A BARGAINING APPROACH TO THE PROVISION OF PUBLIC GOODS*

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Abstract

What do you prefer sharing equally the surplus or sharing equally the cost? In a context of provision of public goods, and from the axiomatic point of view, the difference is exactly one axiom. A full characterization of both solution is provided which differ in forbidding transfers of private good (NPT axiom) or imposing that final welfare should be above inactivity (WIR axiom). In the middle of both solutions we find the characterization of five more solutions that combine NPT and WIR with some of the other axioms proposed and take into account individual contributions to the surplus.

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1 Introduction

In a previous paper, Ginés and Marhuenda (2000), it is shown that Kalai's characterization of the welfare egalitarian solution can be transplanted from classical Bargaining Theory into some meaningful environments, in our case the provision of public goods. And we found that individual contributions are irrelevant, the important data is the total surplus.

The present paper deals with solutions which take into account the individual contributions. In order to compare how different are those solutions we provide the following example: Two agents with quasi-linear utilities (it is provided the non-linear part) $b_1(y) = 2 + 2y$, $b_2(y) = 10y$ and a technology represented by a cost function, $c(y) = 8 + y^2$. The surplus is $V(N) = 30$ and the efficient bundle of public goods is $y = 6$. Now if we share equally the total surplus we get a utility distribution $U = (15, 15)$. But this solution implies a transfer of private goods between the agents $b_1(6) = 14$. In the other extreme, if we believe that costs should be shared equally independently of the individual contributions, the equal loss solution is $U = (-8, 38) = (2+2*6-44/2, 10*6-44/2)$. The utility given to agent 1 is less that consuming $y = 0$ and paying all the fixed costs $u = 2-8 = -6$. The characterization proposed for both solutions are obtained using some axioms from the literature such as Pareto optimality, independence of preference contraction and independence of technological contraction, independence of cost function's zero and equal translation invariance. Independence of preference contraction and independence of technological contraction both assume that any change in preferences or technology that does not affect to the solution is allowed. Independence of cost function's zero implies equal share of the fixed costs and equal translation invariance means that any equal change in the inactivity point (related to preference) should be translated to the final solution. Both solutions, equal division of the surplus and equal share of the cost satisfy all the axioms above mentioned, but they differ in the final axiom needed to characterize both solution. Equal share of the surplus imposes that everybody should be above its inactivity situation, and equal division of the costs forbids direct transfers of private good. Clearly, all the axioms together are not compatible. But if we analyze the roll of each axiom in the characterization we find that eliminating one axiom each time of the seven proposed it is possible to characterize a different solution. The other solutions characterized are A)the lexicographic extension of the equal share of the surplus, B)lexicographic extension preserving equal share of the fixed costs C)rational equal loss (extension of the equal share of the costs) D) rational equal loss preserving the disagreement point and finally E) disagreement solution that gives everybody the utility of consuming zero and paying half of the fixed costs.. In our example these solutions corresponds to the following utility distributions, the lexicographic extension of the equal share gives rise to $U = (14, 16)$. Sharing fixed cost equally and applying the lexicographic extension, modifies the last solution to $U = (10, 20)$. If costs should be allocate such that everybody should pay, at most, his valuation of the public good then $U = (0, 30)$. Finally, if we think that $b(0)$ and $c(0)$ are important, the disagreement solution is $(-2, -4)$ and the solution rational equal loss from inactivity give us a distribution $U = (-2, 32) = (2 - 8/2 + 12 - 12, 0 - 8/2 + 60 - 24)$.

The intersection of these solutions with the classical bargaining theory comes from the idea that they have their correspondence in the bargaining theory with claims where the claims are defined as the individual contributions to the surplus.

The lexicographic extension of the Nash solution appears when Nash solution does not fit with the claims. It equalizes gains from inactivity point as much as claims allow. The non-envy or equal-loss solution prescribes equal share of the cost of the public good provided. It corresponds to the equal-loss from the claims point. The rational equal loss avoids situations where the equal-loss prescribes a payment bigger than the individual valuation of the public good. That is, utilities below the inactivity point.

The next table summarizes the results of the paper for the case of two agents, seven solutions characterized by combinations of seven properties.

