3.3. Some post-Solowian growth models

3.3.1. The Frankel[1] (1962) model

The Frankel model tries to get the best of the Solow-Swan and the Harrod-Domar model: the first operates at the micro level (to determine allocation) and the second at the macro level (to generate growth).

The economy has m firms. Each firm i is endowed with a Cobb-Douglas technology

$$Y_i = A \cdot H \cdot K_i{}^{\alpha} \cdot L_i{}^{1-\alpha}$$

that includes a term H interpreted as a development modifier. The value of H represents the level of development of the economy. This level embodies a positive externality for the production Y_i made by firm i when the firm makes uses of K_i units of capital and L_i units of labour. The term A captures the common technology available to all the firms. From the standpoint of a firm H is treated as a parameter because, just by itself, the firm is not capable to alter the level of development of the economy.

Let firm i use the proportion π_i of all the factors: $K_i = \pi_i \cdot K$ and $L_i = \pi_i \cdot L$, where K is the total stock of capital in the economy and L is the total amount of labour. Therefore, total ouput Y is

$$Y = \sum_{i=1}^{m} Y_i = \sum_{i=1}^{m} A \cdot H \cdot K_i{}^{\alpha} \cdot L_i{}^{1-\alpha} = A \cdot H \cdot \sum_{i=1}^{m} K_i{}^{\alpha} \cdot L_i{}^{1-\alpha} =$$

$$= A \cdot H \cdot \sum_{i=1}^{m} \pi_i{}^{\alpha} \cdot K^{\alpha} \cdot \pi_i{}^{1-\alpha} \cdot L^{1-\alpha} = A \cdot H \cdot \sum_{i=1}^{m} \pi_i \cdot K^{\alpha} \cdot L^{1-\alpha} =$$

$$A \cdot H \cdot K^{\alpha} \cdot L^{1-\alpha} \cdot \sum_{i=1}^{m} \pi_i = A \cdot H \cdot K^{\alpha} \cdot L^{1-\alpha}.$$

- **Aggregate production function** $\quad Y = A \cdot H \cdot K^{\alpha} \cdot L^{1-\alpha}$, with $0 < \alpha < 1$, $A > 0$ and $H > 0$

The development modifier H is assumed to depend on the level of capital per capita, which can be viewed as a proxy for development: the higher the amount of capital per worker in an economy, the more developed the economy.

- **Level of development** $\quad H = \left(\frac{K}{L}\right)^{\gamma}$

Inserting this definition of the modifier into the production function,

$$Y = A \cdot H \cdot K^{\alpha} \cdot L^{1-\alpha} = A \cdot \left(\frac{K}{L}\right)^{\gamma} \cdot K^{\alpha} \cdot L^{1-\alpha} = A \cdot K^{\alpha+\gamma} \cdot L^{1-\alpha-\gamma}$$

[1] Frankel, Marvin (1962): "The production function in allocation of growth: A synthesis", *American Economic Review* 52, 995-1022.

When $\alpha + \gamma = 1$, the resulting production function is the one from the model: $Y = A \cdot K$. In this case, each firms is endowed with a Cobb-Douglas production function, as the one in the Solow-Swan model, but the economy as a whole is endowed with a fixed coeficient function, like the one in the Harrod-Domar model.

The rest of the Frankel model is conventional. For simplicity, let $\delta = 0$.

- **Aggregate savings function** $S = \mathbb{s} \cdot Y$, where $0 < \mathbb{s} < 1$
- **Macroeconomic equilibrium** $S = I$
- **Capital accumulation** $K(t+1) = I(t) + K(t)$ or $I = \Delta K$

Combining the preceding equations, with " ' " designating a variable referring to the next period,

$$I = \Delta K \Rightarrow S = \Delta K \Rightarrow \mathbb{s} \cdot Y = \Delta K \Rightarrow \frac{\mathbb{s} \cdot Y}{L'} = \frac{K' - K}{L'} \Rightarrow \frac{\mathbb{s} \cdot Y}{(1+n) \cdot L} = \frac{K'}{L'} - \frac{K}{(1+n) \cdot L}.$$

In sum,

$$\frac{\mathbb{s}}{1+n} \cdot y = k' - \frac{1}{1+n} \cdot k\,.$$

Therefore,

$$\frac{\mathbb{s}}{1+n} \cdot y = k' - k + k - \frac{1}{1+n} \cdot k = \Delta k + \frac{n}{1+n} \cdot k.$$

