3.1. Some pre-Solowian growth models

3.3.1. The Harrod[1] (1939) - Domar[2] (1946) model

The production function in the Harrod-Domar (HD) model is

$$Y = F(K, L) = min\{A \cdot K, B \cdot L\},$$

where A, $B > 0$ are fixed coefficients. If <u>K is the limiting factor</u> (that is, $A \cdot K < B \cdot L$), then $Y = A \cdot K$. Hence, $y = A \cdot k = f(k)$. In this case,

$$\frac{1}{v} = \frac{1}{\frac{K}{Y}} = \frac{Y}{K} = A\,.$$

Define the effective growth rate g (of output per capita) as the one resulting from the equilibrium condition $S = I$. Therefore,

$$\mathbb{s} \cdot Y = S = I = \Delta K + \delta \cdot K$$

Since $Y = A \cdot K$,

$$\mathbb{s} \cdot A \cdot K = \Delta K + \delta \cdot K$$

Dividing by K,

$$\mathbb{s} \cdot A = \frac{\Delta K}{K} + \delta$$

or

$$\mathbb{s} \cdot A = g_K + \delta$$

In sum,

$$g_K = \mathbb{s} \cdot A - \delta$$

Ginve that $Y = A \cdot K$, it follows that $g_Y = g_K = \mathbb{s} \cdot A - \delta$. All in all, the effective growth rate g coincides with the growth rate g_Y ensuring the full employment of capital.

The growth rate $\boldsymbol{g}_Y$ that could be considered socially optimal is the one ensuring the full employment of labour. It can be defined as the growth rate $g_{B \cdot L}$ of effective labour $B \cdot L$ (assuming that technology progress is labour augmenting). Letting n designate the population (labour) growth rate and b represent the rate of technological progress,

$$g_{B \cdot L} = \frac{B' \cdot L'}{B \cdot L} - 1 = \frac{B'}{B} \cdot \frac{L'}{L} - 1 = (g_B + 1) \cdot (g_L + 1) - 1 = (b + 1) \cdot (n + 1) - 1 = n + b \cdot (1 + n)$$

Hence, set $\boldsymbol{g}_Y = g_{B \cdot L} = n + b \cdot (1 + n)$.

[1] Harrod, Roy F. (1939): "An essay in dynamic theory", *Economic Journal* 49, 14-33.

[2] Domar, Evsey D. (1946): "Capital expansion, rate of growth, and employment", *Econometrica* 14, 137-147.

If $g_Y < \boldsymbol{g}_Y$, then the effective growth rate of the economy is insufficient to ensure the full employment of labour, so structural unemployment arises. This occurs when

$$\mathbb{s} \cdot A - \delta < n + b \cdot (1 + n)$$

To recap, if

$$\mathbb{s} < \frac{n + b \cdot (1 + n) + \delta}{A}$$

then the economy generates structural unemployment because of an insufficient saving rate. Even if the effective labour remains constant ($n = b = 0$), the saving rate guaranteeing full employment of capital does not ensure the full employment of labour (structural unemployment would occur if $\mathbb{s} < \delta/A$).

When $\mathbb{s} < \delta/A$, equality between g_Y and $\boldsymbol{g}_Y$ could be obtained by lowering δ/A. This could be achieved by increasing A, which amounts to improving the productivity of capital (the output to capital ratio Y/K). But this approach has a limit, because it has been assumed that K is the limiting factor, in the sense that $A \cdot K < B \cdot L$.

Developed economies can be characterized by the condition $g_Y > \boldsymbol{g}_Y$. One explanation for this fact is having a small population growth rate n. An implication of this condition is that savings are excessive or that investment is not enough to meet savings plans.

In non-developed economies the opposite occurs: $g_Y < \boldsymbol{g}_Y$. This is consistent with a small saving rate $\mathbb{s}$, which manifests itself in an imbalance between the population growth and the rate at capital accumulation. In fact, as $g_Y = g_K$, it follows from $g_Y < \boldsymbol{g}_Y$ that $g_K < \boldsymbol{g}_Y$. Thus, $g_Y < \boldsymbol{g}_Y$ implies

$$g_K < n + b \cdot (1 + n)$$

In particular, if the productivity B of labour remains constant, so $b = 0$, having $g_Y < \boldsymbol{g}_Y$ amounts to having $g_K < n$.

In any case, in the HD model growth can be permanent, whereas in the SS model growth is temporary (it is a by-product of convergence to a steady state and, to be sustained, must be exogenously induced).

