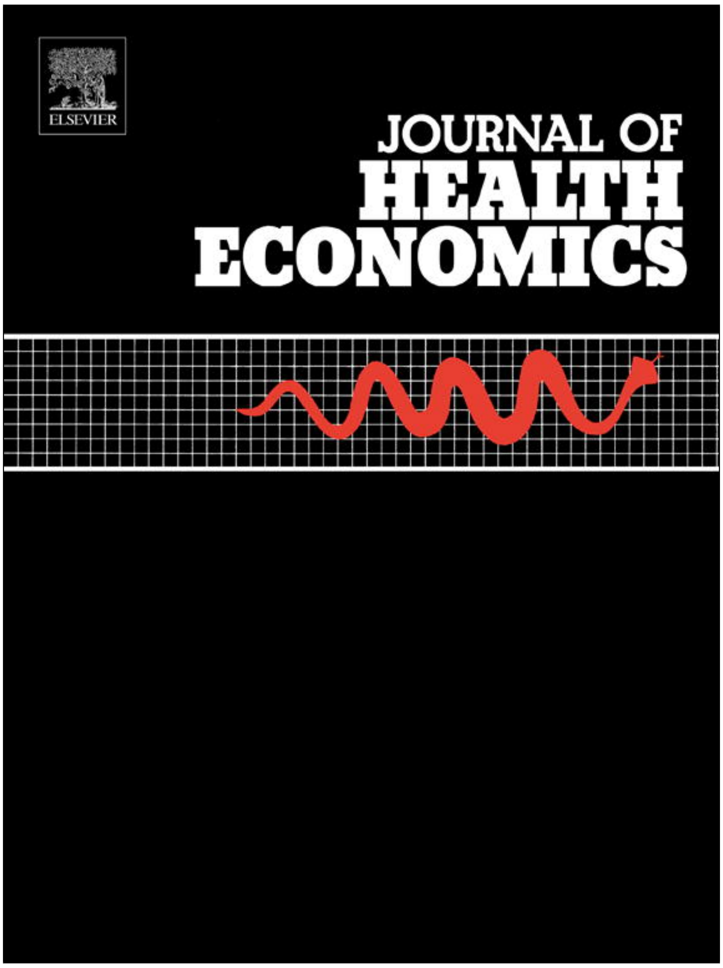
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Proposing indicators to measure achievement and shortfall inequality consistently☆

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ABSTRACT

In several economic fields, such as those related to health or education, the individuals' characteristics are measured by bounded variables. Accordingly, these characteristics may be indistinctly represented by achievements or shortfalls. A difficulty arises when inequality needs to be assessed. One may focus either on achievements or on shortfalls but the respective inequality rankings may lead to contradictory results. In this note we propose a procedure to define indicators that measure equally the achievement and shortfall inequality. Specifically, we derive measures which are invariant under ratio-scale or translation transformations, and a decomposable measure is also obtained. As the indicators proposed depend on the distribution bounds, families of indices that guarantee the same inequality rankings regardless of the distribution maximal levels are identified.

1. Introduction

A number of recent papers have highlighted the difficulties in measuring inequality of a distribution that can be described either in terms of achievements or shortfalls (among them Clarke et al., 2002; Erreygers, 2009c; Lambert and Zheng, 2011). This situation arises in different economic fields in which bounded variables are involved, particularly in the measurement of health inequality. As stressed in the mentioned papers, the choice between achievement and shortfall inequality measurement is not innocuous, since different choices may lead to contradictory results.

Erreygers (2009c) characterizes two indicators, appropriate normalizations of the absolute Gini index and the coefficient of variation, respectively, both depending on the distribution bounds, which measure achievement and shortfall inequality identically. The square of the latter is decomposable in the sense that the overall inequality can be expressed as a weighted sum of the inequality levels computed for population subgroups plus inequality arising from the differences among subgroup means. In turn, Lambert and Zheng (2011) introduce a weaker property to measure achievement and shortfall inequality consistently, and show that all relative and intermediate standard inequality indices fail their requirement. They also identify two classes of absolute inequality indices according to which the measure of achievement and shortfall inequality is identical, and show that, among them only the variance is subgroup decomposable.

All these results rightly consider that achievements and shortfalls are different sides of the same coin and, consequently, inequality of shortfalls and inequality of achievements should mirror each other. Our starting point is slightly different. In fact, this paper proposes considering a unified framework where the achievement and the shortfall distributions can be jointly analyzed. One simple way to do this, given any inequality measure, is to aggregate the respective achievement and shortfall inequality levels in a single indicator. Section 3 shows that taking a generalized mean of these two values allows us to transform any inequality measure into an indicator which is able to capture the achievement and the shortfall inequality consistently. In addition, some

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of the properties enjoyed by the original index are inherited by its transformation. Accordingly, measures both ratio-scale and translation invariant, may be obtained and a decomposable index is also identified.

When a standard inequality index is used to measure shortfall inequality, the results depend on the bounds of the distribution. The same happens if the indicators we suggest are applied. Most times these levels are fixed values, for instance, if attainment is measured by a variable in percentage terms. However, it may be the case that the bounds change. Then the procedure proposed will introduce a source of arbitrariness in the measurement since inequality orderings may change when the bounds vary. Hence, Section 4 is devoted to obtaining inequality indicators that are bound-consistent, that is, they lead to the same orderings regardless of the bounds. A family of decomposable indices which gauges shortfall inequality bound-consistently is characterized. We also identify indices for which the geometric mean aggregator rankings are independent of the bounds. Finally we show that, in a decomposable setting, only absolute measures can be aggregated through the arithmetic mean indicator so that the inequality orderings remain unchanged when the bounds vary.

