3. Overlapping generations models with money

1. Fiat money

Given the failure of the First Welfare Theorem in overlapping generations models, there is no guarantee that the consumption allocation of a general competitive equilibrium be Pareto efficient. This observation leads to the consideration of mechanisms allowing the economy to reach a Pareto efficient consumption allocation.

One such mechanism may be called "fiat money", an intrinsically worthless asset accepted in exchange for the good under the belief that it will be also accepted in exchange for the good in the future.

Example 1.1. Consider the economy described Bhattacharya (2008); see also Champ et al. (2011, ch. 1) or McCandless and Wallace (1991, ch. 10). All generations are identical, each generation grows at a constant rate $n > 0$, and old people have no endowment. Specifically: (i) for every generation t, $N(t) = (1 + n) \cdot N(t - 1)$; (ii) for every individual $i \in N(t)$, $u_t^i\left(c_t^i(t), c_t^i(t+1)\right) = c_t^i(t) \cdot c_t^i(t+1)$ represents i's utlity when young and $w_t^i = (w, 0)$ is i's lifetime endowment; and (iii) for every individual $i \in N(t)$, $c_t^i(t+1)$ represents the utility of i when old (in $t + 1$).

Since all young individuals are identical, inside money (loans) is not possible, because there is no loan market: all young individuals would like to lend and no one would like to borrow. Therefore, there is no trade and consumers must consume their endowments. The consequence of this autarkic state of the economy' is that the old starve.

As all members of a generation have the same characteristics, it is possible to measure the welfare of a generation in terms of the common utility function.

Proposition 1.2. *The feasible consumption bundle that maximizes the welfare of each generation in Example 1.1 is* $\left(\frac{w}{2}, (1+n) \cdot \frac{w}{2}\right)$.

Proof. The consumption bundle that maximizes generation t's welfare is obtained by maximizing the common utility function $u_t^i\left(c_t^i(t), c_t^i(t+1)\right)$ subject to the resource constraint in period t t

$$N(t) \cdot c_t^i(t) + N(t-1) \cdot c_{t-1}^i(t) = N(t) \cdot w$$

where w is the young individual's endowment. The right-hand side represents the total supply of the good available in period t, whereas the left-hand side stands for the total demand for the good in t, namely, the amount $N(t) \cdot c_t^i(t)$ that the young demand plus the amount $N(t-1) \cdot c_{t-1}^i(t)$ that the old demand.

As $N(t) = (1+n) \cdot N(t-1) > 0$ and $c_{t-1}^i(t) = c_t^i(t+1)$,

$$c_t^i(t) + \frac{c_t^i(t+1)}{1+n} = w$$

where n (the population growth rate) can be interpreted as a short of "biological interest rate". Given that $u_t^i\left(c_t^i(t), c_t^i(t+1)\right) = c_t^i(t) \cdot c_t^i(t+1)$, then the solution satisfies

$$1 + n = \frac{c_t^i(t+1)}{c_t^i(t)}$$

$$c_t^i(t+1) = \left[w - c_t^i(t)\right] \cdot (1+n).$$

Consequently, $c_t^i(t) = \frac{w}{2}$ and $c_t^i(t+1) = (1+n) \cdot \frac{w}{2}$. ■

In autarky, the utility of each young individual is $u_t^i(w, 0) = 0$ and the utility of each old individual is $c_t^i(t+1) = w_t^i(t+1) = 0$ (being unable to borrow when young, the fact that an old individual has no endowment means that he cannot consume).

In the welfare-maximizing solution, every individual in t gets positive utility: when the individual is young, he obtains $u_t^i\left(\frac{w}{2}, (1+n) \cdot \frac{w}{2}\right) = \frac{(1+n) \cdot w^2}{4} > 0$; when old, he obtains $c_t^i(t+1) = \frac{(1+n) \cdot w}{2} > 0$.

The welfare-maximizing solution could be regarded as the one that a social planner would choose. Is this solution achievable through a money market?

To answer this question, imagine that, in the economy of Example 1.1, the old invent fiat money in the initial period 1: a worthless asset intended to be generally accepted in exchange for the good in a competitive market.

Let M be the amount of fiat money created in $t = 1$. This amount can be bought and sold every period t in a money market.

Definition 1.3. For all t, $p(t)$ designates the price of the good in terms of money: in t, one unit of good is worth $p(t)$ units of money.

Definition 1.3 suggests that $p(t)$ can be interpreted as the price level in the economy, whereas $v(t) = 1/p(t)$ would be the price, value or purchasing power of money. Hence, $v(t)$ would measure the amount of the good that one unit of money can purchase in period t.

