8. Representative agent models

1. The neoclassical growth model (Frank Ramsey 1928)

Time is discrete. There is only one good in each period. Expressing variables in per capita terms, in each period t, production in t equals consumption in t plus investment in t.

$$y_t = c_t + i_t \tag{1}$$

Output can only be consumed or saved: $y_t = c_t + s_t$. Therefore, $i_t = s_t$.

In each period a fraction $0 < \delta < 1$ of capital depreciates. Capital in $t+1$ is investment in t plus the remaining capital from period t.

$$k_{t+1} = i_t + (1-\delta) \cdot k_t \tag{2}$$

The production function f makes output per capita depend on capital per capita.

$$y_t = f(k_t) \tag{3}$$

The production function f satisfies the typical properties: $f \geq 0$, $f' > 0$, $f'' < 0$, $\lim_{k_t \to 0} f'(k_t) = \infty$, and $\lim_{k_t \to \infty} f'(k_t) = 0$.

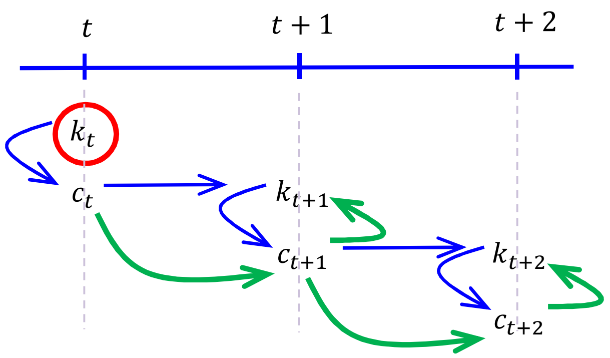
Combining (1), (2), and (3),

$$f(k_t) = c_t + k_{t+1} - (1-\delta) \cdot k_t \tag{4}$$

or, defining $\Delta k_{t+1} = k_{t+1} - k_t$,

$$f(k_t) = c_t + \Delta k_{t+1} + \delta \cdot k_t. \tag{5}$$

Equation (5) establishes the dynamic constraint the economy faces. There are two interpretations of this constraint. Interpretation 1: given k_t, c_t and k_{t+1} are determined; given k_{t+1}, c_{t+1} and k_{t+2} are determined… Interpretation 2: given k_t, decision is over c_t, c_{t+1}, c_{t+2}…



There is a representative agent. If population is constant, then variables can be seen as per capita variables (c_t is what the agent consumes in period t).

Suppose that the aim of the agent is to maximize consumption each period (no discounting). The problem can be solved first by considering the steady state (the long run of the economy). Let c and k designate the steady state values.

It follows from (5) that $f(k) = c + \delta \cdot k$; that is,

$$c = f(k) - \delta \cdot k.$$

This corresponds to the familiar idea that steady-state consumption is the output that remains once taken the output necessary to replace the lost capital, so that capital remains constant.

The first-order condition to maximize c is $\frac{\partial c}{\partial k} = 0$; therefore, $f'(k) = \delta$. Since $f'' < 0$, the second-order condition $\left(\frac{\partial^2 c}{\partial k^2} < 0\right)$ holds.

The condition $f'(k) = \delta$ states that the marginal product of capital equals its depreciation rate. This solution is known as "the golden rule". If $f'(k) < \delta$, c can be increased by rising k. If $f'(k) > \delta$, c can be increased by lowering k.

Shocks and the golden rule

Denote by (c_G, k_G) the golden rule solution. Suppose that capital is exogenously reduced to $k < k_G$ but the agent tries to maintain c_G.

In this case $c_G = f(k_G) - \delta \cdot k_G$ and $c = f(k) - \delta \cdot k - \Delta k$. If $c_G = c$, then $f(k_G) - \delta \cdot k_G = f(k) - \delta \cdot k - \Delta k$. Solving for Δk,

$$\Delta k = (f(k) - \delta \cdot k) - (f(k_G) - \delta \cdot k_G).$$

As (c_G, k_G) is the golden rule solution $f(k_G) - \delta \cdot k_G > f(k) - \delta \cdot k$. In sum, $\Delta k < 0$.

In words, with less capital than the golden value k_G, future output would be smaller. The attempt to keep c_G will further decrease the stock of capital, making the consumption level c_G eventually untenable. The lesson is that "too much" consumption sooner or later exhausts the capital stock, so the economy will be unable to sustain that consumption level.

The solution to the negative shock on k consists in diverting consumption temporarily to rebuild the capital stock. Once k_G is restored, c can be increased to reach level c_G.