	PO	IPC	ITC	ICFZ	ETI	WIR	NPT
WE	X	X	X	X	X	X	
EL	X	X	X	X	X		X
LWE	X	X	X	X		X	X
REL	X	X	X		X	X	X
LWE0	X	X		X	X	X	X
RELd	X		X	X	X	X	X
d		X	X	X	X	X	X

2 Model

Let $N = \{1, 2, 3, ..., n\}$ denote the set of agents. The space of public goods is $Y = \mathbb{R}^m_+ = \{y \in \mathbb{R}^m : y \geq 0\}$. The technology to produce those goods is jointly owned by all agents and it is described by a function $c : Y \to \mathbb{R}$ which measures the cost of producing each bundle of public goods in terms of the single private good of the economy. The set $X_i = \mathbb{R}$ represents the possible payments, in terms of the private good, made by agent $i \in N$. Let $X^n = \prod_{i=1}^n X_i$.

The preference relation of agent $i \in N$ is represented by a quasi-linear utility function $u_i(y;t_i) = b_i(y) - t_i$, with $(y;t_i) \in Y \times X_i$, which represents the utility obtained by agent $i \in N$ when bundle $y \in Y$ of public goods is implemented and he has to contribute the amount t_i towards its financing.

Assumption 2.1 *The cost function* $c : Y \rightarrow \mathbb{R}_+$ *is continuous and non-decreasing.*

Assumption 2.2 *For each* $i \in N$, $b_i : Y \rightarrow \mathbb{R}$ *is a continuous, non-decreasing function satisfying* $\lim sup_{||y|| \rightarrow \infty} b_i(y)/c(y) = 0$.

Given $N = \{1, ..., n\}$, $b_i(y) - t_i$ is interpreted as the net benefit agent $i \in N$ obtains when he has to contribute t_i units of his private good in order to enjoy the bundle y of public goods. Thus, $b_i(y) - t_i$ is the net contribution that agent $i \in N$ makes towards the net surplus $b_1(y) + ... + b_n(y) - c(y)$ that the society obtains from the consumption of a bundle $y \in Y$ of public goods. Since the utilities of the agents are always quasi-linear, it is identified the utility function with its non-linear part.

Let $b = (b_1, ..., b_n)$ be a vector of utilities, $y \in Y$ be a bundle of public goods and $t = (t_1, \ldots, t_n) \in X^n$ a vector of contributions, I will use the following notation $u_b(y;t) = (b_1(y) - t_1, ..., b_n(y) - t_n)$. The utility profile resulting from $b = (b_1, ..., b_i, ..., b_n)$ when utility function b_i is replaced by a new utility function v_i is denoted by $(b_{-i}, v_i) = (b_1, ..., b_{i-1}, v_i, b_{i+1}, ..., b_n)$. Given two different utility profiles b and v defined on Y, denote $b \geq v$ whenever $b(y) \geq v(y)$ for every $y \in Y$.

An economy $e = (N_0; b, c)$ is defined by a finite set of agents $N_0 \subset N = \{1, ..., n\}$, a utility profile b and a cost function c. Let E be the set of economies satisfying 2.1 and 2.2 and such that if $e \in E$ implies there is $y \in Y$ with $\sum_{i \in N_0}(b_i(y) - b_i(0)) - c(y) \geq 0$. The last condition implies that the nine solutions are well defined.

An allocation $(y;t) = (y; t_1, ., t_{n_0}) \in Y \times X^{n_0}$ is feasible in economy $e = (N_0; b, c) \in E$ whenever $c(y) \leq \sum_{i=1}^{n_0} t_i$. $F(e) = \{(y; t_1, ., t_{n_0}) : c(y) \leq \sum_{i=1}^{n_0} t_i\}$ is the set of all feasible allocations and $U(e) = \{u_b(y;t) : (y;t) \in F(e)\}$ denotes the set of feasible utilities. Denote by $PO(e)$ the set of Pareto optimal allocations, that is, those feasible allocations $(y;t) \in F(e)$ for which if $u_b(y;t) < u_b(z;r)$ then $(z;r)$ is not feasible. And $UPO(e) = \{u_b(y;t) : (y;t) \in PO(e)\}$ the set of vectors of utilities provided by the Pareto optimal allocations of this economy.

Now, for each economy $e \in E$, I define $V(e) = \text{Max}_{y \in Y} \sum_{i=1}^{n_0} b_i(y) - c(y)$ as the total surplus to share among the agents. Since agents have quasi-linear preferences, $UPO(e) = \{(a_1, ..., a_{n_0}) \in \mathbb{R}^{n_0} : \sum_{i \in N} a_i = V(e)\}$. And let $\bar{y} \in \arg max_{y \in Y} \sum_{i=1}^{n_0} b_i(y) - c(y)$ denote an optimal bundle of public goods.