Using the money market a young individual could save purchasing power for the future by holding money. Let $m^i(t)$ represent the number of money units bought by an individual i who is young in period t. As a result, when young, i's budget constraint becomes

$$c_t^i(t) + \frac{m^i(t)}{p(t)} = w \tag{1}$$

and, when old, his budget constraint is

$$c_t^i(t+1) = \frac{m^i(t)}{p(t+1)}\,. \tag{2}$$

Remark 1.4. Individuals are assumed to have perfect foresight: every young individual in t knows the price level $p(t+1)$ in the next period.

Money demand per person in t is $m^i(t) = p(t) \cdot [w - c_t^i(t)]$. Total money demand in t is then $N(t) \cdot m^i(t)$. In a money market equilibrium total money demand in t equals total money supply in t. That is, $N(t) \cdot m^i(t) = M$. Consequently,

$$p(t) = \frac{M}{N(t) \cdot [w - c_t^i(t)]}\,.$$

This relationship is also valid for $t+1$:

$$p(t+1) = \frac{M}{N(t+1) \cdot [w - c_{t+1}^i(t+1)]}\,.$$

Given that all generations are identical, $c_{t+1}^i(t+1) = c_t^i(t)$. Thus, $N(t+1) = (1+n) \cdot N(t)$ implies

$$\frac{p(t)}{p(t+1)} = \frac{N(t+1)}{N(t)} = 1 + n\,.$$

The above equation (see also (3) below) defines the equilibrium condition in the money market.

Definition 1.5. The ratio

$$P = \frac{p(t)}{p(t+1)}$$

is the gross return of fiat money: it is the amount of good earned in $t+1$ by investing in t one unit of good in money.

One unit of the good in t can get (can be exchanged for) $p(t)$ units of money in t. As each unit of money in $t+1$ buys $v(t+1) = 1/p(t+1)$ units of the good in $t+1$, $p(t)$ can buy $P = p(t)/p(t+1)$ units of the good. As a result, one unit of the good invested in money in period t yields P units of the good in period $t+1$. The following sketch summarizes the explanation.

$$\text{1 unit of the good in } t \;\rightarrow\; p(t) \text{ units of money in } t \;\rightarrow\; \frac{p(t)}{p(t+1)} \text{ units of good in } t+1$$

The interpretation of n as a (net, real) interest rate can be taken to mean that the future (real) value of an asset is $(1+n)$ times the present (real) value of the asset. In the case at hand, the asset is money and the real value in period t of one unit of money is $\frac{1}{p(t)}$. This just follows from the fact that $\frac{1}{p(t)}$ measures the purchasing power in t of one unit of money.

With gross real interest rate assumed to be $1+n$, the real value in $t+1$ of $\frac{1}{p(t)}$ is $\frac{1}{p(t)} \cdot (1+n)$. This has to be the real value the asset "money" in $t+1$. By definition, the real value of money in $t+1$ is its purchasing power $\frac{1}{p(t+1)}$. Summarizing, it must be that

$$\frac{1}{p(t+1)} = \frac{1}{p(t)} \cdot (1+n)$$

or, rearranging,

$$\boxed{\frac{p(t)}{p(t+1)} = 1+n} \tag{3}$$

which is the equilibrium condition in the money market. This condition can then be understood as sustained by arbitrage: the return (in real terms) $1+n$ of making a hypothetical loan of the good should the same as the return $\frac{p(t)}{p(t+1)}$ of investing in money.

Proposition 1.6. *The introduction of fiat money and a market for fiat money in the economy of Example 1.1 makes it possible to attain the feasible consumption bundle* $\left(\frac{w}{2}, (1+n) \cdot \frac{w}{2}\right)$ *that maximizes the welfare of each generation.*

Proof. Consider the money market so far discussed that originates in the decision of the old from period to create the amount M of fiat money. The demand for money by a young individual i is obtained by maximizing $u_t^i\left(c_t^i(t), c_t^i(t+1)\right) = c_t^i(t) \cdot c_t^i(t+1)$ subject to (1) and (2). That is, i chooses $m^i(t)$ to

$$\text{maximize } \left(w - \frac{m^i(t)}{p(t)}\right) \cdot \frac{m^i(t)}{p(t+1)}$$

where, since the money market is assumed competitive, the prices $p(t)$ and $p(t+1)$ are taken as given. After equating to zero the derivative with respect to $m^i(t)$, real money demand is

$$\frac{m^i(t)}{p(t)} = \frac{w}{2}.$$

The resulting consumption bundle for every individual coincides with the welfare maximizing consumption bundle from Proposition 1.2,

$$c_t^i(t) = w - \frac{m^i(t)}{p(t)} = \frac{w}{2}$$

$$c_t^i(t+1) = \frac{m^i(t)}{p(t+1)} = \frac{w \cdot \frac{p(t)}{2}}{p(t+1)} = \frac{w \cdot (1+n)}{2},$$

which is what had to be shown.■

Proposition 1.6 shows that fiat money (i) generates the consumption levels that maximize welfare and (ii) improves upon the no trade situation.

