THE GENERALIZED PROBABILISTIC WEIGHTED AVERAGING OPERATOR AND ITS APPLICATION IN STRATEGIC DECISION MAKING

José M. Merigó

University of Barcelona

Abstract

We analyze different types of aggregation operators based on the use of probabilities, weighted averages (WAs) and generalized aggregation operators. We present the generalized probabilistic weighted averaging (GPWA) operator. It is a new aggregation operator that unifies the WA and the probability in the same formulation considering the degree of importance that each concept has in the analysis. Moreover, it also uses generalized means providing a more complete formulation. Its main advantage is that it includes a wide range of particular cases such as the probabilistic weighted averaging (PWA) operator, the probabilistic weighted geometric averaging (PWGA) operator, the probabilistic weighted quadratic averaging (PWQA) operator, the arithmetic weighted average, the arithmetic probabilistic aggregation, and a lot of other cases. We further generalize this approach by using quasi-arithmetic means obtaining the Quasi-PWA operator. We end the paper with an application of the GPWA in a decision making problem about selection of production strategies.

Keywords: Aggregation operators, Weighted average, Probabilities, Generalized mean, Decision making, Selection of production strategies.

INTRODUCTION

The weighted average (WA) is one of the most common aggregation operators found in the literature. Another interesting concept that can be used as an aggregation operator is the probability. These two concepts have been used in a lot of applications concerning statistics, economics, engineering, physics, etc. Probably, these two concepts are the most relevant in statistics. However, there are a lot of other aggregation operators such as the ordered weighted averaging (OWA) operator and others (Beliakov, 2005; Beliakov et al., 2007; Calvo et al., 2002; Fodor et al., 1995; Karayiannis, 2000; Kaufmann and Gil-Aluja, 1987; Merigó, 2008a; 2009a; 2009b; 2009c; Merigó and Casanovas, 2009a; 2009b; Merigó and Gil-Lafuente, 2009; Torra, 1997; Torra and Narukawa, 2007; Xu, 2007; Xu and Da, 2003; Yager, 1988; 1993; 1996; 2002; 2004; 2007; Yager and Filev, 1994; Yager and Kacprzyk, 1997). A very interesting type of generalization can be obtained by using generalized and quasi-arithmetic means. Then, we get the generalized weighted average (GWA) and the generalized probabilistic aggregation operator.

Recently, Merigó has suggested a new model that unifies the weighted average with the probability (Merigó, 2009c). He called it the probabilistic weighted averaging (PWA) operator. The main
advantage of the PWA is that it is able to unify the probability and the weighted average in the same formulation considering the degree of importance that each concept has in the aggregation.

In this paper, we present a new approach that unifies the weighted average and the probability in the same formulation considering the importance that each concept has in the aggregation and using a more general framework by using generalized and quasi-arithmetic means. Thus, we will present two new general aggregation operators, the generalized PWA (GPWA) and the quasi-arithmetic PWA (Quasi-PWA) operator. Their main advantage against the PWA is that they are more robust and general because they include the PWA as a particular case and a lot of other particular cases not included in the PWA. For example, they include the geometric PWA (PWGA) operator, the quadratic PWA (PWQA) operator, the harmonic PWA (PWHA) operator, the generalized probability and GWA. We also study some of the main properties of the GPWA operator.

We study the applicability of the GPWA and we see that it is extremely broad because all the studies that use the WA or the probability can be revised and extended with this new approach. For example, we could use it a lot in statistics, in economics, in engineering and in decision theory. In this paper we focus on a decision making problem about strategic decision making about the selection of the optimal production strategy. We see that depending on the importance that we give to the probability and the WA, the results may lead to different decisions.

This paper is organized as follows. In Section 2, we briefly comment some basic concepts about the weighted aggregation operators, the probabilistic aggregation operators and the generalized aggregation operators. Section 3 presents the GPWA operator. Section 4 analyzes some of its main particular cases and Section 5 develops a further generalization by using quasi-arithmetic means. Section 6 presents a numerical example of the new approach.