A solution for the problem of the optimal provision of public goods is a function $S : E \rightarrow Y \times X$ which assigns to every economy $e \in E$ a feasible allocation $S(e) = (y;t)$ such that $c(y) = \sum_{i=1}^{n_0} t_i$. Our feasible solution does not waste resources.

Axiom 2.3 *(PO): A solution S satisfies Pareto optimality if $S(e) \in PO(e)$ for each $e \in E$.*

In order to analyze how changes in preferences and technology influence in the solution, two different axioms derived from the independence of irrelevant alternatives axiom are stated.

Axiom 2.4 *(IPC): A solution S satisfies Independence of preference contraction whenever, given $e = (b,c)$ and $e' = (v,c)$ such that $b_i \geq v_i$ for each $i \in N_0$ then $u_v(S(e)) = u_b(S(e))$ implies that $S(e') = S(e)$.*

Axiom 2.5 *(ITC): A solution S satisfies Independence of Technological Contraction whenever, given $e = (b,c)$, $e' = (v,c')$ such that $c' \leq c$, then $S(e) = (y;t)$ with $c'(y) = c(y)$ implies $S(e') = S(e)$.*

Another axiom standard in the classical bargainig theory is equal translation invariance. With this property the solution is invariant under equal translation of the inactivity point.

Axiom 2.6 *(ETI): A solution S satisfies equal translation invariance if given an economy $e = (N; b, c) \in E^n$ and a real number $k \leq V(e)/n$, then $u_{b-k}(S(N; b-k, c)) = u_b(S(N; b, c)) - k$, where $b - k$ denotes the utility profile $(b_1(y) - k, ..., b_n(y) - k)$.*

Since fixed costs are allowed, the next property states how to share them. The independence of cost function's zero axiom (ICFZ) says that any fixed costs ($c^*(0) > 0$) or any subsidies ($c^*(0) < 0$) are shared equally among the agents.

Axiom 2.7 *(ICFZ): A solution S satisfies independence of cost function's zero whenever given an economy $e = (N; b, c) \in E^n$ with $S(e) = (\bar{y}; t)$ and a scalar $\beta \leq n \min_{i \in N} b_i(\bar{y})$ and $\beta \leq V(e)$, such that $c^* = c + \beta$, and $h = (\beta/n, ..., \beta/n)$ then $u_b(S(e)) - h = u_b(S(N; b, c^*))$.*

In order to characterize the welfare egalitarian solution, first it is imposed that a solution should provide utilities bigger than the inactivity point and this property is called weak individual rationality

Axiom 2.8 *(WIR): A solution S satisfies weak individual rationality if $u_b(S(e)) \geq 0$ for each economy $e = (N; b, c) \in E^n$ with $c(0) = 0$ and $b(0) = 0$.*

Now, the formal definition of the Nash or welfare egalitarian solution is stated.

Definition 2.9 *The welfare egalitarian solution is defined as follows: Given an economy $e \in E^n$, $WE(e) = \{(y;t) \in PO(e) : \forall i \in N \quad b_i(y) - t_i = b_1(y) - t_1\}$.*

A new characterization of the welfare egalitarian solution is provided, without using the solidarity axiom (Ginés and Marhuenda (2000)).

Theorem 2.10 *A solution S satisfies PO, IPC, ITC, ETI, ICFZ and WIR axioms if and only if $S(e) \in WE(e)$.*

Because sometimes it is important to take into account the individual contributions to the surplus, an axiom called No Private Transfers (NPT) is introduced. This axiom reflects the idea that the individual contributions to the surplus matter and cannot be sumarized by total surplus. NPT axiom was used in Moulin (1987) in order to characterize the egalitarian equivalent solution in the case of one public good. Although private transfers are allowed, nobody will receive in the solution more utility than its contribution to the surplus. This axiom provides a bound on the possible claims.

Axiom 2.11 *(NPT): A mechanism S satisfies the axiom of No Private Transfers if for every $e \in E$, $S(e) = (y; t_1, ..., t_n)$ is such that $t_i \geq 0$ for each $i = 1, ..., n$.*

Clearly this axioms is not satisfied by the welfare egalitarian solutions. If NPT axiom is added to the characterization then some other axiom should be dropped. If WIR axiom is the one dropped, the characterization of the Equal-loss solution (EL) appears. If the chosen one is ICFZ axiom, the characterization of the lexicographic extension of the welfare egalitarian solution (LWe) is obtained. . If ETI axiom is picked, it is obtained the characterization of the rational equal-loss solution (REL). Similarly replacing axiom ITC we found the lexicographic extension of the welfare egalitarian preserving fixed costs (LWE0). The solution that satisfies the seven properties except IPC is the rational equal loss preserving the disagreement point (RELd). Finally all the properties except optimaly characterize the disagreement point based on inactivity and equal share of the fixed costs.