Remark 1.7. The effectivity of fiat money as a mechanism to improve efficiency depends on its ability to be a deposit of value, namely, that individuals believe that the money bought when young will be accepted when old in exchange for goods. If the young in t no longer expect that the young in $t+1$ will be willing to accept money, then the young in t will not accept money. In this case, money is actually absolutely worthless and turns out to have no economic function.

Remark 1.8. The individuals' maximization problem in the proof of Proposition 1.6 has been solved by direct substitution of (1) and (2) into the utility function. The alternative procedure is to maximize utility subject to the lifetime budget constraint obtained by putting together (1) and (2) so that $m^i(t)$ cancels out:

$$\begin{aligned} &maximize_{\{c_t^i(t),\, c_t^i(t+1)\}} \;\; u_t^i\left(c_t^i(t), c_t^i(t+1)\right) = c_t^i(t)\cdot c_t^i(t+1) \\ &subject\ to \;\; c_t^i(t) + \frac{p(t+1)}{p(t)}\cdot c_t^i(t+1) = w_t^i(t)\,. \end{aligned} \tag{4}$$

By setting $R = 1+n$, the money market equilibrium condition (3) implies that $\frac{p(t+1)}{p(t)} = \frac{1}{R}$. As a consequence, the lifetime budget constraint is actually the familiar requirement $c_t^i(t) + \frac{c_t^i(t+1)}{R(t)} = w_t^i(t) + \frac{w_t^i(t+1)}{R(t)}$ in disguise.

Remark 1.9. If the inflation rate in period t is defined as

$$\pi(t) = \frac{p(t)-p(t-1)}{p(t-1)} = \frac{p(t)}{p(t-1)} - 1\,,$$

then, by (3),

$$1+\pi(t) = \frac{p(t)}{p(t-1)} = \frac{p(t+1)}{p(t)} = \frac{1}{1+n}$$

or, equivalently,

$$\boxed{\pi(t) = -\frac{n}{1+n}.}$$

This says that the economy of Example 1.1 with the fixed supply M of fiat money experiences deflation at a constant rate. Since

$$\pi(t) = -\frac{1}{1+\frac{1}{n}}$$

deflation is more intense the higher the rate n of population growth (conversely, $\pi > 0$ if $n < 0$):

$$\uparrow n \;\Rightarrow \downarrow \tfrac{1}{n} \Rightarrow \uparrow \tfrac{1}{1+\frac{1}{n}} \;\Rightarrow \uparrow |\pi|\,.$$

In addition, $c_t^i(t+1) = \frac{w/2}{1+\pi(t+1)}$: the old consume half of the (inflation-based) present value of the young's endowment.

Remark 1.10. The analysis in the proof of Proposition 1.6 has been carried out in real terms. Since the economy in that proof is monetary, budget constraints could be set in monetary terms. In particular, for a young individual,

$$p(t) \cdot c_t^i(t) + m(t) = p(t) \cdot w_t^i(t)$$

whereas, for an old individual,

$$p(t+1) \cdot c_t^i(t+1) = m(t).$$

By combining the two constraints,

$$p(t) \cdot c_t^i(t) + p(t+1) \cdot c_t^i(t+1) = p(t) \cdot w_t^i(t)$$

or, dividing by $p(t)$,

$$c_t^i(t) + \frac{p(t+1)}{p(t)} \cdot c_t^i(t+1) = w_t^i(t),$$

which is the lifetime budget constraint expressed in real terms from (4).

2. Money and the government

Remark 2.1. The creation in $t = 1$ of the amount M of money in the economy of Example 1.1 could be attributed to the government. The government can create fiat money and give that money to the old in $t = 1$ as a kind of pension. A money market can next be set in motion if the government declares the money given to the old to be legal tender, that is, the government ensures the buyers of money that money will be accepted in exchange for the good in the next period.

If the government is given the faculty to create money for the old in $t = 1$, it seems natural to consider the possibility of giving money for the old of all generations.

Example 2.2. The economy is that of Example 1.1 with the difference that the government gives, each period, the amount M of money to the old in that period. Money is not destroyed. In addition, the government sets a money market in which money for the current and previous periods can be bought and sold. What is in this case the dynamics of the price level $p(t)$ when the money market is assumed to be in equilibrium?

In the economy of Example 2.2 the budget constraints of an individual of generation t are still given by (1) and (2). Consequently, money demand per person in t is $m^i(t) = p(t) \cdot [w - c_t^i(t)]$ and total money demand in t is $N(t) \cdot m^i(t) = N(t) \cdot p(t) \cdot [w - c_t^i(t)]$.