AGGREGATION OPERATORS

In this Section, we briefly describe the weighted aggregation functions, the probabilistic aggregation functions, the generalized aggregation functions and the PWA operator.

WEIGHTED AGGREGATION FUNCTIONS

Weighted aggregation functions are those functions that weight the aggregation process by using the weighted average. The weighted average can be defined as follows.
**Definition 1.** A WA operator of dimension $n$ is a mapping $\text{WA}: \mathbb{R}^n \to \mathbb{R}$ that has an associated weighting vector $W$, with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, such that

$$\text{WA}(a_1, \ldots, a_n) = \sum_{i=1}^{n} w_i a_i$$  \hfill (1)$$

where $a_i$ represents the $i$th argument variable.

Other extensions of the weighted average are those that use it with the OWA operator such as the WOWA operator (Torra, 1997; Torra and Narukawa, 2007) and the hybrid averaging (HA) operator (Xu and Da, 2003). Recently, Merigó (2009a) suggested another approach called the OWA weighted average (OWAWA) operator. Its main advantage against the WOWA and the HA is that it includes the OWA and the WA considering the degree of importance that each concept have in the aggregation. It can be defined as follows.

**Definition 2.** An OWAWA operator of dimension $n$ is a mapping $\text{OWAWA}: \mathbb{R}^n \to \mathbb{R}$ that has an associated weighting vector $W$ of dimension $n$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$\text{OWAWA}(a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j$$  \hfill (2)$$

where $b_j$ is the $j$th largest of the $a_i$, each argument $a_i$ has an associated weight (WA) $v_i$ with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, $\hat{v}_j = \beta w_j + (1 - \beta) w_j$ with $\beta \in [0, 1]$ and $v_j$ is the weight (WA) $v_j$ ordered according to $b_j$, that is, according to the $j$th largest of the $a_i$.

Note that other approaches for unifying the OWA and the WA are possible as it was suggested in (Merigó, 2008) such as a similar approach than the immediate probability. Thus, in the WA we get the immediate weighted OWA (IWOWA) operator that could be defined, for example, by using

$$\hat{v}_j = (w_j v_j / \sum_{j=1}^{n} w_j v_j) \text{ or by using } \hat{v}_j = [w_j + v_j / \sum_{j=1}^{n} (w_j + v_j)] .$$

Note that in the literature we find a lot of extensions of weighted aggregation functions such as those that use uncertain information represented in the form of interval numbers, FNs or linguistic variables (Merigó, 2008a).
PROBABILISTIC AGGREGATION FUNCTIONS

Probabilistic aggregation functions (or operators) are those functions that use probabilistic information in the aggregation process. Some examples are the aggregation with simple probabilities, the aggregation with belief structures (Merigó, 2008a; Casanovas and Merigó, 2008; Merigó and Casanovas, 2009), the concept of immediate probabilities (Engemann et al, 1996; Merigó 2008b; Yager, 1995) and the probabilistic OWA operator (Merigó, 2009b). The immediate probability is an approach that uses OWAs and probabilities in the same formulation. It can be defined as follows.

**Definition 3.** An IPOWA operator of dimension $n$ is a mapping $\text{IPOWA}: \mathbb{R}^n \rightarrow \mathbb{R}$ that has an associated weighting vector $W$ of dimension $n$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$\text{IPOWA} (a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j$$

where $b_j$ is the $j$th largest of the $a_i$, each argument $a_i$ has a probability $v_i$ with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, $\hat{v}_j = (w_j v_j / \sum_{j=1}^{n} w_j v_j)$ and $v_j$ is the probability $v_i$ ordered according to $b_j$, that is, according to the $j$th largest of the $a_i$.

Note that the IPOWA operator is a good approach for unifying probabilities and OWAs in some particular situations. But it is not always useful, especially in situations where we want to give more importance to the probabilities or to the OWA operators. In order to see why this unification does not seem to be a final model is considering other ways of representing $\hat{v}_j$. For example, we could also use $$\hat{v}_j = [w_j + v_j / \sum_{j=1}^{n} (w_j + v_j)]$$ or other similar approaches.