If consumption in different periods is valued differently, the agent may choose to maximize the present value of the infinite sequence of consumption $(c_0, c_1, c_2, \ldots)$ or, given a utility function u common for each t, the present value of $(u(c_0), u(c_1), u(c_2), \ldots)$.

$$max_{\{c_t,\, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$\text{subject to } \; c_t + k_{t+1} = f(k_t) + (1 - \delta) \cdot k_t$$

The usual assumptions on u hold: $u \geq 0$, $u' > 0$, and $u'' < 0$. The parameter $\beta \in (0, 1)$ is the discount factor.

Using the method of Lagrange multipliers, define the Lagrangian as

$$\mathcal{L}_t = \sum_{t=0}^{\infty}[\beta^t \cdot u(c_t) + \lambda_t(f(k_t) + (1-\delta) \cdot k_t - c_t - k_{t+1})]$$

which is maximized with respect to c_t, k_{t+1}, and λ_t ($\mathcal{L}_t$ is not maximized with respect to k_t because k_t is known in period t).

First-order conditions (FOCs)

$$0 = \frac{\partial \mathcal{L}_t}{\partial c_t} = \beta^t \cdot u'(c_t) - \lambda_t$$

$$0 = \frac{\partial \mathcal{L}_t}{\partial k_{t+1}} = \lambda_{t+1}(f'(k_{t+1}) + 1 - \delta) - \lambda_t$$

$$0 = \frac{\partial \mathcal{L}_t}{\partial \lambda_t} = f(k_t) + (1-\delta) \cdot k_t - c_t - k_{t+1}$$

Transversality condition (TC)

$$lim_{t\to\infty}\ \beta^t \cdot u'(c_t) \cdot k_{t+1} = 0$$

Interpretation of the transversality condition

To interpret TC, suppose t is the last period. If $k_{t+1} > 0$ (some capital is left in the last period), then $u'(c_t) = 0$: consuming that capital should have no impact on utility.

If $u'(c_t) > 0$, then it cannot be that some capital is saved for the next (non-existent) period, because utility would be increased by consuming that capital now. Therefore, it must be that $k_{t+1} = 0$.

From the first FOC, $\lambda_t = \beta^t \cdot u'(c_t)$ and $\lambda_{t+1} = \beta^{t+1} \cdot u'(c_{t+1})$. Substituting for λ_t and λ_{t-1} in the second FOC,

$$\beta^{t+1} \cdot u'(c_{t+1}) \cdot [f'(k_{t+1}) + 1 - \delta] = \beta^t \cdot u'(c_t)$$

The result is the so-called Euler equation (6).

$$\beta \cdot u'(c_{t+1}) \cdot [f'(k_{t+1}) + 1 - \delta] = u'(c_t) \tag{6}$$

Interpretation of the Euler equation

Consider the following question: How much additional c_{t+1} can be obtained by just reducing c_t while leaving total utility (and everything else beyond period $t+1$) constant? Since periods after $t+1$ are unaffected, attention can be restricted to $u(c_t) + \beta \cdot u(c_{t+1})$, which must remain constant. Taking the total differential,

$$0 = du(c_t) + d[\beta \cdot u(c_{t+1})] = du(c_t) + \beta \cdot du(c_{t+1}) = u'(c_t) \cdot dc_t + \beta \cdot u'(c_{t+1}) \cdot dc_{t+1}$$

or

$$-\frac{dc_{t+1}}{dc_t} = \frac{u'(c_t)}{\beta \cdot u'(c_{t+1})}. \tag{7}$$

Equation (7) is the marginal rate of substitution: how much consumption from one period should be given up to increase consumption in the other period keeping utility constant. Given that the resource constraints in t and $t+1$ must hold,

$$dc_t + dk_{t+1} = df(k_t) + (1-\delta) \cdot dk_t$$

$$dc_{t+1} + dk_{t+2} = df(k_{t+1}) + (1-\delta) \cdot dk_{t+1}.$$

Equivalently,

$$dc_t + dk_{t+1} = f'(k_t) \cdot dk_t + (1-\delta) \cdot dk_t \tag{8}$$

$$dc_{t+1} + dk_{t+2} = f'(k_{t+1}) \cdot dk_{t+1} + (1-\delta) \cdot dk_{t+1} \tag{9}$$