An axiom called separability (Sp) is introduced to generalize some characterizations from two agents to a generic n. Under Sp it is possible to reduce the problem from n agent to $n-1$ agents under the condition that agent n, with the lower valuation of the public good should always end up with utility level $u_n(S(e))$.

Axiom 2.12 *(Sp): A solution S satisfies the separability axiom if given an economy $e = (N_0; b, c) \in E$ with $N_0 = \{1, \ldots, n_0\}$ $b_1 \geq b_2 \geq \ldots \geq b_{n_0-1} \geq b_{n_0}$, then $u_{b_{-(n_0)}}(S(N_0; b, c)) = u_b(N_1; b_{-(n_0)}, c')$ where $c'(y) = c(y) - b_{n_0}(y) + u_{n_0}(S(e))$ for all $y \in Y$ and $N_1 = \{1, ..., n_0 - 1\}$.*

The equal-loss solution or non-envy solution, in the present setting, means that all the agents pay the same amount of private good in order to provide the public goods. In the Axiomatic Bargaining literature it corresponds to the equal-loss solution from the claims point.

Definition 2.13 *Given an economy $e \in E$, the non-envy or equal-loss solution consists of all Pareto efficient allocations that prescribe equal contributions to the cost of the public goods. And then, $EL(e) = \{(y; t) \in PO(e) : t_i = t_1$ for all $i = 1, \ldots, n\}$*

The next result compares both solutions, welfare egalitarian and equal-loss solution.

Theorem 2.14 *A solution S satisfies*

a) $PO, IPC, ITC, ETI, ICFZ$ and WIR axioms if and only if $S(e) \in WE(e)$.

b) $PO, IPC, ITC, ETI, ICFZ$ and NPT axioms if and only if $S(e) \in EL(e)$

It is remarkable that the axiom that distinguishes welfarism from liberalism (equal share of the costs) is based in assuring a minimum level above inactivity while the solution equal-loss changes to an axiom that only avoids direct compensations in terms of private good. The normative distinction between both solution yields to assure a minimum level of utility in the egalitarian rule while the liberal rule avoids direct subsidies.

Usually, in the axiomatic bargaining literature, one should be sure of which is the roll of each axiom in the characterization and if it exists a solution satisfying all the axioms except the one analyzed. In this paper we not only do that but we also characterize these alternative solutions with the axioms presented in the above theorem. Clearly, the seven axioms postulated are incompatible but eliminating one each time we characterize a different solution. The definition of the other five solutions and the characterization results are stated properly.

The lexicographic extension of the Welfare egalitarian solution coincides with the welfare egalitarian solution in the case last solution is compatible with NPT axiom, alternatively some agents do not pay any money and the rest pay in order to attain the same utility. Agents are equalized as much as the axiom NPT allows.

Definition 2.15 *Given an economy $e \in E$, the lexicographic extension of the welfare egalitarian solution is defined by*

$$LWE(e) = \arg max_{\{(y;t) \in PO(e) \cap NPT(e)\}} \{\prod_{i=1}^{n} (b_i(y) - t_i)\}^1$$

If fixed costs should be assigned apart from the variable costs, the next solution extents lexicographic extension of the welfare egalitarian preserving equal share of the fixed costs.

Definition 2.16 *Given an economy $e = (N_0; b, c) \in E$, take $c' = c - c(0)$, the lexicographic extension of the welfare egalitarian solution preserving fixed costs is defined by $LWE0(e) = \{(y; t + c(0)/n) \in PO(e) : (y, t) \in LWE(c')\}$*

Clearly if $c(0) = 0$ then $LWE0 = LWE$.

[1]Since the utilities are quasi-linear the set $PO(e) \cap NPT(e) = \{(y;t) \in PO(e)/t_i \geq 0 for i \in N\}$ is always non-empty.

To avoid situations where the equal-loss solution prescribes a result where contributions to the cost are bigger than the valuation of the public goods,it appears the rational equal-loss.