Another approach for unifying probabilities and OWAs in the same formulation is the probabilistic OWA (POWA) operator (Merigó, 2008; 2009b). Its main advantage is that it is able to include both concepts considering the degree of importance of each case in the problem. It is defined as follows.

**Definition 4.** A POWA operator of dimension $n$ is a mapping $\text{POWA}: \mathbb{R}^n \rightarrow \mathbb{R}$ that has an associated weighting vector $W$ of dimension $n$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:
POWA \((a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{p}_j b_j \)  

(4)

where \(b_j\) is the \(j\)th largest of the \(a_i\) each argument \(a_i\) has an associated probability \(p_i\) with \(\sum_{i=1}^{n} p_i = 1\) and \(p_i \in [0, 1]\), \(\hat{p}_j = \beta w_j + (1 - \beta) p_j\) with \(\beta \in [0, 1]\) and \(p_j\) is the probability \(p_i\) ordered according to the \(j\)th largest of the \(a_i\).

**PWA operator**

The probabilistic weighted averaging (PWA) operator is an aggregation operator that unifies the probability and the weighted average in the same formulation considering the degree of importance that each concept has in the aggregation. It is defined as follows.

**Definition 5.** A PWA operator of dimension \(n\) is a mapping \(\text{PWA}: \mathbb{R}^n \rightarrow \mathbb{R}\) such that:

\[
\text{PWA} (a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j a_i
\]

(5)

where the \(a_i\) are the argument variables, each argument \(a_i\) has an associated weight (WA) \(v_i\) with \(\sum_{i=1}^{n} v_i = 1\) and \(v_i \in [0, 1]\), and a probabilistic weight \(p_i\) with \(\sum_{i=1}^{n} p_i = 1\) and \(p_i \in [0, 1]\), \(\hat{v}_j = \beta v_j + (1 - \beta) v_i\) with \(\beta \in [0, 1]\) and \(\hat{v}_j\) is the weight that unifies probabilities and WAs in the same formulation.

Note that it is also possible to formulate the PWA operator separating the part that strictly affects the probabilistic information and the part that affects the WAs. This representation is useful to see both models in the same formulation but it does not seem to be as a unique equation that unifies both models. Note that if the weighting vector of probabilities or WAs is not normalized, i.e., \(P = \sum_{i=1}^{n} p_i \neq 1\), or \(V = \sum_{i=1}^{n} v_i \neq 1\), then, the PWA operator can be expressed as:

\[
f (a_1, \ldots, a_n) = \frac{\beta}{P} \sum_{j=1}^{n} p_j a_i + \frac{(1 - \beta)}{V} \sum_{i=1}^{n} v_j a_i
\]

(6)

The PWA is monotonic, commutative, bounded and idempotent. For further reading on the PWA, see (Merigó, 2008).
GENERALIZED AGGREGATION OPERATORS

There are a lot of aggregation operators that use the generalized mean such as the weighted generalized mean, the generalized OWA operator, the Minkowski distance and a lot of other cases. In this section we will give an example of how to use the generalized mean based on the OWA operator.

The generalized OWA (GOWA) operator was introduced by Yager in (2004). It generalizes a wide range of aggregation operators that includes the OWA operator with its particular cases, the ordered weighted geometric (OWG) operator, the ordered weighted harmonic averaging (OWHA) operator and the ordered weighted quadratic averaging (OWQA) operator. It can be defined as follows.

**Definition 6.** A GOWA operator of dimension \( n \) is a mapping \( \text{GOWA}: \mathbb{R}^n \rightarrow \mathbb{R} \) that has an associated weighting vector \( W \) of dimension \( n \) such that the sum of the weights is 1 and \( w_j \in [0,1] \), then:

\[
\text{GOWA}(a_1, a_2, ..., a_n) = \left( \sum_{j=1}^{n} w_j b_j^\lambda \right)^{1/\lambda}
\]

where \( b_j \) is the \( j \)th largest of the \( a_i \), and \( \lambda \) is a parameter such that \( \lambda \in (-\infty, \infty) \).