As a result,

$$\Delta k = \frac{\mathbb{s}}{1+n} \cdot y - \frac{n}{1+n} \cdot k$$

and

$$g_k = \frac{\Delta k}{k} = \frac{\mathbb{s}}{1+n} \cdot \frac{y}{k} - \frac{n}{1+n}.$$

On the other hand,

$$y = \frac{Y}{L} = \frac{A \cdot K^{\alpha+\gamma} \cdot L^{1-\alpha-\gamma}}{L} = A \cdot \frac{K^{\alpha+\gamma}}{L^{\alpha+\gamma}} \cdot \frac{L^{1-\alpha-\gamma}}{L^{1-\alpha-\gamma}} = A \cdot \left(\frac{K}{L}\right)^{\alpha+\gamma} = A \cdot k^{\alpha+\gamma}.$$

In view of this,

$$g_k = \frac{\mathbb{s}}{1+n} \cdot \frac{y}{k} - \frac{n}{1+n} = \frac{\mathbb{s} \cdot A}{1+n} \cdot k^{\alpha+\gamma-1} - \frac{n}{1+n}.$$

When $\delta = 0$, the first part of the above expression holds in the Solow-Swan model:

$$\boldsymbol{g}_k = \frac{\mathbb{s}}{1+n} \cdot \frac{y}{k} - \frac{n}{1+n} = \frac{\mathbb{s}}{1+n} \cdot \frac{f(k)}{k} - \frac{n}{1+n}.$$

In the latter case,

$$\frac{\partial \boldsymbol{g}_k}{\partial k} = \frac{\mathbb{s}}{1+n} \cdot \frac{\partial (f(k)/k)}{\partial k} < 0$$

because $\frac{\partial (f(k)/k)}{\partial k} < 0$ (remember that $f(k)/k$ is a decreasing function). The fact that $\frac{\partial \boldsymbol{g}_k}{\partial k} < 0$ means that, as k grows, the rate at which k is each time smaller and converges to zero (the steady state).

As distinguished from this result, in the Frankel model,

$$\frac{\partial g_k}{\partial k} = (\alpha + \gamma - 1) \cdot \frac{\mathbb{s} \cdot A}{1+n} \cdot k^{\alpha+\gamma-2} \, .$$

Only if $\alpha + \gamma < 1$, the dynamics of k is the same as in the Solow-Swan model, because in this case $\frac{\partial g_k}{\partial k} < 0$. If $\alpha + \gamma = 1$, then k accumulates at a constant rate (not necessarily zero). Finally, if $\alpha + \gamma > 1$, the more capital per capita is accumulaed, the higher the rate at which it accumulates.

Since $y = A \cdot k^{\alpha+\gamma}$, and A, α, and γ are constants,

$$g_y \approx (\alpha + \gamma) \cdot g_k \, .$$

Consequently, $g_k > 0$ implies $g_y > 0$. Hence, sustained growth in output per capita is possible, as in the Harrod-Domar model but unlike the Solow-Swan model.

3.3.2. The Mankiw-Romer-Weil[2] (1992) model

For any variable, V will be written instead of $V(t)$, whereas V' will be written instead of $V(t+1)$. With $\alpha + \hat{\alpha} < 1$, H designating human capital, and technological progress assumed labour-augmenting, the production function is

$$Y = K^{\alpha} \cdot H^{\hat{\alpha}} \cdot (A \cdot L)^{1-\alpha-\hat{\alpha}}.$$

Notation

- $N =$ gross rate of growth of L
- $\mathbb{s} =$ propensity to accumulate K
- $\delta =$ rate at which K depreciates
- $G =$ gross rate of growth of A
- $\hat{\mathbb{s}} =$ propensity to accumulate human capital
- $\hat{\delta} =$ rate at which human capital depreciates

Per capita variables are defined in effective labour units: $k = \frac{K}{A \cdot L}$, $h = \frac{H}{A \cdot L}$, and $y = \frac{Y}{A \cdot L}$.