2. Notation and basic definitions

We consider a population consisting of $n \geq 2$ individuals. An *achievement distribution* is represented by a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in D^n$, with $D^n = \mathbb{R}^n_+$ or $D^n = \mathbb{R}^n_{++}$, where x_i represents individual i's achievement. We assume that the variables are ratio-scale and are lower bounded by 0. The set of all feasible distributions is $D = \bigcup_{n \geq 2} D^n$. The positive part will be denoted by D_+. For any $\mathbf{x} \in D$, $\mu_{\mathbf{x}} = \mu(\mathbf{x})$ and $n_{\mathbf{x}} = n(\mathbf{x})$ stand, respectively, for the mean and population size of the distribution $\mathbf{x}$.

For each $\alpha > 0$ we let D^α represent the set of distributions for which α is an upper bound and denote as $D^\alpha_+ = \{\mathbf{x} \in D_+ / x_i < \alpha\}$. Note that if $\alpha' > \alpha$, then $D^{\alpha'} \supset D^\alpha$ and D can be decomposed as $D = \bigcup_{\alpha > 0} D^\alpha$. The *shortfall distribution* associated with $\boldsymbol{x} \in D^\alpha$ is denoted as $\boldsymbol{s} = (s_1, s_2, \ldots, s_n) \in \mathbb{R}^n_+$, where $s_i = \alpha - x_i$ represents individual i's shortfall. We use the notation $\mathbf{1} = (1, \ldots, 1)$ and $\lambda\mathbf{1} = (\lambda, \ldots, \lambda)$. Hence the shortfall distribution can be equivalently denoted by $\boldsymbol{s} = \alpha\mathbf{1} - \boldsymbol{x}$.

Given two distributions $\boldsymbol{x}, \boldsymbol{x}' \in D$, we say that $\boldsymbol{x}'$ is obtained from $\boldsymbol{x}$ by a *progressive transfer* if there exist two individuals $i, j \in \{1, \ldots, n\}$ and $h > 0$ such that $x'_i = x_i + h \leq x_j - h = x'_j$ and $y'_k = y_k$ for every $k \neq i, j$.

An inequality index I is a real valued continuous function $I : D \to \mathbb{R}$ which fulfils the following properties.

Pigou-Dalton Transfer Principle (TP). $I(\boldsymbol{x}') < I(\boldsymbol{x})$ whenever $\boldsymbol{x}'$ is obtained from $\boldsymbol{x}$ by a progressive transfer.

Normalization (NOR). $I(\lambda\mathbf{1}) = 0$ for all $\lambda > 0$.

Symmetry (SYM). $I(\boldsymbol{x}) = I(\boldsymbol{x}')$ whenever $\boldsymbol{x} = \Pi\boldsymbol{x}'$ for some permutation matrix Π.

Replication Invariance (RI). $I(\boldsymbol{x}) = I(\boldsymbol{x}')$ whenever $\boldsymbol{x}' = (\boldsymbol{x}, \boldsymbol{x}, \ldots, \boldsymbol{x})$ with $n_{\boldsymbol{x}'} = mn_{\boldsymbol{x}}$ for some positive integer m.

The crucial axiom in inequality measurement is the *Pigou-Dalton transfer principle* which requires that a transfer from a richer person to a poorer one decreases inequality. In addition, the indices are usually assumed to be *normalized* with the inequality level equal to 0 when everybody has exactly the same distribution value. *Symmetry* establishes that the inequality index should be insensitive to a reordering of the individuals. Finally, *replication invariance* allows populations of different sizes to be compared. These four properties are considered to be inherent to the concept of inequality and have come to be accepted as basic properties for an inequality index.

An inequality index $I_R : D_+ \to \mathbb{R}$ is *relative* if proportional changes in all the values do not alter the inequality level, that is, for all $\mathbf{x} \in D_+$ $I(\lambda x) = I(x)$ where $\lambda > 0$.

A relative index is insensitive to variations in the unit in which the variables are measured.

An inequality index $I_A : D \to \mathbb{R}$ is *absolute* if the same increase in all the distribution values does not change the inequality level, that is, for all $\boldsymbol{x} \in D$ $I(\boldsymbol{x} + \eta\mathbf{1}) = I(\boldsymbol{x})$ for all η whenever $\boldsymbol{x} + \eta\mathbf{1} \in D$.

Given an inequality measure I and $\alpha > 0$, $I^S(.;\alpha)$ stands for the *shortfall indicator* defined as $I^S(\boldsymbol{x};\alpha) = I(\alpha\mathbf{1} - \boldsymbol{x})$ for any $\boldsymbol{x} \in D^\alpha$.

3. Proposing perfect complementary indicators.

3.1. The r-indicators associated with an inequality measure.

This paper deals with the problem of evaluating and comparing the inequality level of bounded distributions. In these cases, a person's characteristics can be represented in terms of achievements or in terms of shortfalls. Consequently, the inequality level can be assessed focusing on either of these terms. These two frameworks are linked but nevertheless distinct, and can yield different results. As mentioned above, recent efforts have been made to introduce conditions and to define indicators for which the respective inequality levels mirror each other.

