An OLG model of capital accumulation and fertility decisions

1. Description of the economy

Time is discrete. Only one good Y is produced. Each unit of the good lasts at most two periods. There is no money, so variables are measured in terms of the good. The good is produced using two inputs: capital K and labour L. The production function (1) establishes the amount Y_t of good Y that can be produced in period t with the basket of inputs (K_t, L_t), where α and ϕ are positive constants.

$$Y_t = (K_t)^{\alpha} \cdot (L_t)^{\phi} \tag{1}$$

All individuals in the economy are identical (individuals are presumed to be females since the basic human form appears to be female). Each individual lives for three consecutive periods: childhood, adulthood, and old age. An adult individual chooses how many children to have and makes all decisions for her children. Therefore, in practice, it is as if individuals lived for only two periods.

The goal of each adult individual is to maximize the utility function u_t given by (2), where c_t is the consumption of the good by the individual when adult, $\hat{c}_{t+1}$ is his consumption when old, and β is a positive constant.

$$u_t = c_t \cdot (\hat{c}_{t+1})^{\beta} \tag{2}$$

Every adult in period t has one unit of labour that supplies inelastically in exchange for a wage ω_t (of units of the good per unit of labour). There are three uses for this wage: consumption c_t, capital accumulation k_{t+1}, and supporting the n_{t+1} children that the adult has chosen to have (capital is good that can be transferred without cost or loss to the next period). The cost for an adult of raising a children is $\gamma > 0$. Thus, an adult in t faces the budget constraint (3).

$$c_t + k_{t+1} + \gamma \cdot n_{t+1} = \omega_t \tag{3}$$

An old individual is only interested in consuming as much as possible of the good. Each old individual has two sources of income (old individuals are not endowed with labout). First, the capital k_{t+1} accumulated in t when young can be rented in $t+1$ for a rental price σ_{t+1} (of units of the good per unit of capital). Second, each of the n_{t+1} children the old gave birth in t when young transfer to the old a pension of p_{t+1} units of the good. As a result, an old individual in $t+1$ is subject the budget constraint (4).

$$\hat{c}_{t+1} = \sigma_{t+1} \cdot k_{t+1} + p_{t+1} \cdot n_{t+1} \tag{4}$$

Young individuals could be interpreted as workers. Old individuals could be seen as capitalists that, simultaneously, are pensioneers. As only adults supply labour, the number of adult individuals in period t will be denoted by L_t.

Some social mechanism determines the distribution of income between workers, capitalists, and pensioneers. Equations (5) and (6) summarize the mechanism, where $\tilde{\phi}$ and $\tilde{\alpha}$ are numbers representing proportions, with $\tilde{\phi} + \tilde{\alpha} < 1$.

$$\omega_t \cdot L_t = \tilde{\phi} \cdot Y_t \tag{5}$$
$$\sigma_t \cdot K_t = \tilde{\alpha} \cdot Y_t \tag{6}$$

Viewed as income, Y_t is distributed among individuals according to (7).

$$\omega_t \cdot L_t + \sigma_t \cdot K_t + p_t \cdot L_t = Y_t \tag{7}$$

The pension bill $p_t \cdot L_t$ in period t is generated by L_{t-1} old individuals (in period t) each one receiving p_t from each of her n_t daughters. Hence, the old receive $p_t \cdot n_t \cdot L_{t-1}$. This amounts to $p_t \cdot L_t$ given that $n_t \cdot L_{t-1} = L_t$: the n_t children the L_{t-1} people that were young in $t-1$ become the L_t young in period t.

It follows from (5), (6), and (7) that

$$p_t \cdot L_t = \left(1 - \tilde{\alpha} - \tilde{\phi}\right) \cdot Y_t. \tag{8}$$

From the expenditure point of view, there are four uses of Y_t: two consumption categories (consumption by the adults and by the olds) and two investment categories (investment in capital and in children). (9) represents the four demand components of the economy

$$C_t + \hat{C}_t + K_{t+1} + \gamma \cdot L_{t+1} = Y_t \tag{9}$$

where:

- $C_t = c_t \cdot L_t$ is the total consumption in period t by the L_t adults in t;
- $\hat{C}_t = \hat{c}_t \cdot L_{t-1}$ is the total consumption in period t by the L_{t-1} old in t;
- $K_{t+1} = k_{t+1} \cdot L_t$ is the total stock of capital accumlated in period t (to be used in $t+1$) by the L_t adults in t;
- $\gamma \cdot L_{t+1} = \gamma \cdot n_{t+1} \cdot L_t$ is the investment made in period t by the L_t adults in t when each such adult invests γ units of the good in each of her n_{t+1} children.

2. Solving the model

An old individual maximizes (2), with respect to c_t and $\hat{c}_{t+1}$, subject to (3) and (4). Define

$$y_t = \frac{Y_t}{L_t}.$$

Given (5), (3) can be rewritten as

$$c_t + k_{t+1} + \gamma \cdot n_{t+1} = \tilde{\phi} \cdot y_t. \tag{10}$$

On the other hand, by (5) and (7), (4) can be rewritten as

$$\hat{c}_{t+1} = \sigma_{t+1} \cdot k_{t+1} + p_{t+1} \cdot n_{t+1} = \sigma_{t+1} \cdot \frac{K_{t+1}}{L_t} + p_{t+1} \cdot \frac{L_{t+1}}{L_t} = \frac{Y_{t+1} - \omega_t \cdot Y_{t+1}}{L_t} =$$
$$= \left(1 - \tilde{\phi}\right) \cdot \frac{Y_{t+1}}{L_{t+1}} \cdot \frac{L_{t+1}}{L_t} = \left(1 - \tilde{\phi}\right) \cdot y_{t+1} \cdot n_{t+1}.$$

