The SD-prenucleolus for TU games

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June 7, 2013

Abstract

We introduce and characterize a new solution concept for TU games. The new solution is called SD-prenucleolus and is a lexicographic value although is not a weighted prenucleolus. The SD-prenucleolus satisfies several desirable properties. Itis the only known solution that satisfies core stability, strong aggregate monotonicity and null player out property in the class of balanced games. It is the only known continuous core concept that satisfies monotonicity for games with veto players.

Keywords: TU games, prenucleolus, monotonicity.

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JEL classification: C71, C72.

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1 Introduction

This paper introduces and characterizes a new solution concept for coalitional games with transferable utility (TU games). The new solution is a lexicographic value, so its name (SD-prenucleolus) reflects its connection with the classic, widely-analyzed prenucleolus. The solution also has a relationship with the family of weighted prenucleoli although it is not a member of this family. In particular, the new solution shares some similarities with the per capita prenucleolus. The SD-prenucleolus satisfies several desirable properties. It is the only known solution that satisfies core stability, the Null Player Out property and strong aggregate monotonicity in the class of balanced games. It is the only known continuous core concept satisfying monotonicity in the class of convex games and in the class of veto balanced games.

Given a TU game the prenucleolus, defined as a lexicographic value, selects the vector of excesses of coalitions that lexicographically dominates any other vector of excesses of coalitions. When this vector is selected its associated allocation is automatically selected and this proves to be the prenucleolus of the game. When the excesses of coalitions are weighted by using a system of weights for the size of the coalitions this procedure will generate the different weighted prenucleoli. In the per capita prenucleolus excesses are divided by the size (cardinality) of the coalition.

In this paper we propose a different way of computing the excesses of coalitions given an allocation[1]. Once the vector of excesses is computed for any allocation the SD-prenucleolus arises as the lexicographic optimal value in the set of vectors of excesses of coalitions. We characterize the solution in terms of balanced collections of coalitions, the equivalent of Kohlberg's classic theorem of the prenucleolus (Kohlberg, 1971). This characterization is the main tool for checking whether an allocation is the SD-prenucleolus of the game. Section 5 provides a simple formula for computing the SD-prenucleolus of monotonic games with veto players. In this class, the SD-prenucleolus satisfies coalitional monotonicity.

[1] In Section 3 we explain and interpret the new vector of satisfactions.

2 Preliminaries

2.1 TU Games

A *cooperative n-person game in characteristic function form* is a pair (N, v), where N is a finite set of n elements and $v : 2^N \to \mathbb{R}$ is a real-valued function in the family 2^N of all subsets of N with $v(\emptyset) = 0$. Elements of N are called *players* and the real valued function v the characteristic function of the game. Any subset S of N is called a *coalition*. Singletons are coalitions that contain only one player. A game is *monotonic* if whenever $T \subset S$ then $v(T) \leq v(S)$. The number of players in S is denoted by $|S|$. Given $S \subset N$ we denote by $N \backslash S$ the set of players of N that are not in S. A distribution of $v(N)$ among the players, an allocation, is a real-valued vector $x \in \mathbb{R}^N$ where x_i is the payoff assigned by x to player i. A distribution satisfying $\sum_{i \in N} x_i = v(N)$ is called an *efficient allocation* and the set of efficient allocations is denoted by $X(v)$. We denote $\sum_{i \in S} x_i$ by $x(S)$. The core of a game is the set of imputations that cannot be blocked by any coalition, i.e.

$$C(N, v) = \{x \in X(N, v) : x(S) \geq v(S) \text{ for all } S \subset N\}.$$

It has been shown that a game with a non-empty core is balanced[2] and therefore games with non-empty core are called balanced games. *Player i is a veto player if* $v(S) = 0$ for all S where player i is not present. A balanced game with at least one veto player is called a veto balanced game. We denote by Γ_B the class of balanced games and by Γ_{VB} the class of veto balanced games.