This paper aims to propose a mixed approach in which achievements and shortfalls may be jointly analyzed. We may think of the inequality of a bounded distribution as an aggregate of the inequality of achievements and the inequality of shortfalls. The properties enjoyed by the r-order means make them an appropriate way of aggregation in several economic fields. As will be showed, also in this framework they behave in a satisfactory way.

Consider an inequality measure I, a maximum level of achievements α, and, for a given distribution $\boldsymbol{x} \in D^\alpha$, the inequality values $I(\boldsymbol{x})$ and $I(\alpha\mathbf{1} - \boldsymbol{x})$. If we are interested in analysing simultaneously the achievement and the shortfall inequality, we may think of aggregating these two values. A natural aggregation procedure may be any r-order mean of them. The indicator defined in such a way depends on the distribution $\mathbf{x}$ and on bound α.

Specifically, given $\alpha > 0$ we propose to consider the *r-indicator associated with I*, denoted by I^r that, for each distribution $\boldsymbol{x} \in D^\alpha$, takes the following value

$$I^r(\boldsymbol{x};\alpha) = \begin{cases} \left(\dfrac{I(\boldsymbol{x})^r + I(\alpha\mathbf{1} - \boldsymbol{x})^r}{2}\right)^{1/r} & \text{if } r \neq 0 \\ (I(\boldsymbol{x})I(\alpha\mathbf{1} - \boldsymbol{x}))^{1/2} & \text{if } r = 0 \end{cases}$$

When $r < 0$, the r-order means are defined only for positive values. However, as $I(x) = 0$ implies, by normalization, that $\boldsymbol{x} = k\mathbf{1}$, we will take the convention that whenever $I(\boldsymbol{x}) = I(\alpha\mathbf{1} - \boldsymbol{x}) = 0$, $I^r(\boldsymbol{x};\alpha) = 0$ for any $r < 0$.

Now some properties of the r-order means are mentioned. For any r, $I^r(\boldsymbol{x};\alpha)$ lies between $I(\boldsymbol{x})$ and $I(\alpha\mathbf{1} - \boldsymbol{x})$. Particular members of this family are $I^1(\boldsymbol{x};\alpha)$, which corresponds to the arithmetic mean of the two values and $I^0(\boldsymbol{x};\alpha)$, the geometric mean. The mapping $r \to I^r(\boldsymbol{x};\alpha)$ is a non decreasing continuous function on all of $\mathbb{R}$. The limiting case at one extreme is as $r \to -\infty$, giving $I^r(\boldsymbol{x};\alpha) \to \min\{I(\boldsymbol{x}), I(\alpha\mathbf{1} - \boldsymbol{x})\}$. At the other extreme, as I^r, giving $I^r(\boldsymbol{x};\alpha) \to \max\{I(\boldsymbol{x}), I(\alpha\mathbf{1} - \boldsymbol{x})\}$. Moreover, for a given r, $I^r(\boldsymbol{x};\alpha)$ is non-decreasing in $I(\boldsymbol{x})$ and in $I(\alpha\mathbf{1} - \boldsymbol{x})$.[2]

In what follows we show that some additional properties fulfilled by I are inherited by the r-indicators.

[2] See for example Steele (2004).

Proposition 1. For any $\alpha > 0$ the r-indicator $I^r(.;\alpha)$ associated with an inequality measure I satisfies continuity, TP, NOR, SYM and RI for any r. It also holds that for any $\boldsymbol{x} \in D^\alpha$, $I^r(\boldsymbol{x};\alpha) = I^r(\alpha\mathbf{1} - \boldsymbol{x};\alpha)$. In addition:

(i) If I_R is relative, for any $\boldsymbol{x} \in D^\alpha$ $I_R^r(\boldsymbol{x};\alpha) = I_R^r(\lambda\boldsymbol{x};\lambda\alpha)$ for all $\lambda > 0$.
(ii) If I_A is absolute, for any $\boldsymbol{x} \in D^\alpha$ $I_A^r(\boldsymbol{x};\alpha) = I_A^r(\boldsymbol{x} + \eta\mathbf{1};\alpha + \eta)$ for all $\eta > 0$.

Proof. It is clear that $I^r(.;\alpha)$ satisfies continuity, NOR, SYM and RI as I does. To prove that $I^r(.;\alpha)$ also fulfils TP, let us assume that $\boldsymbol{x}'$ is derived from $\mathbf{x}$ by a progressive transfer. Then, $\boldsymbol{s}' = \alpha\mathbf{1} - \boldsymbol{x}'$ is also derived from $\boldsymbol{s} = \alpha\mathbf{1} - \boldsymbol{x}$ by a progressive transfer. In fact, a progressive transfer among two individuals' achievements leads to an increment in the richer person's shortfall, whereas the poorer person's shortfall decreases. Since the richer person's shortfall is smaller than the poorer one's, a progressive transfer of achievements is equivalent to a progressive transfer of shortfalls. Consequently, under a progressive transfer both $I(\boldsymbol{x})$ and $I(\alpha\mathbf{1} - \boldsymbol{x})$ are bound to decrease and so does $I^r(.;\alpha)$. Finally, from the definitions, it is clear that $I^r(.;\alpha) = I^r(\alpha\mathbf{1} - \boldsymbol{x};\alpha)$ and that statements (i) and (ii) hold. □

I^r is not a standard inequality measure since it depends on α. Nevertheless, it fulfils all the properties that are usually assumed for an inequality index, mainly TP. So it is able to capture the distribution inequality.