The total money supply is no longer constant: in $t = 1$ it is M; in $t = 2$ it is $M + M = 2 \cdot M$; in $t = 3$ it is $2 \cdot M + M = 3 \cdot M$; and, in general, in period t, it is $t \cdot M$. In view of this, the money market equilibrium condition in period t holds that

$$t \cdot M = N(t) \cdot p(t) \cdot \left[w - c_t^i(t)\right].$$

Thus,

$$p(t) = \frac{t \cdot M}{N(t) \cdot \left[w - c_t^i(t)\right]}$$

and

$$p(t+1) = \frac{(t+1) \cdot M}{N(t+1) \cdot \left[w - c_{t+1}^i(t+1)\right]}.$$

Using the relationship $N(t+1) = (1+n) \cdot N(t)$,

$$\frac{p(t)}{p(t+1)} = \frac{\dfrac{t \cdot M}{N(t) \cdot \left[w - c_t^i(t)\right]}}{\dfrac{(t+1) \cdot M}{(1+n) \cdot N(t) \cdot \left[w - c_{t+1}^i(t+1)\right]}} = (1+n) \cdot \frac{t}{t+1} \cdot \frac{w - c_{t+1}^i(t+1)}{w - c_t^i(t)}. \qquad (5)$$

The individual's decision problem is still represented by (4). The solution to this problem is such that $c_t^i(t) = w/2$. Fortunately, this solution is independent of t. This means that $c_{t+1}^i(t+1) = w/2$. Therefore, (5) is transformed into (6).

$$\frac{p(t)}{p(t+1)} = (1+n) \cdot \frac{t}{t+1} \qquad (6)$$

The inflation rate arising from (6) is (where $t \geq 1$)

$$\pi(t+1) = \frac{p(t+1) - p(t)}{p(t)} = \frac{1}{1+n} \cdot \frac{t+1}{t} - 1 = \frac{1 - t \cdot n}{t + t \cdot n} = \frac{\dfrac{1}{t} - n}{1+n}. \qquad (7)$$

Proposition 2.3. *In the economy of Example 2.2, given a fixed positive value of the rate n of population growth, deflation eventually occurs.*

Proof. As just shown, the inflation rate in period $t+1$ is given by (7). Since $n > 0$, this rate is negative if and only if $\frac{1}{t} < n$. But, no matter the value of n, for a sufficiently large t, $\frac{1}{t}$ will be smaller than n. In fact, as $t \to \infty$, $\frac{1}{t} \to 0$; that is, the inflation rate converges to the inflation rate $-\frac{n}{1+n}$ of the economy of Example 1.1 (an economy where the money supply does not grow). ■

3. Exercicis

Exercici 1. Diner fiduciari en expansió i població constant. Considera l'economia monetària basada en l'Exemple 1.1 amb l'única diferència que la població no creix ($n = 0$) i la quantitat de diner creix cap període a una taxa constant $m > 0$. Calcula el valor de la taxa d'inflació i, en concret, el seu signe.

Exercici 2. Diner fiduciari i població en expansió. Considera l'economia monetària basada en l'Exemple 1.1 amb l'única diferència que la quantitat de diner creix cap període a una taxa constant $m > 0$. Expressa $\frac{p(t)}{p(t+1)}$ i la taxa d'inflació $\pi(t)$ en funció d'n i d'm.

Exercici 3. Teoria quantitativa del diner. En l'economia monetària basada en l'Exemple 1.1, és certa la implicació de la teoria quantitativa del diner segons la qual el nivell de preus és proporcional a l'estoc de diner? En concret, que passaria amb el nivell de preus en un determinat periode t si l'estock de diner és dupliqués?

Exercici 4. Equilibri amb diner i préstecs. La funció d'utilitat de cada consumidor i és $u_t^i\left(c_t^i(t), c_t^i(t+1)\right) = c_t^i(t) \cdot c_t^i(t+1)$. Cada generació està formada per 100 individus, 60 amb dotació $(1, 0)$ i els altres 40 amb dotació $(0, 1)$.

(i) Calcula l'equilibri general si només hi ha un mercat de préstecs.

(ii) Calcula l'equilibri general si, a més del mercat de préstecs (del bé), hi ha un mercat de diner on es compra i ven una unitat (perfectament divisible) de diner fiduciari creada i venuda en el període inicial per la gent gran. Determina la funció de demanda de diner dels joves. Compara la utilitat que cada individu obté en aquest i en l'anterior apartat.

Exercici 5. Equilibri amb diner i bons. (i) Basant-te en una economia com la descrita en l'Exemple 1.1, construeix i soluciona un model on el govern transfereix τ unitats del bé a cada individu gran del període inicial, finança la transferència amb emissió de bons i, en el període següent, paga el valor real dels bons amb diner fiduciari indestructible. (ii) Torna a respondre l'apartat (i) suposant que el diner creat pel govern és destructible (només dura una quantitat finita de períodes).

4. References

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