$$y = \frac{Y}{A \cdot L} = \frac{K^{\alpha} \cdot H^{\hat{\alpha}} \cdot (A \cdot L)^{1-\alpha-\hat{\alpha}}}{A \cdot L} = \frac{K^{\alpha} \cdot H^{\hat{\alpha}}}{(A \cdot L)^{\alpha+\hat{\alpha}}} = \frac{K^{\alpha}}{(A \cdot L)^{\alpha}} \cdot \frac{H^{\hat{\alpha}}}{(A \cdot L)^{\hat{\alpha}}} = \left(\frac{K}{A \cdot L}\right)^{\alpha} \left(\frac{H}{A \cdot L}\right)^{\hat{\alpha}} = k^{\alpha} \cdot h^{\hat{\alpha}}.$$

As in the SS model, $K' = \mathbb{s} \cdot Y + (1-\delta)K$. After dividing both sides by $A' \cdot L'$,

$$k' = \frac{K'}{A' \cdot L'} = \frac{\mathbb{s} \cdot Y}{A' \cdot L'} + \frac{(1-\delta) \cdot K}{A' \cdot L'} = \mathbb{s} \cdot \frac{K^{\alpha} \cdot H^{\hat{\alpha}} \cdot (A \cdot L)^{1-\alpha-\hat{\alpha}}}{(G \cdot A) \cdot (N \cdot L)} + \frac{(1-\delta) \cdot K}{(G \cdot A) \cdot (N \cdot L)} =$$

$$= \frac{\mathbb{s}}{G \cdot N} \cdot \frac{K^{\alpha} \cdot H^{\hat{\alpha}} \cdot (A \cdot L)^{1-\alpha-\hat{\alpha}}}{A \cdot L} + \frac{1-\delta}{G \cdot N} \cdot \frac{K}{A \cdot L} = \frac{\mathbb{s}}{G \cdot N} \cdot \frac{K^{\alpha} \cdot H^{\hat{\alpha}}}{(A \cdot L)^{\alpha+\hat{\alpha}}} + \left(\frac{1-\delta}{G \cdot N}\right) \cdot k =$$

$$= \frac{\mathbb{s}}{G \cdot N} \cdot \frac{K^{\alpha}}{(A \cdot L)^{\alpha}} \frac{H^{\hat{\alpha}}}{(A \cdot L)^{\hat{\alpha}}} + \frac{1-\delta}{G \cdot N} \cdot k = \frac{\mathbb{s}}{G \cdot N} \cdot k^{\alpha} \cdot h^{\hat{\alpha}} + \frac{1-\delta}{G \cdot N} \cdot k = \frac{\mathbb{s}}{G \cdot N} \cdot y + \frac{1-\delta}{G \cdot N} \cdot k.$$

By substracting k from both sides,

$$\Delta k = \frac{\mathbb{s}}{G \cdot N} \cdot y - \frac{\delta + G \cdot N - 1}{G \cdot N} \cdot k.$$

This equation coincides with the one from the SS model (with $G = 1 + a$ and $N = 1 + n$). For $\Delta k = 0$ it must be that $\mathbb{s} \cdot y = (\delta + G \cdot N - 1) \cdot k$. That is,

$$\mathbb{s} \cdot k^{\alpha} \cdot h^{\hat{\alpha}} = (\delta + G \cdot N - 1) \cdot k$$

Solving for h,

$$h(t) = \left(\frac{\delta + GN - 1}{\mathbb{s}}\right)^{1/\hat{\alpha}} \cdot k(t)^{(1-\alpha)/\hat{\alpha}}.$$

The above equation represents the condition $\Delta k(t) = 0$, represented in Fig. 1.

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[2] Mankiw, N. Gregory; Paul Romer; David N. Weil. (1992): "A contribution to the empirics of economic growth", *Quarterly Journal of Economics* 107, 407-437.