Since k_t is given in t, $dk_t = 0$. Then (8) becomes $dk_{t+1} = -dc_t$: the extra capital in $t+1$ comes from the consumption cut in t. By assumption, $dk_{t+2} = 0$. As $dk_{t+1} = -dc_t$, (9) is equivalent to

$$dc_{t+1} = -f'(k_{t+1}) \cdot dc_t - (1-\delta) \cdot dc_t$$

or

$$-\frac{dc_{t+1}}{dc_t} = f'(k_{t+1}) + (1-\delta). \tag{10}$$

From (10) and (7) the Euler equation obtains. The interpretation is as follows. The output dc_t not consumed in t yields a utility loss in t of $|u'(c_t) \cdot dc_t|$. This output is invested in $t+1$, as dk_{t+1}, to increase output in $t+1$. The additional output $|f'(k_{t+1}) \cdot dc_t|$ and the undepreciated part $(1-\delta) \cdot dk_{t+1} = |(1-\delta) \cdot dc_t|$ of the extra capital are consumed at $t+1$. All in all,

$$dc_{t+1} = [f'(k_{t+1}) + (1-\delta)] \cdot |dc_t|.$$

The discounted utility of dc_{t+1} is

$$\beta \cdot u'(c_{t+1}) \cdot dc_{t+1} = \beta \cdot u'(c_{t+1}) \cdot [f'(k_{t+1}) + (1-\delta)] \cdot |dc_t|.$$

But to keep utility constant, the utility $\beta \cdot u'(c_{t+1}) \cdot dc_{t+1}$ gained in $t+1$ must equal the utility $u'(c_t) \cdot |dc_t|$ lost in t. As a result,

$$u'(c_t) \cdot |dc_t| = \beta \cdot u'(c_{t+1}) \cdot [f'(k_{t+1}) + (1-\delta)] \cdot |dc_t|$$

which is the Euler equation once the common term $|dc_t|$ is cancelled out.

The steady state solution of the model is determined as follows. For steady-state values c and k, the Euler equation can be written as

$$\beta \cdot u'(c) \cdot [f'(k) + 1 - \delta] = u'(c)$$

so

$$f'(k) = \delta + \frac{1}{\beta} - 1 .$$

The golden rule solution is $f'(k_G) = \delta$. Since $\frac{1}{\beta} - 1 > 0$, $f'(k) > f'(k_G)$. As $f'' < 0$, $k < k_G$. There is less capital than under the golden rule because now future utility is discounted at a rate $\frac{1}{\beta} - 1$. Moreover, $k < k_G$ yields $c < c_G$: discounting lowers consumption.

Concerning the dynamic analysis, it relies on the two equ-tions giving the solution in each t: Euler equation (6) and the resource constraint (5).

$$\beta \cdot \frac{u'(c_{t+1})}{u'(c_t)} \cdot [f'(k_{t+1}) + 1 - \delta] = 1$$

$$\Delta k_{t+1} = f(k_t) - c_t - \delta \cdot k_t \qquad (11)$$

Linearizing the Euler equation by taking a Taylor series expansion of $u'(c_{t+1})$ around c_t,

$$u'(c_{t+1}) \approx u'(c_t) + \Delta c_{t+1} \cdot u''(c_t)$$

or

$$\frac{u'(c_{t+1})}{u'(c_t)} \approx 1 + \Delta c_{t+1} \cdot \frac{u''(c_t)}{u'(c_t)} .$$

Inserting the previous approximation into the Euler equation yields (12), where $\frac{u''}{u'} < 0$.

$$\Delta c_{t+1} = \frac{u''(c_t)}{u'(c_t)} \left(\frac{1}{\beta \cdot [f'(k_{t+1}) + 1 - \delta]} - 1 \right) \qquad (12)$$

Equations (11) and (12) establish the changes in the capital stock and consumption.

Let c and k be the steady-state values, that is, the solutions of (11) and (12) if $\Delta k_{t+1} = \Delta c_{t+1} = 0$. If $k_{t+1} < k$, then $f'(k_{t+1}) > f'(k)$. Hence,

$$\beta \cdot [f'(k_{t+1}) + 1 - \delta] > \beta \cdot [f'(k) + 1 - \delta] .$$

As shown above on this page, $f'(k) = \delta + \frac{1}{\beta} - 1$. Thus, $\beta \cdot [f'(k) + 1 - \delta] = 1$. Consequently, $\beta \cdot [f'(k_{t+1}) + 1 - \delta] > 1$ and, in (9), $\frac{1}{\beta \cdot [f'(k_{t+1}) + 1 - \delta]} < 1$. Since $\frac{u''}{u'} < 0$, the final conclusion is that

$$k_{t+1} < k \Rightarrow \Delta c_{t+1} > 0 .$$

A similar reasoning proves that

$$k_{t+1} > k \Rightarrow \Delta c_{t+1} < 0$$

$$k_{t+1} = k \Rightarrow \Delta c_{t+1} < 0.$$

This consumption dynamics is represented in Fig. 1: for capital stock to the left of the steady-state value k, consumption increases; for stock to the right of k, consumption decreases.