Because $I^r(\boldsymbol{x};\alpha) = I^r(\alpha\mathbf{1} - \boldsymbol{x};\alpha)$, Proposition 1 opens up a wide range of possibilities to derive *perfect complementary* indices, both rank-dependent and rank-independent. Firstly, if I is a *perfect complementary* indicator, i.e. $I(\boldsymbol{x}) = I(\alpha\mathbf{1} - \boldsymbol{x})$, then $I^r(\boldsymbol{x};\alpha) = I(\boldsymbol{x})$.

The invariance conditions fulfilled by I^r depend on the conditions satisfied by I. Specifically all the relative measures such as the Gini-coefficient, the S-Gini family (Donaldson and Weymark, 1980), the coefficient of variation, the Generalized Entropy family (Shorrocks, 1980) or the Atkinson family (Atkinson, 1970) generate r-indicators that are insensitive to changes in the measurement unit, and so unit-free.

For instance, the family of r-indicators associated with the Gini coefficient, taking into account that $G(\alpha\mathbf{1} - \boldsymbol{x}) = (\mu_{\boldsymbol{x}}/\alpha - \mu_{\boldsymbol{x}})G(\boldsymbol{x})$, is as follows

$$G^r(\boldsymbol{x};\alpha) = \begin{cases} G(\boldsymbol{x})\left(\dfrac{\mu_{\boldsymbol{x}}^r + (\alpha - \mu_{\boldsymbol{x}})^r}{2(\alpha - \mu_{\boldsymbol{x}})^r}\right)^{1/r} & \text{if } r \neq 0 \\ G(\boldsymbol{x})\left(\dfrac{\mu_{\boldsymbol{x}}}{\alpha - \mu_{\boldsymbol{x}}}\right)^{1/2} & \text{if } r = 0 \end{cases}$$

Similar expressions hold for the coefficient of variation. As $CV(\alpha\mathbf{1} - \boldsymbol{x}) = (\mu_{\boldsymbol{x}}/\alpha - \mu_{\boldsymbol{x}})CV(\boldsymbol{x})$, then

$$CV^r(\boldsymbol{x};\alpha) = \begin{cases} CV(\boldsymbol{x})\left(\dfrac{\mu_{\boldsymbol{x}}^r + (\alpha - \mu_{\boldsymbol{x}})^r}{2(\alpha - \mu_{\boldsymbol{x}})^r}\right)^{1/r} & \text{if } r \neq 0 \\ CV(\boldsymbol{x})\left(\dfrac{\mu_{\boldsymbol{x}}}{\alpha - \mu_{\boldsymbol{x}}}\right)^{1/2} & \text{if } r = 0 \end{cases}$$

Moreover, it may be interesting to note that the r-indicators associated with a relative measure show a different behaviour when the values of the distribution decrease proportionally, depending on whether $r > 0$ or $r \leq 0$. For a given $\alpha > 0$, consider an α-bounded distribution $\mathbf{x}$, $\lambda \in (0, 1]$ and the distribution $\boldsymbol{x}' = \lambda\boldsymbol{x}$, where everyone's achievement decreases gradually. In the end, when λ is almost 0, the achievement and the shortfall distributions are almost egalitarian. One could expect that the indicator level be equal to 0. Nevertheless this will not be always the case. Since I_R is a relative measure $I_R(\lambda\boldsymbol{x}) = I_R(\boldsymbol{x})$. Then

$$I_R^r(\lambda\boldsymbol{x}) = \left(\frac{I_R(\lambda\boldsymbol{x})^r + I_R(\alpha\mathbf{1} - \lambda\boldsymbol{x})^r}{2}\right)^{1/r} = \left(\frac{I_R(\boldsymbol{x})^r + I_R(\alpha\mathbf{1} - \lambda\boldsymbol{x})^r}{2}\right)^{1/r}$$

As $\lambda \to 0$, when $r > 0$, $I_R^r(\lambda\boldsymbol{x}) \to I_R(\boldsymbol{x})/2^{1/r}$, whereas when $r \leq 0$ $I_R^r(\lambda\boldsymbol{x}) \to 0$.[3]

The r-indicators associated with absolute measures are also worth computing. One interesting feature of these indicators that will be used in the ensuing section is that $I_A^r(\boldsymbol{x};\alpha) = I_A^r(\boldsymbol{x};\alpha')$ for any α, α' upper bounds of the distribution. In other words, the value of the inequality does not depend on the upper-bound chosen.

3.2. Sensitivity conditions

Transfer sensitivity conditions (Kolm, 1976; Shorrocks and Foster, 1987) demand that the inequality measure be more sensitive to transfers lower down the distribution. Lambert and Zheng (2011) establishes that no consistent inequality measure exists that satisfies the transfer sensitivity axiom. The same simple example they introduce to prove this result may be used in our setting. Hence no r-indicator associated with any inequality measure fulfils the transfer sensitive axiom.