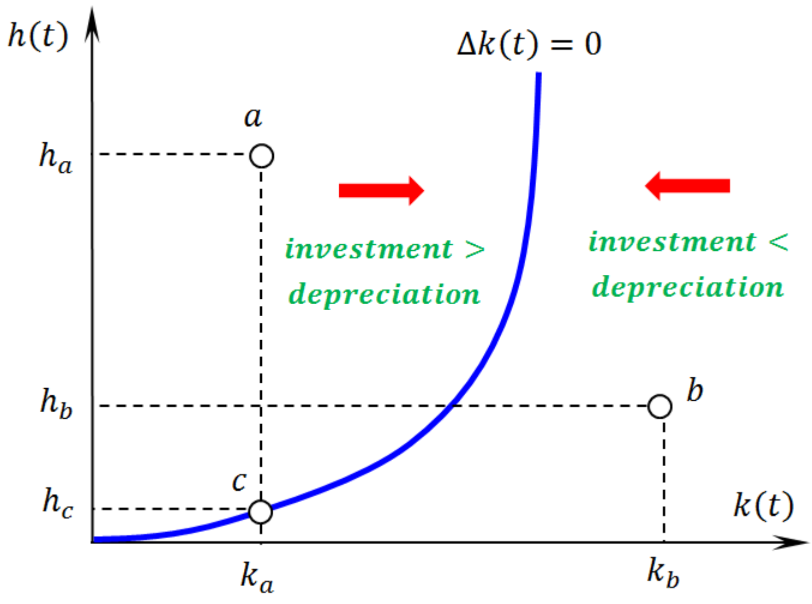

Fig. 1. Graphical representation of $\Delta k(t) = 0$

At points like a in Fig. 1, investment in k is higher than depreciation of k, so k increases.

Investment is higher than depreciation at a because, given k_a, it is enough to have per capita human capital equal to h_c for investment to equal depreciacion (that is, for $\Delta k(t) = 0$ to hold).

As $h_b > h_c$, there is a human capital excess creating too much output, which generates too much investment (in comparison with the depreciation corresponding to k_a).

Similarly, k decreases at points to the right of the curve $\Delta k(t) = 0$ (like point b in Fig. 1).

On the other hand, starting with

$$H' = \hat{\mathbb{s}} \cdot Y + \left(1 - \hat{\delta}\right) H$$

a similar procedure leads to

$$\Delta h = \frac{\hat{\mathbb{s}}}{G \cdot N} \cdot y - \frac{\hat{\delta} + G \cdot N - 1}{G \cdot N} \cdot h \, .$$

For $\Delta h = 0$ it must be that

$$\hat{\mathbb{s}} \cdot k^{\alpha} \cdot h^{\hat{\alpha}} = \left(\hat{\delta} + G \cdot N - 1\right) \cdot h \, .$$

Solving for h, the condition representing $\Delta h(t) = 0$, represented in Fig. 2, is

$$h(t) = \left(\frac{\hat{\mathbb{s}}}{\hat{\delta} + GN - 1}\right)^{1/(1-\hat{\alpha})} \cdot k(t)^{\alpha/(1-\hat{\alpha})} \, .$$

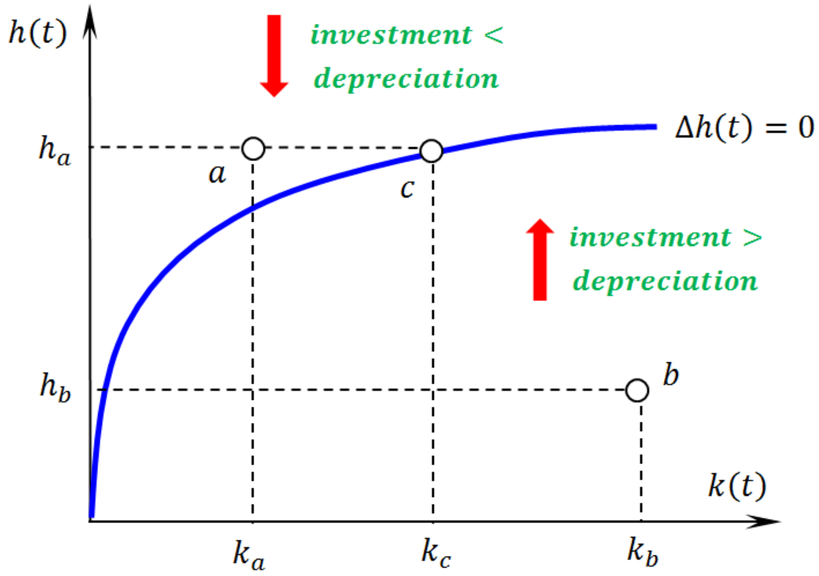

Fig. 2. Graphical representation of $\Delta h(t) = 0$

<u>At points like a</u> in Fig. 2, investment in h is smaller than depreciation of h, so <u>h decreases</u>.