Specifically, since $Y = A \cdot K$, it is plain that $y = A \cdot k = f(k)$ and $g_y = g_k$. As in the SS model,

$$\Delta k = \mathbb{s} \cdot f(k) - \delta \cdot k.$$

Therefore,

$$g_y = g_k = \frac{\Delta k}{k} = \mathbb{s} \cdot \frac{f(k)}{k} - \delta = \mathbb{s} \cdot A - \delta$$

With population growing at rate n,

$$g_y = g_k = \frac{\mathbb{s} \cdot A}{1 + n} - \frac{\delta + n}{1 + n}.$$

The main lesson of these results is that a sustained increase in output per capita ($g_y > 0$ permanently) is possible with a sufficiently high saving rate:

$$g_y > 0 \quad \Leftrightarrow \quad \mathbb{s} \cdot A > \delta + n$$

That is, sustained growth can be achieved when

$$\mathbb{s} > \frac{\delta + n}{A}.$$

The corresponding policy recommendation for prosperity is simple: "save more".

Define the golden age as the state in which $g_Y = \boldsymbol{g}_Y$. This is equivalent to

$$\mathbb{s} = \frac{n + b \cdot (1 + n) + \delta}{A}$$

But there is no mechanism in the model to ensure this equality. First, $\mathbb{s}$ is an exogenous decision (by savers, the families). Second, n is also exogenous and determined by an unmodelled demographic dynamics. Third, the capital to output ratio $\frac{K}{Y} = \frac{1}{A}$ arises from the expectations of investors (assumed to be correct). Finally, b is also a technological exogenous parameter.

The basic conclusions obtained from the model are:

(i) that the economy may be considered unstable in the sense that there is no guaranteed convergence to an equilibrium (the existence of a stationary state is not ensured); and

(ii) that it is highly unlikely that the effective rate of growth of output is exactly the rate that generates the full employment of labour.

The preceding results should be qualified, because they rely on the presumption that K is the limiting factor. But if $g_k > 0$, then, eventually, K will no longer be the limiting factor.

That K is the limiting factor means that $A \cdot K < B \cdot L$; equivalently, $k = K/L < B/A$. Hence, if k grows, the condition $k < B/A$ expressing the fact that K is the limiting factor will eventually no longer hold: at some point in time, K/L will exceed B/A. In this case, the model should be solved again with production function $Y = B \cdot L$.

When $Y = B \cdot L$, $g_Y = n$ and, if B does not grow, Y/L cannot grow. If L is the limiting factor,

$$g_k = \mathbb{s} \cdot B \cdot \frac{1}{k} - \delta.$$

In sum:

(i) with K being the limiting factor, the HD model could account for a (limited) sustained growth of output per capita (long-run growth), whereas the Solow-Swan (SS) model cannot;

(ii) conversely, the HD model fails to account for convergence among economies, which is a phenomenon the SS model can explain.

3.1.2. The AK model

It is like the Harrod-Domar (or the Solow-Swan) model with $Y = A \cdot K$ always. This model can generate sustained long-run growth. In fact,

$$k(t+1) = (\mathbb{s} \cdot A + 1 - \delta) \cdot k(t).$$

Hence, if $\mathbb{s} \cdot A + 1 - \delta > 1$ (that is, $\mathbb{s} \cdot A > \delta$), then k grows forever.

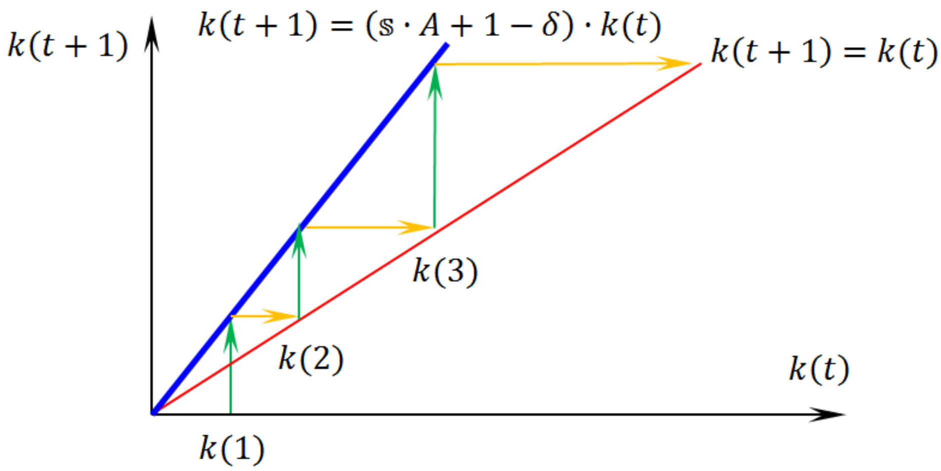

Fig. 4. Sustained growth in the AK model

Justification of the AK model: the accumulation of capital generates, through learning by doing, technical progress that prevents the productivity of capital from falling. Knowledge externalities can also explain sustained long-run growth.