As it is demonstrated in (Yager, 2004), the GOWA operator is a mean operator. This is a reflection of the fact that the operator is commutative, monotonic, bounded and idempotent. It can also be demonstrated that the GOWA operator has as special cases the maximum, the minimum, the generalized mean and weighted generalized mean. Other families of GOWA operators can be found in (Yager, 2004).

THE GENERALIZED PROBABILISTIC WEIGHTED AVERAGING OPERATOR

The generalized probabilistic weighted averaging (GPWA) operator is an aggregation operator that unifies the probability and the weighted average in the same formulation considering the degree of importance that each concept has in the aggregation. Moreover, in this approach we use generalized means in order to include a wide range of aggregation operators such as the PWA, the probabilistic weighted geometric average (PGWA), the probabilistic weighted harmonic average (PHWA), the probabilistic weighted quadratic average (PQWA), and a lot of other cases. In this case, we unify the weighted generalized average (WGA) with the probabilistic generalized mean (PGM), and we are able to include other unifications, for example, the weighted geometric mean (WGM) with the probabilistic
geometric mean (PGM), the weighted quadratic mean (WQM) with the probabilistic quadratic mean (PQM), and so on. It is defined as follows.

**Definition 7.** A GPWA operator of dimension $n$ is a mapping $GPWA: \mathbb{R}^n \rightarrow \mathbb{R}$ such that:

$$GPWA(a_1, \ldots, a_n) = \left( \sum_{i=1}^{n} \hat{v}_i a_i^{\lambda} \right)^{1/\lambda}$$

where the $a_i$ are the argument variables, each argument $a_i$ has an associated weight (PWA) $v_i$ with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, and a probabilistic weight $p_i$ with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0, 1]$, $\hat{v}_i = \beta p_i + (1-\beta) v_i$ with $\beta \in [0, 1]$, $\hat{v}_i$ is the weight that unifies probabilities and WAs in the same formulation and $\lambda$ is a parameter such that $\lambda \in (-\infty, \infty)$.

Note that it is also possible to formulate the GPWA operator separating the part that strictly affects the probabilistic information and the part that affects the WA. This representation is useful to see both models in the same formulation but it does not seem to be as a unique equation that unifies both models.

**Definition 8.** A GPWA operator is a mapping $GPWA: \mathbb{R}^n \rightarrow \mathbb{R}$ of dimension $n$, if it has an associated probabilistic vector $P$, with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0, 1]$ and a weighting vector $V$ that affects the WA, with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, such that:

$$GPWA(a_1, \ldots, a_n) = \beta \left( \frac{\sum_{j=1}^{n} p_j a_j^{\lambda}}{P} \right)^{1/\lambda} + (1-\beta) \left( \frac{\sum_{j=1}^{n} v_j a_j^{\lambda}}{V} \right)^{1/\lambda}$$

where the $a_i$ are the argument variables, $\beta \in [0, 1]$ and $\lambda$ is a parameter such that $\lambda \in (-\infty, \infty)$.

Note that if the weighting vector of probabilities or WAs is not normalized, i.e., $P = \sum_{i=1}^{n} p_i \neq 1$, or $V = \sum_{i=1}^{n} v_i \neq 1$, then, the GPWA operator can be expressed as:

$$GPWA(a_1, \ldots, a_n) = \frac{\beta}{P} \left( \frac{\sum_{j=1}^{n} p_j a_j^{\lambda}}{P} \right)^{1/\lambda} + \frac{(1-\beta)}{V} \left( \frac{\sum_{j=1}^{n} v_j a_j^{\lambda}}{V} \right)^{1/\lambda}$$

(10)
If $B$ is a vector corresponding to the ordered arguments $b_j$, we shall call this the ordered argument vector and $W^T$ is the transpose of the weighting vector, then, the GPWA operator can be expressed as:

$$GPWA(a_1, a_2, \ldots, a_n) = W^T B$$  \hspace{1cm} (11)

The GPWA is monotonic, commutative, bounded and idempotent. It is monotonic because if $a_i \geq u_i$, for all $a_i$, then, $GPWA(a_1, \ldots, a_n) \geq GPWA(u_1, u_2, \ldots, u_n)$. It is commutative because any permutation of the arguments has the same evaluation. It is bounded because the GPWA aggregation is delimited by the minimum and the maximum. That is, $\text{Min}\{a_i\} \leq GPWA(a_1, \ldots, a_n) \leq \text{Max}\{a_i\}$. It is idempotent because if $a_i = a$, for all $a_i$, then, $GPWA(a_1, \ldots, a_n) = a$.