Definition 2.17 *Let* $e = (N; b, c) \in E$ *be an economy,* $(\bar{y}; t_1, ..., t_n) \in REL(e)$ *if it is defined in the following way: Let* $\bar{y}$ *be a Pareto optimal bundle of public goods. Suppose, without loss of generality, that* $b_1(\bar{y}) \leq b_2(\bar{y}) \leq \ldots \leq b_n(\bar{y})$*.*
Denote by $j = \min\{i \in N : b_i(\bar{y}) \geq (c(\bar{y}) - \sum_{h=1}^{j-1} b_h(\bar{y}))/(n - j + 1)\}$
If $j = 1$ *then* $t_i = c(\bar{y})/n$ *for all* $i = 1, \ldots, n$*. (That is equal share of the cost).*
If $j \geq 2$*, for* $1 \leq p \leq j - 1$ *define* $t_p = b_p(\bar{y})$ *and for* $n \geq p \geq j$ *assign* $t_p = (c(\bar{y}) - \sum_{h=1}^{j-1} b_h(\bar{y}))/(n - j + 1)$*.*

Every agent pays as much as its valuation of the public goods allows. The solution tends to equalize the contributions of the agents to the cost of the public goods.

If we think that fixed costs and inactivity are important and should not be trade or put together with the surplus. That is, if we consider the status quo paying the equal share of the cost and everybody owns its inactivity subsidies a new solution can be defined.

Definition 2.18 *Let $e = (N; b, c) \in E$ be an economy, the rational equal-loss preserving the equal share of the fixed costs and the consumption of 0 is defined $RELd(e) = \{(y; t + c(0)/n) : (y; t) \in REL(N; b - b(0), c - c(0))\}$.*

Every agent pays as much as its net valuation of the public goods allows. The solution tends to equalize the contributions of the agents to the net cost of the public goods (fixed costs equally shared).

Finally the solution that each time assigns the disagreement point defined as inactivity and equal share of the fixed costs.

Definition 2.19 *Let $e = (N; b, c) \in E$ be an economy, the disagreement solution is such that $D(e) = (b_1(0) - c(0)/n, \ldots, b_n(0) - c(0)/n)$.*

Before presenting the main result just add that in order to generalize the result from two agents on, we need a weaker version of ITC and IPC that preserve cost of zero and $b(0)$.

Axiom 2.20 *(IPC0): A solution S satisfies Independence of preference contraction preserving $b(0)$ whenever, given $e = (b, c)$ and $e' = (v, c)$ such that $b_i \geq v_i$ and $b_i(0) = v_i(0)$ for each $i \in N_0$ then $u_v(S(e)) = u_b(S(e))$ implies that $S(e') = S(e)$.*

Axiom 2.21 *(ITC0): A solution S satisfies Independence of Technological Contraction preserving fixed costs whenever, given $e = (b, c)$, $e' = (v, c')$ such that $c' \leq c$, then $S(e) = (y; t)$ with $c'(y) = c(y)$ and $c'(0) = c(0)$ implies $S(e') = S(e)$.*

Now, the main result is stated.

Theorem 2.22 *A solution S satisfies*

1)$PO, IPC, ITC, ETI, ICFZ$ and WIR axioms if and only if $S(e) \in WE(e)$.

2)$PO, IPC, ITC, ETI, ICFZ$ and NPT axioms if and only if $S(e) \in EL(e)$.

3)$PO, IPC, ITC, ETI, WIR, NPT$ and SP axioms if and only if $S(e) \in LWE(e)$.

4)$PO, IPC, ITC, ICFZ, WIR, NPT$ and SP axioms if and only if $S(e) \in REL(e)$.

5)$PO, IPC, ITC0, ETI, ICFZ, WIR, NPT$ and SP axioms if and only if $S(e) \in LWE0(e)$.

6)$PO, IPC0, ITC, ETI, ICFZ, WIR, NPT$ and SP axioms if and only if $S(e) \in RELd(e)$.

7)$IPC, ITC, ETI, ICFZ, WIR$ and NPT axioms if and only if $S(e) \in D(e)$.

3 Appendix

The following lemma will be used in all the following results.

Lemma 3.1 *Given $e = (N; b, c) \in E^n$ an economy and S a solution satisfying PO, IPC and ITC, there is a function $\bar{v} : Y \to \mathbb{R}$, scalars $(\alpha_1, ..., \alpha_n) \in \mathbb{R}^m_+$ and a cost function c' such that if $v = (v_1, ..., v_n)$ is a utility profile with $v_i = \alpha_i \bar{v}$ the economy $e' = (N; v, c')$ satisfies $S(e) = S(e')$*

Proof

Let $e = (N; b, c) \in E^n$ an economy and S a solution satisfying PO, IPC and ITC and let $S(e) = (\bar{y}; t)$.