The AK model fails to account for convergence, as it suggests different growth rates and does not distinguish between capital accumulation and technological progress.

The HD model suggests that the processes of accumulation and growth very likely generate instability: the discrepancy between effective capacity and desired capacity, between actual growth and expectations by investors, leads to growth with persistent unemployment (in the model, the constant $v = K/Y$ captures the investors' expectations).

The SS model claims the opposite: accumulation and growth do not generate instability. Long-run growth is only sustained with correct expectations and full employment is achieved thanks to a variable capital to output ratio v.

As a summary of the SS, HD, and AK models, the common framework adopted in the three models is the following (all three models rely on the conceptual sequence **profitability (rate of return on capital) $\rightarrow$ capital accumulation $\rightarrow$ economic growth**).

- Equilibrium condition $S = I$
- Saving function $S = \mathbb{s} \cdot Y$ constant saving rate $0 < \mathbb{s} < 1$
- Gross investment $I = \Delta K + \delta \cdot K$
- Net investment $I - \delta \cdot K = \Delta K$

Inserting the equilibrium condition and the saving function into net investment,

$$\Delta K = \mathbb{s} \cdot Y - \delta \cdot K$$

or, dividing both sides by K,

$$\frac{\Delta K}{K} = \mathbb{s} \cdot \frac{Y}{K} - \delta \,. \tag{1}$$

Define the capital to output ratio as

$$v = \frac{K}{Y}$$

and the growh rate of the stock of capital as

$$g_K = \frac{\Delta K}{K} \,.$$

Then (1) can be equivalently expressed as

$$g_K = \frac{\mathbb{s}}{v} - \delta \,.$$

For the stock of capital K to grow it must be that $g_K > 0$; that is, $\mathbb{s} > \delta \cdot v$.

Now consider the capital to labour ratio $k = \frac{K}{L}$. Then the growth rate g_k of k is

$$g_k = \frac{\Delta k}{k} = \frac{k' - k}{k} = \frac{k'}{k} - 1 = \frac{\frac{K'}{L'}}{\frac{K}{L}} - 1 = \frac{\frac{K'}{K}}{\frac{L'}{L}} - 1 = \frac{g_K + 1}{g_L + 1} - 1 = \frac{g_K - g_L}{g_L + 1} \,.$$

Accordingly, if population grows at a constant rate $n > 0$,

$$g_k = \frac{g_K - g_L}{g_L + 1} = \frac{\frac{\mathbb{s}}{v} - \delta - n}{n + 1} \,.$$

The capital to labour ratio k grows if $g_k > 0$; that is, if

$$\frac{\mathbb{s}}{v} - \delta - n > 0$$

or

$$\frac{\mathbb{s}}{v} > \delta + n\,.$$

If technological progress A is labour augmenting, labour is defined as effective labour $A \cdot L$, and the capital to (effective) labour ratio $k = \frac{K}{A \cdot L}$, then the growth rate g_k of k is

$$g_k = \frac{\Delta k}{k} = \frac{k'}{k} - 1 = \frac{\frac{K'}{A' \cdot L'}}{\frac{K}{A \cdot L}} - 1 = \frac{\frac{K'}{K}}{\frac{L'}{L} \cdot \frac{A'}{A}} - 1 = \frac{g_K + 1}{(g_L + 1) \cdot (g_A + 1)} - 1 = \frac{g_K - g_L - g_A \cdot (g_L + 1)}{(g_L + 1) \cdot (g_A + 1)}\,.$$

In view of this, if population grows at a constant rate $n > 0$ and technology progresses at a constant rate $a > 0$,

$$g_k = \frac{g_K - g_L - g_A \cdot (g_L + 1)}{(g_L + 1) \cdot (g_A + 1)} = \frac{\frac{\mathbb{s}}{v} - \delta - n - a \cdot (1 + n)}{(1 + a) \cdot (1 + n)}\,.$$

As a result,

$$g_k > 0 \quad \Leftrightarrow \quad \frac{\mathbb{s}}{v} > \delta + n + a \cdot (1 + n)\,.$$

The kind of growth models considered in this course can be classified according to two basic choices, concerning the saving rate $\mathbb{s}$ and the capital to output ratio v.

$\mathbb{s}$ exogenous / endogenous

- $\mathbb{s}$ exogenous — Solow-Swan model, Harrod-Domar model, AK model
- $\mathbb{s}$ endogenous — overlapping generations model, neoclassical growth model

v constant / variable

- v constant — Harrod-Domar model, AK model
- v variable — Solow-Swan model, overlapping generations, neoclassical growth model