3.3. Decomposability

In many applied analyses, the population is split into groups according to social characteristics as region, race, gender, and so on. In these cases it is quite useful to invoke properties which allow the inequality in each group to be related to overall inequality. An often used requirement proposed by Shorrocks (1980) is to demand that the overall inequality may be decomposed as the sum of the between- and the within-group components. The between-group component is defined as the inequality level of a hypothetical distribution in which each person's distribution values are replaced by the mean of their subgroup. The within-group component is a weighted sum of the subgroup inequality levels.

If this axiom is fulfilled, it is possible not only to identify subgroups where inequality is particularly high, but also to evaluate their contribution to overall inequality. Thus it is quite useful in applied analysis since it allows policy makers to target these groups in order to achieve a maximum reduction in inequality levels.

To formalize this decomposition assumption, suppose that a population of n individuals is split into $J \geq 2$ mutually exclusive subgroups with distribution $\boldsymbol{x}^j = (\boldsymbol{x}_1^j, \ldots, \boldsymbol{x}_{n_j}^j)$, where $\mu_j = \mu(\boldsymbol{x}^j)$ denotes the mean of the jth subgroup and $n_j = n(\boldsymbol{x}^j)$ represents its size for all $j = 1, \ldots, n$. Let inequality in group j be written $I_j = I(\boldsymbol{x}^j)$. Let us denote by $\boldsymbol{x}^B = (\mu_1\mathbf{1}_{n_1}, \ldots, \mu_J\mathbf{1}_{n_J})$ the distribution in which each person's distribution value is replaced by their subgroup mean.

Decomposability. An index I is *decomposable* if the following relationship holds

$$I(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^1) = I(\boldsymbol{x}^W) + I(\boldsymbol{x}^B) = \sum_{j=1}^{J} w_j(\mu_1, \ldots, \mu_J; n_1, \ldots, n_J)I(\boldsymbol{x}^j) + I(\boldsymbol{x}^B)$$

[3] We are very grateful to one of the referees for having raised this point, closely related to the drawback of the concentration index proposed by Wagstaff (2005) and modified by Erreygers (2009a). See also Wagstaff (2009) and Erreygers (2009b).

where $w_j(\mu_1, \ldots, \mu_J; n_1, \ldots, n_J) \geq 0$ is the weight on subgroup j's inequality level $I_j = I(\boldsymbol{x}^j)$ in the within-group term $j = 1, \ldots, J$.

One implication of this property is the subgroup consistency property (Shorrocks, 1984) which requires that if inequality in one group increases, overall inequality should also increase. Both Erreygers and Lambert–Zheng seek decomposable consistent indices in their respective frameworks. Whereas the Gini-type index characterized by the former is not decomposable, the square of the second satisfies decomposability. In turn, Lambert–Zheng shows that the only consistent inequality index which is decomposable is the variance. These results fit the standard inequality field.

Now consider the first Theil measure (Theil, 1967), belonging to the Generalized Entropy family, defined by $T(x) = \sum_{1\leq i\leq n} \log(\mu/x_i)/n$ for any $\mathbf{x} \in D_+$. The arithmetic mean indicator associated with T is defined as[4]

$$T^1(x;\alpha) = \sum\nolimits_{1\leq i\leq n} \log\left(\frac{\mu_{\boldsymbol{x}}\mu_{(\alpha-\boldsymbol{x})}}{x_i(\alpha - x_i)}\right)/2n \quad \text{for any } \boldsymbol{x} \in D_+^{\alpha}.$$

The following proposition shows that T^1 is decomposable.[5]

Proposition 2. The arithmetic mean index associated with the first Theil measure, T^1, is a decomposable measure for which the following decomposition holds

$$T^1(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^J; \alpha) = T^1(\boldsymbol{x}^W; \alpha) + T^1(\boldsymbol{x}^B; \alpha) = \sum_{j=1}^{J} \frac{n_j}{n} T_j^1(\boldsymbol{x}^j; \alpha) + T^1(\boldsymbol{x}^B; \alpha)$$

Proof.

$$\begin{aligned}
T^1(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^1; \alpha) &= \frac{T(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^1) + T(\boldsymbol{s}^1, \ldots, \boldsymbol{s}^1)}{2} && \text{by definition}\\
&= \frac{1}{2}\left(\sum_{j=1}^{J} \frac{n_j}{n} T_j(\boldsymbol{x}^j) + T(\boldsymbol{x}^B) + \sum_{j=1}^{J} \frac{n_j}{n} T_j(\boldsymbol{s}^j) + T(\boldsymbol{s}^B)\right) && \text{since } T \text{ is discomposable}\\
&= \frac{1}{2}\left(\sum_{j=1}^{J} \frac{n_j}{n} (T_j(\boldsymbol{x}^j) + T_j(\boldsymbol{s}^j)) + T(\boldsymbol{x}^B) + T(\boldsymbol{s}^B)\right) && \text{operating}\\
&= T^1(\boldsymbol{x}^W; \alpha) + T^1(\boldsymbol{x}^B; \alpha) && \text{by definition}
\end{aligned}$$

Since the weights in the within-group component depend only on the subgroup population shares, this decomposition also satisfies the path independent property proposed by Foster and Shneyerov (2000). Contrary to what happens to most of the decompositions, the variations in between-group inequality as measured by this index do not affect the within-group term. In addition, this decomposition allows the policy makers to easily compute the contribution of each group inequality to the overall inequality.