**FAMILIES OF GPWA OPERATORS**

First of all, we are going to consider the two main cases of the PGWA operator that are found by analyzing the coefficient $\beta$. Basically, if $\beta = 0$, then, we get the weighted generalized mean (WGM) and if $\beta = 1$, the generalized probabilistic aggregation (GPA) operator. Note that if $v_i = 1/n$, for all $i$, then, we get the unification between the generalized mean and the GPA operator (arithmetic generalized probabilistic aggregation (AGPA) operator). And if $p_j = 1/n$, for all $j$, then, we get the unification between the generalized mean and the GWA operator (arithmetic generalized weighted average (AGWA) operator).

If we analyze different values of the parameter $\lambda$, we obtain another group of particular cases such as the usual PWA operator, the PWGA operator, the PWQA operator and the PWHA operator.

**Remark 1.** When $\lambda = 1$, the GPWA operator becomes the PWA operator.

$$GPWA (a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j a_j$$  \hspace{1cm} (12)

Note that if $p_j = 1/n$, for all $a_i$, we get the arithmetic weighted average (AWA) and if $v_i = 1/n$, for all $a_i$, we get the arithmetic probabilistic aggregation (APA) operator.

**Remark 2.** When $\lambda = 0$, the GPWA operator becomes the probabilistic weighted geometric averaging (PWGA) operator.
\[ \text{GPWA} \left( a_1, a_2, \ldots, a_n \right) = \prod_{j=1}^{n} a_j^{\frac{v_j}{p_j}} \]  

(13)

Note that if \( p_j = 1/n \), for all \( a_i \), we get the probabilistic geometric arithmetic weighted geometric average (PGAWGA) and if \( v_j = 1/n \), for all \( a_i \), we get the probabilistic geometric arithmetic mean (PGAM) operator. Note that if \( \beta = 1 \), we get the probabilistic geometric aggregation (PGA).

**Remark 3.** When \( \lambda = -1 \), we get the probabilistic weighted harmonic averaging (PWHA) operator.

\[ \text{GPWA} \left( a_1, a_2, \ldots, a_n \right) = \frac{1}{\sum_{j=1}^{n} a_j^{v_j}} \]  

(14)

Note that if \( p_j = 1/n \), for all \( a_i \), we get the probabilistic harmonic arithmetic weighted harmonic average (PHAWHA) and if \( v_j = 1/n \), for all \( a_i \), we get the probabilistic harmonic arithmetic mean (PHAM) operator. Note that if \( \beta = 1 \), we get the probabilistic harmonic aggregation (PHA).

**Remark 4.** When \( \lambda = 2 \), we get the probabilistic weighted quadratic averaging (PWQA) operator.

\[ \text{GPWA} \left( a_1, a_2, \ldots, a_n \right) = \left( \frac{n}{\sum_{j=1}^{n} a_j^{v_j}} \right)^{1/2} \]  

(15)

Note that if \( p_j = 1/n \), for all \( a_i \), we get the probabilistic quadratic arithmetic weighted quadratic average (PQAWQA) and if \( v_j = 1/n \), for all \( a_i \), we get the probabilistic quadratic arithmetic mean (PQAM) operator. Note that if \( \beta = 1 \), we get the probabilistic quadratic aggregation (PQA).