A solution φ on a class of games Γ_0 is a correspondence that associates a set $\varphi(N, v)$ in $\mathbb{R}^N$ with each game (N, v) in Γ_0 such that $x(N) \leq v(N)$ for all $x \in \varphi(N, v)$. This solution is *efficient* if this inequality holds with equality. The solution is *single-valued* if the set contains a *single* element for each game in the class.

Given $x \in \mathbb{R}^N$ the *excess of a coalition S with respect to x* in a game v is defined as $e(S, x) := v(S) - x(S)$. Let $\theta(x)$ be the vector of all excesses at x

[2]See Peleg and Südholter (2007).

arranged in non-increasing order. The weak lexicographic order $\preceq_L$ between two vectors x and y is defined by $x \prec_L y$ if there exists an index k such that $x_l = y_l$ for all $l < k$ and $x_k < y_k$ or $x = y$.

Schmeidler (1969) introduced the *prenucleolus* of a game v, denoted by $PN(v)$, as the unique allocation that lexicographically minimizes the vector of non increasingly ordered excesses over the set of allocations. In formula:

$$\{PN(N,v)\} = \{x \in X(N,v) \, | \theta(x) \preceq_L \theta(y) \text{ for all } y \in X(N,v)\} .$$

For any game v the prenucleolus is a single-valued solution, is contained in the prekernel and lies in the core provided that the core is non-empty.

The per capita prenucleolus (Groote, 1970) is defined analogously by using the concept of per capita excess instead of excess. Given S and x the per capita excess of S at x is

$$e^{pc}(S,x) := \frac{v(S) - x(S)}{|S|}$$

Other weighted prenucleoli can be defined in a similar way whenever a weighted excess function is defined. The same solution concepts can be analogously defined using the notion of satisfaction instead of excess. Given $x \in \mathbb{R}^N$ the *excess of a coalition* S *with respect to* x in a game (N,v) is defined as $f(S,x) := x(S) - v(S)$. In this paper we use the notion of satisfaction in defining the new solution.

2.2 Properties

Some convenient and well-known properties of a solution concept φ on Γ_0 are the following.

- φ satisfies **anonymity** if for each (N,v) in Γ_0 and each bijective mapping $\tau : N \longrightarrow N$ such that $(N, \tau v)$ in $\Gamma\Gamma_0$ it holds that $\varphi(N,\tau v) = \tau(\varphi(N,v))$ (where $\tau v(\tau T) = v(T)$, $\tau x_{\tau(j)} = x_j$ $(x \in R^N, j \in N, T \subseteq N)$). In this case v and τv are equivalent games.

- φ satisfies **equal treatment property (ETP)** if for each (N, v) in Γ_0 and for every $x \in \varphi(N, v)$ interchangeable players i, j are treated equally, i.e. $x_i = x_j$. Here, i and j are interchangeable if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \backslash \{i, j\}$.

- φ satisfies **desirability** if for each (N, v) in Γ_0 and for every $x \in \varphi(N, v)$, $x_i \geq x_j$ if i is more desirable than j in v. We say that in a game v a player i is more desirable than a player j if $v(S \cup i) \geq v(S \cup j)$ for all $S \subset N \backslash \{i, j\}$.

- φ satisfies **covariance** if $(N, v), (N, \alpha v + \beta) \in \Gamma_0$ for any $\alpha > 0$ and any $\beta \in R^N$ implies that $\varphi(N, \alpha v + \beta) = \alpha \varphi(N, v) + \beta$ holds.

- φ satisfies **null player property** if for each (N, v) in Γ_0 and for every $x \in \varphi(N, v)$ null players receive 0. Here, a player is a null player if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \backslash \{i\}$.

- φ satisfies **null player out property (NPO)** if for each (N, v) in Γ_0 and for every $x \in \varphi(N, v)$ it holds that $(x_i)_{i \in N \backslash T} \in \varphi(N \backslash T, v)$. Here T is the set of null players in game (N, v).

