A Multiperiod Binomial Model for Pricing Options in an Uncertain World

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The aim of this paper is to price an option in a multiperiod binomial model, when there is uncertainty on the states of the world at each node of the tree.

The starting point is the one period model, where the two states of the world, namely state up, where the market is “bullish” and state down, where the market is “bearish”, are vague. As a consequence, also the stock price at each state takes imprecise values. In a standard one period binomial model, we are required to give two crisp values, one for each state, for the stock price movement in the next time period. This problem boils down to the estimation of the volatility of the underlying asset, that is an unobservable parameter. Let \( u \) and \( d \) be the up and down crisp jump factors respectively, the standard methodology (Cox, Ross, Rubinstein, 1979) leads to set

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\begin{align*}
u &= e^{\sigma \sqrt{\Delta t}} & d &= e^{-\sigma \sqrt{\Delta t}}
\end{align*}
\]

where \( \sigma \) is the volatility of the underlying asset and \( \Delta t \) is length of the time period.

Sometimes it is hard to give a precise estimate of the volatility of the underlying asset and it may be convenient to let it take interval values. Moreover, it may be the case that not all the members of the interval have the same reliability, as central members are more possible than the ones near the borders. Possibility distributions are used to handle this type of problems. Triangular fuzzy numbers, in particular, are used to model the two possibility distributions. Among all the different types of numbers, the choice of using triangular numbers is made for the sake of simplicity, since assuming more complicated shapes may increase the computational
complexity without substantially affecting the significance of the results. Even if the price of the stock is constrained to move up or down in discrete ticks, there is still uncertainty about the number of ticks it will increase or decrease. For most stocks, daily price movement limits are specified by the exchange, these limits may be used as lower or upper bounds for the support of the decrease or increase in the stock price respectively.

This way of modelling the imprecision has an intuitive appeal. In the standard binomial model we say that the stock price at time one may jump up or down to two exactly given values. In our model we just say that if the market is bullish the stock price increases and if the market is bearish it will decrease, whereby the amount of the change is imprecisely given: we fix the peak value of the fuzzy number equal to the crisp value in the standard binomial case and we allow the nearby prices to have some degree of possibility.

In order to derive the artificial probability measure we assume the following: the agents have homogeneous beliefs and act as price takers; the market is frictionless and complete; interest rates are positive and no arbitrage opportunities are allowed.

Under this set of assumptions we can apply the standard methodology to derive the risk neutral probabilities. It is interesting to observe that, differently from the standard binomial option pricing model, we obtain risk neutral probability intervals instead of point estimates. This is clearly a consequence of our assumption on the stock price. The risk neutral probability intervals arise from the opacity of the stock price at $t=1$, even if the real probabilities of the stock price jumps are crisp and known in advance. This is implied by the risk neutral valuation paradigm that states that if the market is complete and there are no arbitrage opportunities, then the real probabilities involved do not play any role in the pricing problem. In other words, the risk neutral probability measure depends only on the amount of decrease or increase in the stock price. If the jump factors are crisp numbers, then we are back to the standard binomial model, with crisp risk neutral probabilities. By contrast, in this setting, the jump factors are fuzzy numbers and the risk-neutral probabilities are weighted intervals. If $\alpha=1$, where $\alpha$ denotes the $\alpha$-cut, the artificial probabilities collapse to the one of the standard binomial model. Thus our model can be seen as a generalisation of the standard binomial option pricing model as the latter is a special case (if $\alpha=1$) of the former.
The artificial probability measures that we get closely resemble the belief and plausibility measures of evidence theory (Dempster, 1967). These measures are obtained by replacing the additivity requirement by super-additivity or subadditivity respectively. The dual relationship between the two types of measures ensures that given a measure of either of the two types, it induces a unique measure of the other type. Since belief measures are always smaller than or equal to the corresponding plausibility measures, they may be seen as lower and upper probabilities respectively.

Expected values can be computed under these measures resulting in an expected value interval. For applications to the pricing problem, this feature represents a drawback given that one needs some additional criterion in order to get a single price. Cherubini (1997) and Della Lunga applied a particular class of fuzzy measures (Wang, Klir, 1992), the g-lambda measures, to the asset pricing problem and found intervals for the derivative price. In this paper, by modelling the stock price in each state of the world as a fuzzy number we obtain a possibility distribution on the risk neutral probability, i.e. a weighted interval of probability. This, by contrast with the theory of evidence, implies a main advantage for pricing problems.

By computing the option price using the artificial probability measure we get a weighted expected value interval for the price and thus we are able to determine a “most likely” value of the option within the interval.

Moreover, it may be convenient for operative purposes to find a crisp number that synthesises the option price interval in one crisp constant that summarises all the information contained. This type of problem is known in the literature as “defuzzification procedure”. There are many methods that, depending on the kind of fuzzy number that we want to defuzzify, provide a scalar that best represents the information contained. In this paper we use a method that is based on the intuitive idea that the best defuzzifier is the scalar that is “closest” to the fuzzy number. We choose this method in particular, for its simplicity and intuitiveness. We can also get an index of the fuzziness present in the option price, that tells us the degree of imprecision intrinsic in the model.

Since for $\alpha=1$ our model collapses to the standard binomial model, it follows that when the defuzzifier is equal to the peak value of the option price the standard binomial model and our model lead to the same result. This happens for example if the derivative price is represented by
a symmetric triangular fuzzy number. In this case the amount of evidence that the derivative price is less than the peak value is equal to the amount of evidence that the derivative price is more than it. As a consequence the peak value is the best representative of the distribution.

The defuzzifier is in general different from the standard binomial price and it is a more reliable price because it takes into account all the information present in the market, regarding the possible amount of increase or decrease of the stock price. Taking into account only a crisp estimate of the parameters, as in a standard binomial model, may result in a loss of information and in a wrong estimate of the option price.

Our methodology offers some advantages:

• First, it provides an intuitive way to look at the uncertainty in the stock price jumps.
• Second, it includes the results of the Standard Binomial Option Pricing Model.
• Third, it traces back the need of using intervals of risk-neutral probabilities, to the opacity in the two possible jumps of the stock.
• Finally, using weighted intervals of probabilities, i.e. possibility distributions on the risk-neutral probabilities, it provides a weighted expected value interval for the option price and thus we are able to determine a “most likely” value of the option within the interval.

High on the research agenda are the development of alternative defuzzification procedures, the implementation of the model with market data, and the extension to a continuous time model.

References


New Logics for the New Economy