Investment is smaller than depreciation at a because, given h_a, it is necessary to have capital stock per capita equal to k_c for investment to equal depreciacion (for $\Delta h(t) = 0$ to hold).

Since $k_a < k_c$, there is a capital shortage creating an output gap that generates insufficient investment in h (in comparison with the depreciation corresponding to h_a).

Analogously, <u>h increases at points below the curve</u> $\Delta h(t) = 0$ (like point b in Fig. 2).

Graphically, the solution of the model is obtained by combining Fig. 1 with Fig. 2; see Fig. 3. The dynamics of k and h ensure convergence to the point e in Fig. 3 where both curves intersect.

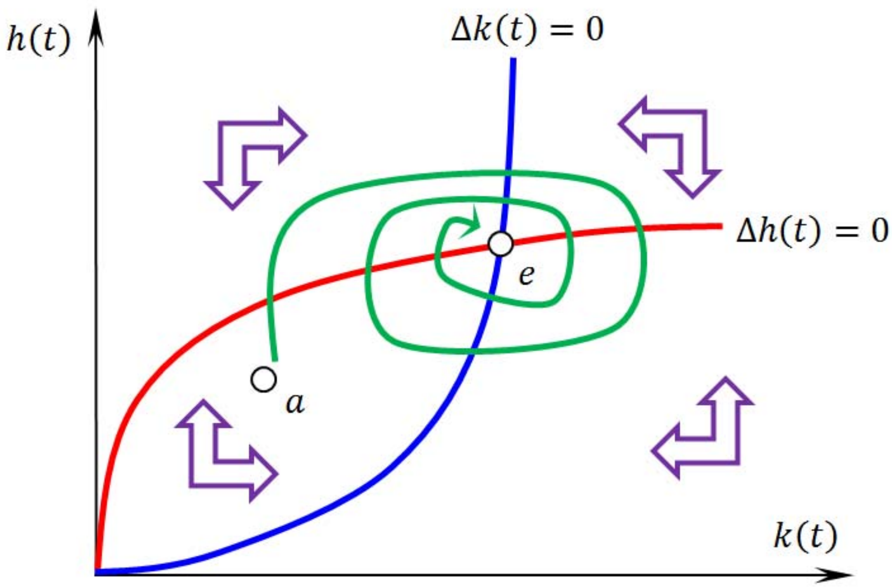

Fig. 3. Solution of the model: $\Delta k(t) = \ \Delta k(t) = 0$

3.3.3. The Barro[3] (1990) model

There is a government that provides a public good, which is interpreted as a positive externality for production. The amount of public good expenditure is denoted by E.

- **Aggregate production function** $Y = A \cdot K^{\alpha} \cdot E^{1-\alpha}$, where $0 < \alpha < 1$ and $A > 0$

Population is assumed to grow at rate $n > 0$. Dividing both sides by population L yields the per capita production function (where $k = K/L$ and $e = E/L$).

$$y = \frac{Y}{L} = \frac{A \cdot K^{\alpha} \cdot E^{1-\alpha}}{L} = A \cdot \frac{K^{\alpha}}{L^{\alpha}} \cdot \frac{E^{1-\alpha}}{L^{1-\alpha}} = A \cdot \left(\frac{K}{L}\right)^{\alpha} \cdot \left(\frac{G}{L}\right)^{1-\alpha} = A \cdot k^{\alpha} \cdot e^{1-\alpha}$$

- **Per capita production function** $y = A \cdot k^{\alpha} \cdot e^{1-\alpha}$

The public good is financed by taxes. The tax rate is $0 < \tau < 1$. Total disposable income Y_d is then $Y_d = (1 - \tau) \cdot Y$.