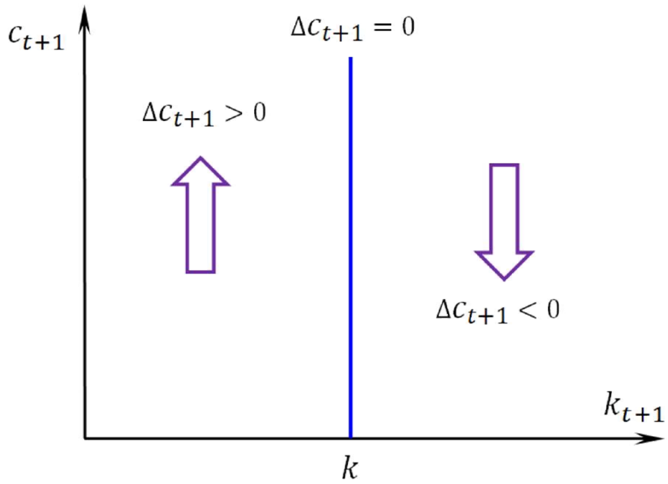


Fig. 1. Consumption dynamics

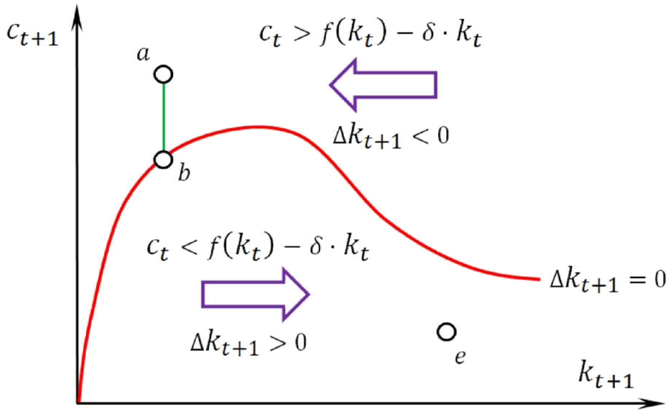


Fig. 2. Capital dynamics

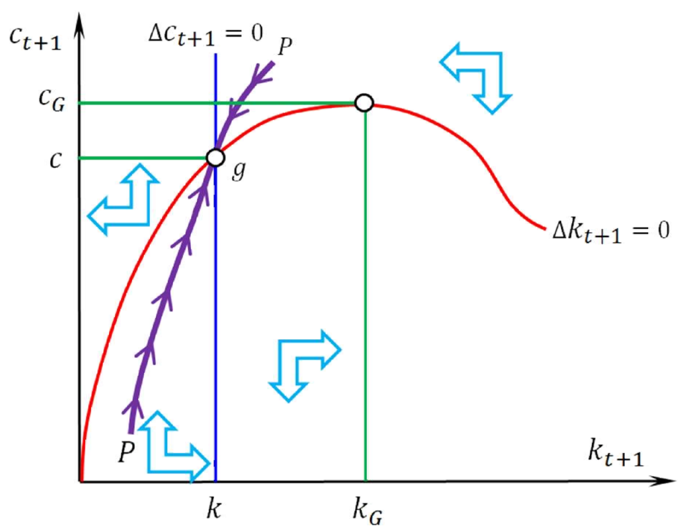
Fig. 2 shows the capital dynamics following (11). Clearly,

$$\Delta k_{t+1} > 0 \ \Leftrightarrow f(k_t) - \delta \cdot k_t > -c_t \, .$$

Above the curve $\Delta k_{t+1} = 0$, consumption is higher that the steady-state consumption, so capital must decumulate. At point a, consumption exceeds the level (given by b) compatible with the steady state (with $\Delta k_{t+1} = 0$). Capital has to decrease to compensate excessive consumption.

Below the curve $\Delta k_{t+1} = 0$, consumption allows capital to accumulate.

When the two preceding figures are put together (Fig. 3), the steady-state solution can be identified as the intersection g of the curves $\Delta k_{t+1} = 0$ and $\Delta c_{t+1} = 0$. The arrows show the dynamics of k_{t+1} and c_{t+1}.



The curve PP (the saddlepath or stable manifold) indicates the only states that are attainable (PP may change when some parameter of the model is modified).

If the economy were outside PP, the dynamics guarantees that the steady state is never reached.

Fig. 3. The phase diagram of the model