First define a function $c^*(y) = \bar{\lambda}(y)c(\bar{y})$ where $\bar{\lambda}(y) \in \arg max\{\lambda \in \mathbb{R}_+ : \lambda\bar{y} \leq y\}$. Now define a cost function $c'(y) = (c \vee c^*)(y) = \text{Max}\{c(y), c^*(y)\}$, which satisfies all the conditions and the function $\bar{v}(y) = 1$ if $y \geq \bar{y}$. Otherwise $\bar{v}(y) = \bar{\lambda}(y)$ where $\bar{\lambda}(y)$ is defined as above.

Also define for each $i \in N$, $\alpha_i = b_i(\bar{y})$ and $v_i(y) = \alpha_i \bar{v}(y)$. Let $v = (v_1, ..., v_n)$. Denote by $(b \wedge v)$ a preference profile such that $(b \wedge v)_i(y) = \min\{b_i(y), v_i(y)\}$. Then $V(N; (b \wedge v), c') \leq V(e)$ because $(b \wedge v) \leq b$ and $c' \geq c$. The fact that $c'(\bar{y}) = c(\bar{y})$, $(b \wedge v)_i(\bar{y}) = b_i(\bar{y})$ for all $i \in N$ clearly concludes that $V(N;(b \wedge v), c') = V(e)$. Applying PO IPC and ITC it is derived $S(N;(b \wedge v), c') = S(N; b, c)$.

Moreover also by applying IPC, ITC and PO $S(N;(b \wedge v), c') = S(N; v, c')$. And, as a result, $S(N; v, c') = S(N; b, c)$.

$\square$

In all the proofs it is easy to check that the different solutions satisfy the axioms which characterize them and, therefore, it is omitted.

Proof of Theorem 2.10

Let $e = (N_0; b, c) \in E$ an economy and S a solution satisfying PO, IPC, ITC, WIR, ICFZ, ETI . Now applying lemma 3.1 $S(N_0; v, c') = S(e)$. Let $V_\epsilon = V(e) - \epsilon$, with $\epsilon \leq V(e)$ a small real number. Defining two new economies $e'_\epsilon = (N_0; v, c' + V_\epsilon)$ and $e''_\epsilon = (N_0; v, c'')$ with $s = \lambda \bar{y}$ defined by $\sum_{i \in N_0} v_i(s) = (c' + V_\epsilon)(s)$ and $c''(y) = \min\{\sum_{i \in N_0} v_i(y), c' + V_\epsilon\}$. Clearly $c'' \leq c' + V_\epsilon$ and by construction $c''(\bar{y}) = (c' + V_\epsilon)(\bar{y})$, (it is only needed that the solution in economy with c'' cost exactly the same as in $c' + V_\epsilon$, since WIR implies positive utilities and surplus) applying ITC and PO, the solution of both economies coincide.

Now, $u_v(S(N_0; v, c')) = u_{v+V_\epsilon/n}(S(N_0; v + V_\epsilon/n, c') - V_\epsilon/n$ by ETI axiom. By ICFZ axiom $u_{v+V_\epsilon/n}(S(N_0; v + V_\epsilon/n, c') - V_\epsilon/n = u_{v+V_\epsilon/n}(S(N_0; v + V_\epsilon/n, c' + V_\epsilon)$. and as a first step with ETI axiom $u_{v+V_\epsilon/n}(S(N_0; v + V_\epsilon/n, c' + V_\epsilon) = u_v(S(N_0; v, c' + V_\epsilon)) + V_\epsilon/n$ and as obtained above equal to $u_v(S(N_0; v, c'') + V_\epsilon/n$.

To sum up $u_v(S(v, c')) = u_v(S(N_0; v, c'') + V_\epsilon/n$. By WIR and NPT the solution on $(N_0; v, c'')$ is bounded (the utility that each agent can attain in the solution) by 0 and $V(N_0; v, c'') = \epsilon$. Then $0 \leq u_v(S(v, c')) - V_\epsilon \leq \epsilon$ for any ϵ small enough. Taking limits when ϵ tends to 0 we obtain that $u_b(S(e)) = V/n$ □