Remark 1. In the next section, it will be useful to have computed the weights of the within-group term in a particular case. Let be $\boldsymbol{x} \in D$, with $\mu_{\boldsymbol{x}} = \mu(\boldsymbol{x})$, $n_{\boldsymbol{x}} = n(\boldsymbol{x})$. Let us consider the distribution $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{x})$. For any inequality measure I, replication invariance implies that $I(\boldsymbol{z}) = I(\boldsymbol{x})$. In addition, if I is decomposable, then $I(\boldsymbol{z}) = 2w_{\boldsymbol{x}}(\mu_{\boldsymbol{x}}, \mu_{\boldsymbol{x}}; n_{\boldsymbol{x}}, n_{\boldsymbol{x}})I(\boldsymbol{x})$. Hence, for any distribution $\boldsymbol{x} \in D$ and for any decomposable measure I, $w_{\boldsymbol{x}}(\mu_{\boldsymbol{x}}, \mu_{\boldsymbol{x}}; n_{\boldsymbol{x}}, n_{\boldsymbol{x}}) = 0.5$.

4. The robustness of the inequality rankings to changes in the upper-bounds

The family of r-indicators introduced in the previous section depends on the α-parameter, which represents the upper-bound of the distribution. And so does any standard inequality index used to assess the shortfall inequality.

When the achievements are measured in percentage terms, the upper-bound is fixed and α can no longer be considered as a parameter. However, sometimes the maximal value from which shortfall is calculated may vary between individuals, or between countries or in different periods of time. A variation in the upper-bound will change the shortfall distributions and then the inequality rankings may be reversed.

Consider for instance the women and the men in a country. Assume that the country's potential permits α and α' to be the respective maximal achievements. Imagine that we are now interested in analysing these two groups as part of a bigger region in which the maximal achievements are higher. It seems natural to think that the inequality rankings are not reversed when we compare the two groups as part of the country or as part of the region.

This section examines under which conditions the shortfall indicators and the r-indicators, which will be denoted indistinctly by I^G, guarantee that the rankings are not reversed when the bounds change. This bound-consistency property is formally defined as follows.

Definition 1. *The indicator I^G is bound-consistent iff for any $\alpha, \alpha' > 0$ and for any two distributions $\boldsymbol{x} \in D^{\alpha}$ and $\boldsymbol{y} \in D^{\alpha'}$*

if $I^G(\boldsymbol{x}; \alpha) \leq I^G(\boldsymbol{y}; \alpha')$ then $I^G(\boldsymbol{x}; \alpha + \theta) \leq I^G(\boldsymbol{y}; \alpha' + \theta)$ for any $\theta >$ 0.

It is clear that as long as absolute indices are involved in the I^G indicator, increments in the bounds do not alter the inequality values. Notice that the definition requires the two maximal levels, α and α', to be increased by the same absolute amount. An alternative formulation would demand a proportional increment of the bounds rather than an absolute one. This possibility is more restrictive and it can be proved that it implies that only absolute measures fulfil the requirement.

The next three propositions determine some restrictive circumstances which guarantee that the shortfall indicator, the arithmetic mean and the geometric mean indicators are bound-consistent. A sketch of the proofs is presented in Appendix A.

Proposition 3 below identifies a family of inequality measures including the absolute indices that implies bound-consistency for the shortfall and the geometric mean indicators.

Proposition 3. Consider an inequality measure I of the form $I(\boldsymbol{x}) = e^{\tau\mu(\boldsymbol{x})}I_A(\boldsymbol{x})$ with $\tau \in \mathbb{R}$ and I_A an absolute measure.

Then the shortfall indicator I^S and the geometric mean indicator I^0 associated with I are bound-consistent.

The following two results rely heavily on the class of decomposable absolute measures given by[6]

$$I_{\beta}(\boldsymbol{x}) = \begin{cases} \dfrac{1}{n(\boldsymbol{x})}\displaystyle\sum_{i=1}^{n(\boldsymbol{x})} \{e^{\beta(x_i - \mu(\boldsymbol{x}))} - 1\} & \text{if } \beta \neq 0 \\ \dfrac{1}{n(\boldsymbol{x})}\displaystyle\sum_{i=1}^{n(\boldsymbol{x})} (x_i - \mu(\boldsymbol{x}))^2 & \text{if } \beta = 0 \end{cases}$$

[4] It should be noted that since $\boldsymbol{x} \in D_+^{\alpha}$, then both $\mathbf{x}$ and $\mathbf{s}$ take positive values, as required to the logarithm function may be computed.

[5] All the Generalized Entropy measures are decomposable (Shorrocks, 1980). The decomposition of the first Theil measure is expressed as follows: $T(y^1, \ldots, y^J) = T(y^W) + T(y^B) = \sum_{1\leq j\leq J} (n_j/n)T_j + T(y^B)$.

[6] This class has been characterized by Bosmans and Cowell (2010) in a continuous framework, and by Chakravarty and Tyagarupananda (1998) in a differentiable setting.