**Remark 5.** Note that other families could be obtained by using different values in the parameter \( \lambda \). And mixing different classes for each part of the aggregation, for example, we could obtain \( \lambda = 1 \) for the probabilities and \( \lambda = 2 \) for the WA. Thus, we would get the probabilistic weighted quadratic average (PWQA). And in a similar way we could form a lot of other cases such as the probabilistic quadratic weighted geometric average (PQWGA), the probabilistic geometric weighted average (PGWA), the probabilistic geometric weighted quadratic average (PGWQA), and so on.
QUASI-ARITHMETIC MEANS IN THE PWA OPERATOR

As it is explained in (Beliakov, 2005), a further generalization of the GOWA operator is possible by using quasi-arithmetic means. Following the same methodology as in (Fodor et al., 1995), we can suggest a similar generalization for the GPWA operator by using quasi-arithmetic means. We will call this generalization the Quasi-PWA operator. It can be defined as follows.

**Definition 9.** A Quasi-PWA operator of dimension \( n \) is a mapping \( \text{QPWA} : \mathbb{R}^n \rightarrow \mathbb{R} \) that has an associated weighting vector \( W \) of dimension \( n \) such that the sum of the weights is 1 and \( w_j \in [0, 1] \), then:

\[
\text{Quasi-PWA} \left( a_1, a_2, \ldots, a_n \right) = g^{-1} \left( \sum_{i=1}^{n} \hat{w}_i g(a_i) \right)
\]

where the \( a_i \) are the argument variables, each argument \( a_i \) has an associated weight (PWA) \( v_i \) with \( \sum_{i=1}^{n} v_i = 1 \) and \( v_i \in [0, 1] \), and a probabilistic weight \( p_i \) with \( \sum_{i=1}^{n} p_i = 1 \) and \( p_i \in [0, 1] \),

\[
\hat{v}_i = \beta p_i + (1 - \beta) v_i \quad \text{with} \quad \beta \in [0, 1],
\]

\( \hat{v}_i \) is the weight that unifies probabilities and WAs in the same formulation and \( g(h) \) is a strictly continuous monotone function.

Note that if the weighting vector of probabilities or WAs is not normalized, i.e., \( P = \sum_{i=1}^{n} p_i \neq 1 \), or \( V = \sum_{i=1}^{n} v_i \neq 1 \), then, the Quasi-PWA operator can be expressed as:

\[
\text{Quasi-PWA} \left( a_1, \ldots, a_n \right) = \frac{\beta}{P} \left( g^{-1} \left( \sum_{i=1}^{n} p_i g(a_i) \right) \right) + \frac{(1 - \beta)}{V} \left( g^{-1} \left( \sum_{i=1}^{n} v_i g(a_i) \right) \right)
\]

Note that all the properties and particular cases commented in the GPWA operator, are also included in this generalization. Thus, we could study all the particular case of Section 4 and a lot of other cases.

**ILLUSTRATIVE EXAMPLE**

In the following, we are going to develop a brief illustrative example of the new approach in a decision making problem about the selection of the optimal production strategy.

Assume a decision maker wants to create a new product and he is analyzing the optimal target in order to obtain the highest benefits. After analyzing the market he considers five possible alternatives.
• Create a new product oriented to the rich customers: \( A_1 \).
• Create a new product oriented to the mid-level customers: \( A_2 \).
• Create a new product oriented to the low-level customers: \( A_3 \).
• Create a new product adapted to all the customers: \( A_4 \).
• Do not create any product: \( A_5 \).

After careful review of the information, the decision maker establishes the following general information about the production strategy. He has summarized the information of the strategies in five general characteristics \( C = \{ C_1, C_2, C_3, C_4, C_5 \} \).

- \( C_1 \): Benefits in the short term.
- \( C_2 \): Benefits in the mid term.
- \( C_3 \): Benefits in the long term.
- \( C_4 \): Risk of the production strategy.
- \( C_5 \): Other factors.

The results are shown in Table 1. Note that the results are numbers between 0 and 100, being 100 the best result.

Table 1. Characteristics of the production strategy.