Proof of Theorem 2.22

2) Equal-loss solution

Let $e = (N; b, c)$ an economy and S a solution satisfying PO, IPC, ITC, NPT, ETI and ICFZ. Using lemma 3.1, $S(N; v, c') = S(e) = (\bar{y}; t)$. Denote $C = c'(\bar{y})$ Now, $u_{v+C/n}(S(N_0; v + C/n, c')) = u_v(S(e')) + C/n$ by ETI axiom. On the other hand applying lemma 3.1 to the economy $(N_0; v + C/n, c')$ a new preference profile v' is obtained such that $v'(0) = 0$ and a cost function c''

such that $S(N_0; v', c'') = S(N_0; v + C/n, c')$. Define $c'''(y) = \max\{c''(y), C\}$. By IPC and ITC axiom, $S(N_0; v', c''') = S(N_0; v', c'')$. Apply ICFZ axiom to the economy $(N_0; v', c''')$ with $\beta = C$, it is derived that $u_{v'}(S(N_0; v', c''' - C)) - C/n = u_{v'}(S(N_0; v', c'''))$ Since $\bar{y}$ is still the bundle which maximizes the surplus of $(N_0; v', c''' - C)$ and its cost is zero, by NPT axiom we derive that in this economy all payoff are zero. This concludes the proof once all the equalities are put together.

3) Lexicographic extension of the welfare egalitarian solution

Clearly, in the case of two agents, there are two possibilities for the lexicographic extension of the welfare egalitarian solution. If the welfare egalitarian solution exists and prescribes no transfers, both coincide. If that is not the case, one of the agents pays the entire cost of the efficient bundle of public goods. This can be generalized to n agents.

Let $e = (N_0; b, c) \in E$ an economy and S a solution satisfying PO, ITC, IPC, WIR, NPT, ETI and Sp. First of all, applying lemma 3.1, $S(N_0; v, c') = S(e)$. Suppose, without loss of generality, that agent n is the agent with the lower valuation of the public goods. In our case because all utility functions in v are comparable by construction, this corresponds to the lower α_i.

If $\alpha_n \geq V(e)/n$ then $u_{v-V(e)}(S(N_0; v - V(e)/n, c')) = u_v(S(e')) - V(e')/n$ by ETI. Now, define $v'(y) = \max\{v(y) - V(e)/n, 0\}$, and $e'' = (N_0; v', c')$ a new economy. By construction, $V(e'') = 0$ and $v(0) = 0$. Then by WIR it is obtained that $u_{v'}(S(e'')) = 0$. Applying IPC and ITC to $(N_0; v - V(e), c')$ and e'' it is derived that $S(N_0; v - V(e)/n, c') = S(e'')$. Finally, $0 = u_{v'}(S(e'')) = u_{v-V(e)}(S(N_0; v - V(e)/n, c')) = u_v(S(e')) - V(e')/n$. And clearly, $u_b(S(e)) = u_v(S(e')) = V(e')/n$

In other case, denote $k = v_n(\bar{y})$. Now apply ETI to this economy $e' = (N_0; v, c')$ with k. Then $u_{v-k}(S(N_0; v - k, c')) = u_v(S(e')) - k$. Next it is defined a preference profile v' similarly to the definition of v, such that $v'_i(0) = 0$ and $v'_i(\bar{y}) = v_i(\bar{y}) - k$ for each $i \in N_0$. Applying PO, IPC and ITC $S(N_0; v', c') = S(N_0; v - k, c')$. And, in particular, $v'_n(\bar{y}) = 0$. By NPT and WIR $u_n(S(N_0; , v', c')) = 0$. Then $u_n(S(N_0; v - k, c')) = 0$ and $u_n(S(N_0; v, c')) = u_n(S(N_0; v - k, c')) + k = v_n(\bar{y})$. Let $N_1 = \{1, ..., n - 1\}$. Applying consistency reduce the problem from n agents to $n - 1$. The new economy is $e'' = (N_1; v, c'')$ where $c''(y) = c'(y) - v_n(y) + u_n(S(N_0; v, c'))$ Now repeat the process from the begining. All this process continues until there is an economy with equal division of the surplus or it is reduced to an economy with two agents where equal division does not meet the claims, that is, it implies violation of NPT axiom. In this case, applying ETI as above, the solution attained prescribes that the lower agent does not pay any private good. This solution corresponds to the lexicographic extension of the Nash solution compatible with NPT.

4)Rational equal-loss solution

Let $e = (N; b, c) \in E^{n+1}$ be an economy and a solution S satisfying PO, WIIA, NPT, WIR, ICFZ and Sp. Firstly,by PO, IPC and ITC, there is $e' = (N; v, c')$ as in lemma 3.1. Now it is possible to distinguish two cases:

I) if $(c(\bar{y})/(n+1) \leq \min_{i \in N} v_i(\bar{y}))$, that is, if there is a equal-loss solution compatible with the WIR, define

$$\hat{c}(y) = \begin{cases} \hat{\lambda}(y)c'(\bar{y}) & \text{if } y \geq \bar{y} \\ c'(\bar{y}) & \text{otherwise} \end{cases}$$

where $\bar{\lambda} \in \arg max\{\lambda \in \mathbb{R}_+ : \lambda\bar{y} \leq y\}$.