As well known, the absolute family proposed by Kolm (1976) arises for $\beta \neq 0$ and the variance when $\beta = 0$. The arithmetic mean indicator associated with the Kolm indices may be easily computed as

$$I_{\beta}^{1}(\boldsymbol{x};\alpha) = \frac{1}{n(\boldsymbol{x})}\sum_{i=1}^{n(\boldsymbol{x})}\{e^{\beta(x_i-\mu(\boldsymbol{x}))} + e^{\beta(\mu(\boldsymbol{x})-x_i)} - 2\} \quad \text{with } \beta \neq 0$$

Proposition 4 below identifies the decomposable indices that measure the shortfall inequality bound-consistently.

Proposition 4. The shortfall indicator I^S associated with a decomposable inequality measure I is bound-consistent if and only if I is a positive multiple of the form

$$I_{\beta\tau}(\boldsymbol{x}) = e^{\tau\mu(\boldsymbol{x})}I_{\beta}(\boldsymbol{x}) \quad \text{where } \beta, \tau \in \mathbb{R}.$$

The class of inequality measures derived in this proposition is a two-parameter family that contains all the decomposable absolute inequality measures for $\tau = 0$. It is easy to prove that no relative measure belongs to the family. In fact, the only unit-consistent index in this class is the variance.[7]

The following Proposition 5 establishes that in a decomposable framework the arithmetic-indicator is bound-consistent only when it is associated with absolute measures.

Proposition 5. The arithmetic mean indicator I^1 associated with a decomposable inequality measure I is bound-consistent if and only if I is a positive multiple of a decomposable absolute measure, that is, $I(\boldsymbol{x}) = I_{\beta}(\boldsymbol{x})$.

5. Conclusions

This note introduces a procedure to derive indicators that capture simultaneously the achievement and the shortfall inequality. The method proposes aggregating the two inequality levels through any r-order mean. It is shown that this new indicator inherits most of the properties enjoyed by the original index. Its dependence on the maximal level of achievements makes this indicator less than a standard inequality index. Seeking indicators for which this dependence does not affect the inequality rankings, a family of measures is identified for which the shortfall inequality rankings do not depend on the upper bounds. In addition it is shown that, in a decomposable framework, only absolute measures are independent of the maximal level of achievements if the arithmetic mean indicator is chosen to rank bounded distributions.

At the outset we have assumed that the characteristics are measured by ratio-scale variables. However the procedure proposed may also be applied when bounded cardinal variables are involved in the analysis. The main difference is that now the lower bound is likely to be different from zero, and it plays a role in computing the shortfalls. Thus the aggregate of the achievement and shortfall inequalities depends on an additional parameter. It is easy to check that the modified versions of Proposition 1 and Proposition 2 hold in this framework.

Since relatively few consistent indicators exist in the literature related to this field, and inequality measurement depends deeply on the indicator chosen, we hope that this note may contribute to the robustness of the results obtained in empirical applications.

[7] According to Zheng (2005, 2007) the unit-consistency property requires that the inequality orderings are not reverse when the units in which the variable is measured change.

Appendix A.

Proof of Proposition 3. It is clear from the definitions.

Proof of Proposition 4. Before proving this proposition theorem, the following lemma needs to be established:

Lemma 1. The indicator I^S associated with a decomposable inequality measure I is bound-consistent if and only if

$$I(\boldsymbol{x}+\eta\mathbf{1}) = e^{\tau\eta}I(\boldsymbol{x}) \quad \text{for any } \tau \tag{1}$$

Proof. The sufficiency of the lemma is obvious. For the necessity, following the proof of Proposition 1 in Zheng (2007), we may conclude that bound-consistency for I^S implies that there exists a continuous function, increasing in the last argument, such that

$$I((\alpha+\eta)\mathbf{1}-\boldsymbol{x}) = f(\eta, I(\alpha\mathbf{1}-\boldsymbol{x})) \tag{2}$$

Following the proof of Proposition 3 in Zheng (2005) and taking into account Remark 1, it can be shown that Eq. (2) can be written as

$$I((\alpha+\eta)\mathbf{1}-\boldsymbol{x}) = f(\eta, I(\alpha\mathbf{1}-\boldsymbol{x})) = a(\eta)I(\alpha\mathbf{1}-\boldsymbol{x}) \tag{3}$$

for all $\alpha > 0$, $\boldsymbol{x} \in D^{\alpha}$, $\eta \in \mathbb{R}_+$ and some positive function $a(.)$.

The proof is completed noting that, for any two factors $\eta, \delta \in R_+$, and from Eq. (3), we have

$$I((\alpha+(\eta+\delta))\mathbf{1}-\boldsymbol{x}) = a(\eta+\delta)I(\alpha\mathbf{1}-\boldsymbol{x})$$

equivalently $I((\alpha+(\eta+\delta))\mathbf{1}-\boldsymbol{x}) = a(\delta)I((\alpha+\eta)\mathbf{1}-\boldsymbol{x}) = a(\eta)a(\delta)I(\alpha\mathbf{1}-\boldsymbol{x})$

$$\text{therefore} \quad a(\eta+\delta) = a(\eta)a(\delta) \tag{4}$$

The solution to this standard Eq. (4), following Aczél (1966, p. 38), is $a(\eta) = e^{\tau\eta}$ for some constant τ.