<table>
<thead>
<tr>
<th></th>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
<th>( C_4 )</th>
<th>( C_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>60</td>
<td>70</td>
<td>80</td>
<td>40</td>
<td>60</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>20</td>
<td>90</td>
<td>80</td>
<td>60</td>
<td>70</td>
</tr>
<tr>
<td>( A_3 )</td>
<td>30</td>
<td>50</td>
<td>40</td>
<td>80</td>
<td>80</td>
</tr>
<tr>
<td>( A_4 )</td>
<td>70</td>
<td>50</td>
<td>70</td>
<td>10</td>
<td>90</td>
</tr>
<tr>
<td>( A_5 )</td>
<td>50</td>
<td>60</td>
<td>70</td>
<td>40</td>
<td>70</td>
</tr>
</tbody>
</table>

With this information, it is possible to develop different methods based on the GPWA operator for selecting a production strategy. In this example, we will consider the probabilistic aggregation, the weighted average, the arithmetic mean, the arithmetic probabilistic aggregation, the arithmetic weighted average and the PWA operator. We will assume that \( \beta = 0.4 \) and the following weights: \( P = (0.1, 0.2, 0.2, 0.2, 0.3) \) and \( V = (0.1, 0.1, 0.1, 0.3, 0.4) \). The results are shown in Table 2.
Table 2. Aggregated results.

<table>
<thead>
<tr>
<th>Prob.</th>
<th>WA</th>
<th>AM</th>
<th>APA</th>
<th>AWA</th>
<th>PWA</th>
</tr>
</thead>
<tbody>
<tr>
<td>A₁</td>
<td>62</td>
<td>57</td>
<td>62</td>
<td>62</td>
<td>59</td>
</tr>
<tr>
<td>A₂</td>
<td>69</td>
<td>65</td>
<td>64</td>
<td>66</td>
<td>64.6</td>
</tr>
<tr>
<td>A₃</td>
<td>61</td>
<td>68</td>
<td>56</td>
<td>58</td>
<td>63.2</td>
</tr>
<tr>
<td>A₄</td>
<td>60</td>
<td>58</td>
<td>58</td>
<td>58.8</td>
<td>58</td>
</tr>
<tr>
<td>A₅</td>
<td>60</td>
<td>58</td>
<td>58</td>
<td>58.8</td>
<td>58</td>
</tr>
</tbody>
</table>

As we can see, depending on the particular type of GPWA operator used, the optimal choice is different. Therefore, it is interesting to establish an ordering of the production strategy for each particular case.

Table 3. Ordering of the production strategy

<table>
<thead>
<tr>
<th>Ordering</th>
<th>Ordering</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prob.</td>
<td>A₂≺A₁≺A₃≺A₄≺A₅</td>
</tr>
<tr>
<td>WA</td>
<td>A₃≺A₂≺A₄≺A₅≺A₁</td>
</tr>
<tr>
<td>AM</td>
<td>A₂≺A₁≺A₄≺A₅≺A₃</td>
</tr>
</tbody>
</table>

As we can see, depending on the particular type of GPWA used, the results and the decisions may be different.

CONCLUSIONS

We have presented a new approach that unifies the probability and the weighted average in the same formulation considering the degree of importance that each concept has in the analysis. Moreover, we have seen that this model uses generalized means providing a more complete formulation of the aggregation operator. We have called it the generalized probabilistic weighted averaging operator. Its main advantage is that it provides a wide range of aggregation operators such as the PWA, the PGWA, the PQWA, the PHWA, the AWA, the APA, and a lot of other cases. We have further generalized the GPWA by using quasi-arithmetic means, obtaining the Quasi-PWA operator. We have also developed an application of the new approach in a decision making problem. We have seen that depending on the particular type of GPWA operator used, the results may be different. However, we should note that the manipulation of the results is not so flexible here as it was in the OWA operator (Merigò, 2008a, Yager, 1988).
In future research, we expect to develop further extensions to this approach by using more general formulations and considering other characteristics in the problem such as the use of uncertain information represented in the form of interval numbers, fuzzy numbers, linguistic variables, and expertons. We will also consider the use of distance measures in the analysis and a lot of other applications in business decision making and other fields such as statistics and engineering.

REFERENCES