Now $c'' = \hat{c} \vee c'$. Clearly $c' \leq c''$ and by IPC, ITC and PO $S(e'') = S(e')$ where $e'' = (N; v, c'')$. But $e^* = (N; v, c^*)$ with $c^* = c'' - c'(\bar{y})$, satisfies by NPT and PO that $S(e^*) = (\bar{y}; 0)$ and by ICFZ $S(e'') = (\bar{y}; r)$ where $r_1 = ... = r_{n+1} = c'(\bar{y})/(n+1)$. And, finally, $S(e) = S(e') = S(e'')$.

II) Otherwise, assume without loss of generality that $v_1(\bar{y}) \geq v_2(\bar{y}) \geq ... \geq v_{n+1}(\bar{y})$. Pick $y^* = \lambda^*\bar{y}$ with $\lambda^* \leq 1$ such that $v_{n+1}(\bar{y}) = c'(y^*)/(n+1)$.

Define a cost function

$$\hat{c}(y) = \begin{cases} c'(y^*) & \text{if } 0 \leq \lambda \leq \lambda^* \\ \dfrac{\lambda - \lambda^*}{1 - \lambda^*} c(\bar{y}) + \dfrac{1 - \lambda}{1 - \lambda^*} c(y^*) & \text{if } \lambda^* \leq \lambda \leq 1 \\ \lambda c(\bar{y}) & \text{if } \lambda \geq 1 \end{cases}$$

In each case $\lambda \in \arg max\{\lambda_1 \in \mathbb{R}_+ : \lambda_1 \bar{y} \leq y\}$. Define $c'' = c' \vee \hat{c}$, then $c' \leq c''$ and $c''(\bar{y}) = c'(\bar{y})$ by IPC and ITC and PO $S(e'') = S(e')$, where $e'' = (N; v, c'')$. But if a new economy $e^* = (N; v, c^*)$ is constructed with $c^* = c'' - c'(y^*)$, applying ICFZ, if $z = (c'(y^*)/(n+1), ..., c'(y^*)/(n+1)) \in \mathbb{R}^{n+1}$ then $u_b(S(e^*)) - z = u_b(S(e''))$. In particular, $u_{n+1}(S(e'')) = u_{n+1}(S(e^*)) - c'(y^*)/(n+1)$. Moreover by NPT $u_{n+1}(S(e^*)) \leq v_{n+1}(\bar{y}) = c'(y^*)/(n+1)$, then it is possible to conclude by WIR that $u_{n+1}(S(e'')) = 0$. Furthermore, $u_{n+1}(S(e')) = 0$. Applying Sp to economy e' restrict the problem to n agents and with $c''(y) = c'(y) - v_{n+1}(y)$. Now, repeat, if there is the equal-loss consistent with WIR apply this case if not reduce the problem as before until $n = 2$ in which this process uniquely determines the rational equal-loss. □

5) Lexicographic extension of the welfare egalitarian solution preserving fixed costs

For the case of two agents is exactly the same proof only having into account to share equally the costs firstly and with a new technology with fixed costs 0. For more agents repeat the proof for LWE and in order to apply separability (SP) it is needed ITC0 (changes in technology but also preserving the cost of 0) and lemma 3.1 apply since our technology has zero fixed costs.

6)Rational equal-loss solution from disagreement

Again by ETI and IFCZ axioms we can transform our economy in a new one with zero fixed costs and in which b(0)=0. Now applying the proof of REL and with IPC0 axiom that applies when separability is needed.

7) Disagreement solution

Let $e = (N_0; b, c) \in E$ an economy and S a solution satisfying ITC, IPC, ICFZ, WIR, NPT and ETI. Let $S(e) = (y; t)$, such that $y \neq 0$. Since the solution is not disagreement we can apply the proof of equal division of the surplus without optimality and obtain that the surplus generated by y is divided equally, on the other hand if we follow the proof of equal division of the cost we obtain equal share of the cost of y. But both to the same time are incompatible. The only remaining possibility is that $y = 0$. Applying ICFZ we share the fixed costs and we have a new economy with zero fixed costs but again the solution is $y = 0$ and now by NPT axiom everybody should pay zero since total cost of zero public goods is zero.

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