Proof of Proposition 4. Zheng (2005) based on Shorrocks (1984) shows that any continuous decomposable inequality index takes the form

$$I(\boldsymbol{x}) = \frac{1}{n(\boldsymbol{x})\lambda(\mu(\boldsymbol{x}))}\sum_{i=1}^{n(\boldsymbol{x})}(\phi(x_i) - \phi(\mu(\boldsymbol{x})))$$

where $\phi(.)$ is continuous and strictly convex; and $\lambda(.)$ is a continuous function.

Lemma 1 above has shown that I must satisfy $I(\boldsymbol{x}+\eta\mathbf{1}) = e^{\tau\eta}I(\boldsymbol{x})$. Define $J(\boldsymbol{x}) = e^{-\tau\mu(\boldsymbol{x})}I(\boldsymbol{x})$. It is easy to see that $J(\boldsymbol{x})$ is also decomposable and an absolute inequality measure. As such, Bosmans and Cowell (2010) (see also Chakravarty and Tyagarupananda (1998)) can be applied obtaining that $J(\boldsymbol{x})$ is a positive multiple of the following measures: $I(\boldsymbol{x}) = (1/n(\boldsymbol{x}))\sum_{i=1}^{n(\boldsymbol{x})}\{e^{\alpha(x_i-\mu(\boldsymbol{x}))} - 1\}$ or $I(\boldsymbol{x}) = (1/n(\boldsymbol{x}))\sum_{i=1}^{n(\boldsymbol{x})}(x_i - \mu(\boldsymbol{x}))^2$. This proves the necessity of the proposition. The sufficiency of the theorem is obvious.

Proof of Proposition 5. Proposition 5 follows from Lemma 2 below.

Lemma 2. The indicator I^1 associated with a decomposable inequality measure I is bound-consistent if and only if I is a decomposable inequality measure that satisfies

$$I(\boldsymbol{x}) + I((\alpha+\eta)\mathbf{1}-\boldsymbol{x}) = e^{\tau\eta}(I(\boldsymbol{x}) + I(\alpha\mathbf{1}-\boldsymbol{x})) \tag{5}$$

for all $\alpha > 0$, $\boldsymbol{x} \in D^{\alpha}$, $\eta \in \mathbb{R}_{++}$ and some constant τ.

Proof. The sufficiency of the lemma is clear. As regards the necessity we follow the proofs of Proposition 1 in Zheng (2007) and of Proposition 3 in Zheng (2005) taking into account remark 1 regarding the weights in the within-group term. If I^1 is bound-consistent, then there exists a continuous function, increasing in the last argument, such that

$$I(\boldsymbol{x}) + I((\alpha+\eta)\mathbf{1} - \boldsymbol{x}) = f(\eta, I(x) + I(\alpha 1 - x)) \quad (6)$$

As in the proof of Proposition 3 in Zheng (2005) it can be shown that

$$I(\boldsymbol{x}) + I((\alpha+\eta)\mathbf{1} - \boldsymbol{x}) = f(\eta, I(\boldsymbol{x}) + I(\alpha\mathbf{1} - \boldsymbol{x}))$$
$$= a(\eta)(I(\boldsymbol{x}) + I(\alpha\mathbf{1} - \boldsymbol{x})) \quad (7)$$

for all $\boldsymbol{x} \in D$, $\eta \in \mathbb{R}_+$ and some positive function $a(.)$.

Similarly to the proof of Lemma 1 it can be obtained that $a(\eta) = e^{\tau\eta}$ for some constant τ.

Proof of Proposition 5. We are to prove that only absolute measures fulfil equation (5). Indeed, Eq. (5) may be rewritten as

$$I((\alpha+\eta)\mathbf{1} - \boldsymbol{x}) = (e^{\tau\eta} - 1)I(\boldsymbol{x}) + e^{\tau\eta}I(\alpha\mathbf{1} - \boldsymbol{x}) \quad (8)$$

Given $\varepsilon > 0$, we will show that $I(\boldsymbol{x} + \varepsilon\mathbf{1}) = I(\boldsymbol{x})$. Let be $\beta = \alpha + \varepsilon$, and consider the distribution

$$\boldsymbol{z} = \varepsilon\mathbf{1} + \boldsymbol{x} = (\beta - \alpha)\mathbf{1} + \boldsymbol{x} \quad (9)$$

From Eq. (8) we get

$$I((\beta+\eta)\mathbf{1} - \boldsymbol{z}) = (e^{\tau\eta} - 1)I(\boldsymbol{z}) + e^{\tau\eta}I(\beta\mathbf{1} - \boldsymbol{z})$$

Since from (9), $\beta\mathbf{1} - \boldsymbol{z} = \alpha\mathbf{1} - \boldsymbol{x}$, then $I(\beta\mathbf{1} - \boldsymbol{z}) = I(\alpha\mathbf{1} - \boldsymbol{x})$ and $I((\beta+\eta)\mathbf{1} - \boldsymbol{z}) = I((\alpha+\eta)\mathbf{1} - \boldsymbol{x})$. Applying (8) the following also holds:

$$(e^{\tau\eta} - 1)I(\boldsymbol{x}) = (e^{\tau\eta} - 1)I(\boldsymbol{z})$$

Consequently, $\tau = 0$ or $I(\boldsymbol{x}) = I(\boldsymbol{z})$. Substituting this in (8) we find that, in both cases, I is an absolute measure.

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