That is,

$$\hat{c}_{t+1} = \left(1 - \tilde{\phi}\right) \cdot y_{t+1} \cdot n_{t+1}. \tag{11}$$

The maximization of (2) subject to (10) and (11) requires that

$$\frac{\gamma \cdot \hat{c}_{t+1}}{y_{t+1} \cdot \left(1 - \tilde{\phi}\right)} = \beta \cdot c_t. \tag{12}$$

Therefore,

$$c_t = \frac{1}{1+\beta} \cdot \left(\tilde{\phi} \cdot y_t - k_{t+1}\right) \tag{13}$$

and, after combining (11)-(13),

$$n_{t+1} = \frac{\beta}{\gamma \cdot (1+\beta)} \cdot \left(\tilde{\phi} \cdot y_t - k_{t+1}\right). \tag{14}$$

Expressing (14) in aggregate terms (since $n_{t+1} = L_{t+1}/L_t$),

$$L_{t+1} = \frac{\beta}{\gamma \cdot (1+\beta)} \cdot \left(\tilde{\phi} \cdot Y_t - K_{t+1}\right) \tag{15}$$

or, equivalently,

$$Y_t = \frac{1}{\tilde{\phi}} \cdot K_{t+1} + \frac{\gamma \cdot (1+\beta)}{\tilde{\phi} \cdot \beta} \cdot L_{t+1}. \tag{16}$$

The derivative $\frac{\partial n_{t+1}}{\partial y_t}$ indicates how fertility decisions are affected by labour productivity: as $y_t = Y_t/L_t$, y_t is not income per capita but the average income generated by workers (adults). Making abstraction of children, income per capita would rather be given by $\frac{Y_t}{L_t+L_{t-1}}$.

In any case, the derivative $\frac{\partial n_{t+1}}{\partial y_t}$ can be used to establish whether the economy is Malthusian, that is, if $\frac{\partial n_{t+1}}{\partial y_t} > 0$: an increase in the productivity of labour stimulates fertility and population growth. Since

$$\frac{\partial n_{t+1}}{\partial y_t} = \frac{\beta}{\gamma \cdot (1+\beta)} \cdot \left(\tilde{\phi} - \frac{\partial k_{t+1}}{\partial y_t}\right)$$

it follows that the economy is Malthusian if and only if $\tilde{\phi} > \frac{\partial k_{t+1}}{\partial y_t}$, that is, if and only the increase in the capital stock caused by a rise in the labour productivity is not large enough.

Conversely, $\uparrow y_t \Rightarrow \downarrow n_{t+1}$ if $\frac{\partial k_{t+1}}{\partial y_t}$ is sufficiently high. In words, the basic recipe for overcoming the Malthusian constraints is to accumulate enough capital and, specifically, for each additional unit of income y_t, the extra capital that is accumulated must be larger than the fraction $\tilde{\phi}$ that workers receive: $\frac{\partial k_{t+1}}{\partial y_t} > \tilde{\phi}$. Unfortunately, this does not seem possible in the model: a worker (adult individual) receives income $\tilde{\phi} \cdot y_t$ and, accordingly, it is impossible for her to accumulate more than that.

Turning to (15), consider the derivative $\frac{\partial L_{t+1}}{\partial K_t}$: how do changes in yesterday's capital stock affect today's population? Expressing (15) as $\beta \cdot \tilde{\phi} \cdot Y_t = \beta \cdot K_{t+1} + \gamma \cdot (1+\beta) \cdot L_{t+1}$, by taking the differential dK_t with respect to K_t,

$$\beta \cdot \tilde{\phi} \cdot \frac{dY_t}{dK_t} = \beta \cdot \frac{dK_{t+1}}{dK_t} + \gamma \cdot (1+\beta) \cdot \frac{dL_{t+1}}{dK_t}.$$

The term $\frac{dY_t}{dK_t}$ is the marginal productivity of capital MPK_t. Hence, solving for $\frac{dL_{t+1}}{dK_t}$,

$$\frac{dL_{t+1}}{dK_t} = \frac{\beta \cdot \left(\tilde{\phi} \cdot MPK_t - \frac{dK_{t+1}}{dK_t}\right)}{\gamma \cdot (1+\beta)}.$$

Consequently, $\frac{dL_{t+1}}{dK_t} < 0$ if and only if $\frac{dK_{t+1}}{dK_t} > \tilde{\phi} \cdot MPK_t$.

3. Stationary states

Using (16), a stationary state of the economy satisfies $Y = \frac{1}{\tilde{\phi}} \cdot K + \frac{\gamma \cdot (1+\beta)}{\tilde{\phi} \cdot \beta} \cdot L$; that is, $K^{\alpha} \cdot L^{\phi} = \frac{1}{\tilde{\phi}} \cdot K + \frac{\gamma \cdot (1+\beta)}{\tilde{\phi} \cdot \beta} \cdot L$. Dividing by L leads to $k^{\alpha} \cdot \ L^{\alpha+\phi-1} = \frac{1}{\tilde{\phi}} \cdot k + \frac{\gamma \cdot (1+\beta)}{\tilde{\phi} \cdot \beta}$ or, equivalently,

$$k^{\alpha} \cdot \tilde{\phi} \cdot L^{\alpha+\phi-1} = k + \frac{\gamma \cdot (1+\beta)}{\beta} \tag{17}$$

Therefore, a stationary state value of the capital per worker k is geometrically determined by the intersection of the functions $f(k) = k^{\alpha} \cdot \tilde{\phi} \cdot L^{\alpha+\phi-1}$ and $g(k) = k + \frac{\gamma \cdot (1+\beta)}{\beta}$. The latter is a linear function, whereas the former is concave if $\alpha < 1$, convex if $\alpha > 1$, and linear if $\alpha = 1$.