumulate.

When the two preceding figures are put together (see Fig. 8), the steady-state solu-tion can be identified as the intersection g of the curves $\Delta k_{t+1} = 0$ and $\Delta c_{t+1} = 0$. The arrows show the dynamics of k_{t+1} and c_{t+1}.

The curve PP (the saddlepath or stable manifold) indicates the only states that are attainable (PP may change when some parameter of the model is modified). If the economy were outside PP, the dynamics guarantees that the steady state is never reached.

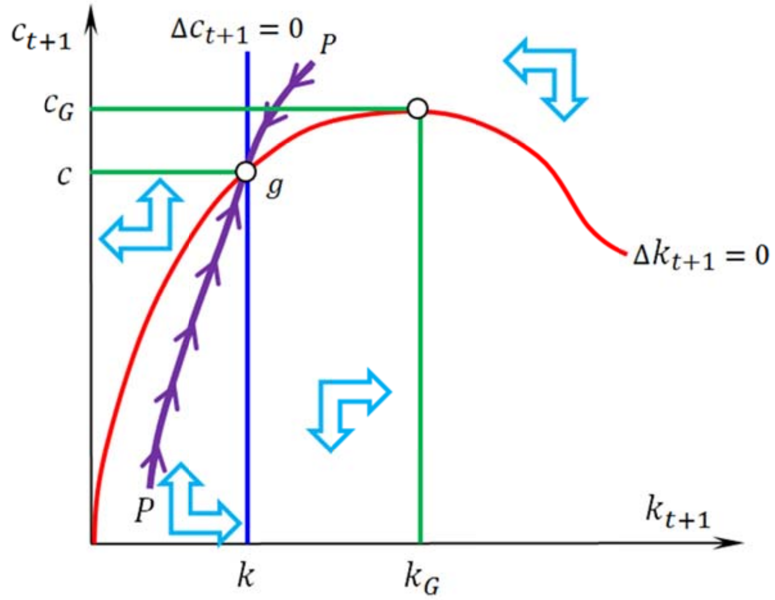

Fig. 8. Phase diagram

Ejercicio 5. Explica la dinámica del consumo y del capital en el punto e de la Fig. 7. ¿Depende la respuesta de dónde se sitúe la recta que representa el estado estacionario del consumo?

Ejercicio 6. Sea

$$u(c_t) = \frac{{c_t}^{1-\sigma} - 1}{1-\sigma}$$

donde $\sigma = -c \cdot u''/u'$ se interpreta como coeficiente de aversión relativa al riesgo. La función de producción es $f(k_t) = A \cdot {k_t}^{\alpha}$. Halla la ecuación de Euler y los valores del capital y del consumo de estado estacionario. Compara los resultados con los de la regla de oro.

Ejercicio 7. Partiendo de la Fig. 8, analiza el efecto sobre la trayectoria de ajuste de la economía (*saddlepath*) y sobre los valores c y k de estado estacionario si: (i) se modifica la tasa de depreciación δ; (ii) se modifica el factor de descuento β; (iii) se produce una mejora tecnológica que afecta a f permanentemente; y (iv) se produce una mejora tecnológica que afecta a f transitoriamente.