- **Aggregate savings function** $S = \mathbb{s} \cdot Y_d = \mathbb{s} \cdot (1 - \tau) \cdot Y$, where $0 < \mathbb{s} < 1$

- **Macroeconomic equilibrium** $S = I$

The stock of capital accumulates as in the Solow-Swan model:

$$K(t+1) = I(t) + (1 - \delta) \cdot K(t).$$

Dividing both sides by $L(t+1)$ and knowing that $L(t+1) = (1+n) \cdot L(t)$,

$$k(t+1) = \frac{K(t+1)}{L(t+1)} = \frac{I(t)}{(1+n) \cdot L(t)} + \frac{(1-\delta) \cdot K(t)}{(1+n) \cdot L(t)}.$$

Using the macroeconomic equilibrium condition,

$$k(t+1) = \frac{\mathbb{s} \cdot (1-\tau) \cdot Y(t)}{(1+n) \cdot L(t)} + \frac{(1-\delta) \cdot K(t)}{(1+n) \cdot L(t)} = \frac{\mathbb{s} \cdot (1-\tau)}{1+n} \cdot y(t) + \frac{1-\delta}{1+n} \cdot k(t).$$

In $k(t)$ is subtracted from both sides,

$$k(t+1) - k(t) = \frac{\mathbb{s} \cdot (1-\tau)}{1+n} \cdot y(t) + \left(\frac{1-\delta}{1+n} - 1\right) \cdot k(t) = \frac{\mathbb{s} \cdot (1-\tau)}{1+n} \cdot y(t) - \left(\frac{\delta + n}{1+n}\right) \cdot k(t)$$

or, in a more compact notation:

[3] Barro, Robert J. (1990): "Government spending in a simple model of endogenous growth", *Journal of Political Economy* 98, 103-125.

$$\Delta k = \frac{\mathbb{s} \cdot (1-\tau)}{1+n} \cdot y - \left(\frac{\delta + n}{1+n}\right) \cdot k \,.$$

- **Dynamics of capital per capita** $\quad g_k = \frac{\Delta k}{k} = \frac{\mathbb{s}\cdot(1-\tau)}{1+n} \cdot \frac{y}{k} - \frac{\delta+n}{1+n}$

Introducing the value of $y = A \cdot k^{\alpha} \cdot e^{1-\alpha}$ leads to

$$g_k = \frac{\mathbb{s} \cdot (1-\tau)}{1+n} \cdot A \cdot \left(\frac{e}{k}\right)^{1-\alpha} - \frac{\delta + n}{1+n}.$$

The government budget is assumed to be always balanced: taxes are equal to the public good expenditure.

- **Government budget** $\quad E = \tau \cdot Y = \tau \cdot A \cdot K^{\alpha} \cdot E^{1-\alpha}$

Expressed in per capita terms, $e = \tau \cdot A \cdot k^{\alpha} \cdot e^{1-\alpha}$. That is,

$$e = k \cdot (\tau \cdot A)^{\frac{1}{\alpha}} \,. \tag{2}$$

If (2) is inserted into the equation describing the dynamics of capital per capita,

$$g_k = \frac{\mathbb{s} \cdot (1-\tau)}{1+n} \cdot A \cdot \left(\frac{e}{k}\right)^{1-\alpha} - \frac{\delta+n}{1+n} = \frac{\mathbb{s} \cdot (1-\tau)}{1+n} \cdot A \cdot (\tau \cdot A)^{\frac{1-\alpha}{\alpha}} - \frac{\delta+n}{1+n} =$$

$$= \frac{\mathbb{s} \cdot (1-\tau)}{1+n} \cdot A^{\frac{1}{\alpha}} \cdot \tau^{\frac{1-\alpha}{\alpha}} - \frac{\delta+n}{1+n}.$$

If (1) is inserted into the equation describing output per capital,

$$y = A \cdot k^{\alpha} \cdot e^{1-\alpha} = A \cdot k^{\alpha} \cdot \left(k \cdot (\tau \cdot A)^{\frac{1}{\alpha}}\right)^{1-\alpha} = A \cdot k \cdot (\tau \cdot A)^{\frac{1-\alpha}{\alpha}} = A^{\frac{1}{\alpha}} \cdot \tau^{\frac{1-\alpha}{\alpha}} \cdot k \,.$$

This expression is equivalent to the per capita production function of an AK model, with the only difference that the constant A in an AK model now takes the form of the constant $A^{\frac{1}{\alpha}} \cdot \tau^{\frac{1-\alpha}{\alpha}}$.

Since y is proportional to k, as in an AK model, it follows that the rate of growth g_y of y is equal to the rate of growth g_k of k. Accordingly,

$$g_y = g_k = \frac{\mathbb{s} \cdot (1-\tau)}{1+n} \cdot A^{\frac{1}{\alpha}} \cdot \tau^{\frac{1-\alpha}{\alpha}} - \frac{\delta+n}{1+n}.$$

An immediate implication of this equation is that $g_y > 0$ if and only if

$$\mathbb{s} \cdot (1-\tau) \cdot A^{\frac{1}{\alpha}} \cdot \tau^{\frac{1-\alpha}{\alpha}} > \delta + n$$

that is, if and only if

$$\boxed{\mathbb{s} > \frac{\delta + n}{(1-\tau) \cdot A^{\frac{1}{\alpha}} \cdot \tau^{\frac{1-\alpha}{\alpha}}}.}$$

The lesson of this result is that a sufficiently high saving rate $\mathbb{s}$ could ensure a sustained growth of output per capita y, in contrast to the eventual convergence of g_y to zero in the Solow-Swan model.

An interesting question concerns the value of τ that maximizes the rate of growth g_y of y. But that is another story that you can try to tell yourself.

3.3.4. The Romer[4] (1990) model

There are four inputs. There is labour, L. There is also a set of (physical) capital goods. The amount of capital good i is K_i. The third input is technology, which is interpreted as a non-rival component in production: when someone makes use of technology everybody else can also make use of it. It can be viewed as freely available knowledge. Technology is represented by the number A of capital goods available. Lastly, there is human capital, H, which is assumed to be a rival component in production. A part H_Y of the human capital H is used to produce the good, whereas the rest $H_A = H - H_Y$ is employed to improve technology in the research sector of the economy.

Each capital good is supposed to have the same production cost and the same productivity. For this reason, it is assumed that the same amount Ꝁ is produced of each capital good. Therefore, the total amount K of capital goods in the economy is

$$K = A \cdot \text{Ꝁ}.$$

The model consists of the equations listed next (it is assumed that $\delta = 0$).

- **Aggregate production function** $Y = {H_Y}^{\alpha} \cdot L^{\beta} \cdot (A \cdot \text{Ꝁ}^{1-\alpha-\beta})$, $0 < \alpha < 1$ and $0 < \beta < 1$
- **Aggregate savings function** $S = \mathbb{s} \cdot Y$, $0 < \mathbb{s} < 1$
- **Macroeconomic equilibrium** $S = I$
- **Capital accumulation** $I = \Delta K$
- **Technological change** $\Delta A = \phi \cdot H_A \cdot A$, $\phi > 0$

The last equation describes the process of creation of generic (non-rival) knowledge. It asserts that the change in technology depends on the productivity ϕ of researchers, the human capital H_A spent in research activities, and the existing techonology A.

It is assumed that the human capital H, the human capital used in production H_Y, labour L, and the amount Ꝁ produced of each capital good remain all constant. In view of this, it follows from $Y = A \cdot {H_Y}^{\alpha} \cdot L^{\beta} \cdot \text{Ꝁ}^{1-\alpha-\beta}$ that the rate of change in output equals the rate of change in technology. That is,

$$g_Y = g_A.$$

The technological change equation $\Delta A = \phi \cdot H_A \cdot A$ implies $g_A = \frac{\Delta A}{A} = \phi \cdot H_A$. The final conclusion is then

[4] Romer, Paul (1990): “Endogenous technological change”, *Journal of Political Economy* 98, 71-102.

$$g_Y = \phi \cdot H_A \,.$$

Interpretation: the output growth rate is proportional to both the human capital H_A devoted to research (to increase the stock of knowledge) and the productivity ϕ of researchers (knowledge generated per researcher).

Given that the population has been assumed constant, $g_y = g_Y$: output and output per capita both grow at the same rate. As a result, $g_y = \phi \cdot H_A$: prosperity can be sustained by just investing in accumulating knowledge.

A sample of textbooks

Acemoglu, Daron (2009): *Introduction to Modern Economic Growth*, Princeton University Press, Princeton, New Jersey.

Aghion, Philippe; Peter W. Howitt (2009): *The Economics of Growth*, MIT Press, Cambridge, Massachusetts.

Barro, Robert J.; Xavier Sala-i-Martin (2009): *Economic Growth*, 2nd edition, MIT Press, Cambridge, Massachusetts.

Jiménez, Félix (2011): *Crecimiento económico: enfoques y modelos*, Fondo Editorial de la Pontificia Universidad Católica del Perú, Lima, Perú.

Jones, Charles I. (2000): *Introducción al crecimiento económico*, Pearson Educación, México.