The NPO property implies the Null Player property. Both properties try to capture the idea that null players should not influence the allocations selected by a solution. However, only the NPO property captures entirely this idea. If the payoff of some players (different than the null player) can be affected for the presence of null players is difficult to conclude that null players are irrelevant players.

- φ satisfies **core stability** if it selects core allocations whenever the game is balanced.

Note that desirability implies ETP. The following two properties are defined for single-valued solutions.

- φ satisfies **coalitional monotonicity**: if for all $v, w \in \Gamma_0$, if for all $S \neq T$, $v(S) = w(S)$ and $v(T) < w(T)$, then for all $i \in T$, $\varphi_i(v) \leq \varphi_i(w)$.

- φ satisfies **aggregate monotonicity**: if for all $v, w \in \Gamma_0$, if for all $S \neq N$, $v(S) = w(S)$ and $v(N) < w(N)$, then for all $i, j \in N$, $\varphi_i(w) - \varphi_i(v) \geq 0$.

- φ satisfies **strong aggregate monotonicity**: if for all $v, w \in \Gamma_0$, if for all $S \neq N$, $v(S) = w(S)$ and $v(N) < w(N)$, then for all $i, j \in N$, $\varphi_i(w) - \varphi_i(v) = \varphi_j(w) - \varphi_j(v) \geq 0$.

Young (1985) proves that no solution satisfies coalitional monotonicity and core stability. However, there are solutions satisfying core stability and the strong aggregate monotonicity. Meggido (1974) proves that the nucleolus does not satisfy aggregate monotonicity. Clearly, strong aggregate monotonicity implies aggregate monotonicity.

3 A vector of satisfactions

3.1 Introduction

The prenucleolus is a lexicographic value that selects a maximal element in the set of vectors of excesses of coalitions. The solution does not change if the vector of satisfaction is taken instead of vectors of excesses. In the definition of the new lexicographic value we use the notion of satisfaction instead of excess. The main change with respect to the classic prenucleolus, the per capita prenucleolus and any other weighted prenucleolus lies in how the vector of satisfactions is defined. The main idea of the new vector of satisfactions is to identify how a coalition divides its surplus (the difference between the payoff received by the coalition and the worth of the coalition) among its members. We also require this distribution to keep some consistency. We argue that the classic prenucleolus has no answer to this question while the per capita prenucleolus provides an unsatisfactory answer.

Before introducing the new vector of satisfactions we illustrate by means of an example the two ideas that support the new solution.

Consider the following 4-player game[3] (N, v):

$$v(S) = \begin{cases} 1 & \text{if } S \in \{\{1,3,4\},\{1,2,4\}\} \\ 4 & \text{if } S = \{1,2,3\} \\ 8 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

Consider the prenucleolus of the game, the allocation $x = (2, 2, 2, 2)$. The satisfaction of coalition $\{1, 2, 3\}$ is 2 and players 1, 2 and 3 share this surplus. That is, if the surplus obtained by player 1 in coalition $\{1, 2, 3\}$ at x is 2 then the surplus obtained by players 2 and 3 in coalition $\{1, 2, 3\}$ at x is 0. However, player 4 owns the entire satisfaction obtained by coalition $\{4\}$ at x. From the point of view of the coalitions it can be asserted that coalitions $\{1, 2, 3\}$ and $\{4\}$ have been treated equally at x but this assertion is not so evident from the point of view of the players.

The per capita prenucleolus apparently solves this question. Consider the per capita prenucleolus of the game, the allocation $y = (2.6, 2.2, 2.2, 1)$.

The per capita satisfaction of coalition $\{1, 2, 3\}$ is 1 and players 1, 2 and 3 share a total surplus of 3. That is, the per capita satisfaction can be seen as how much each player receives from the total surplus. Now the assertion that players in coalition $\{1, 2, 3\}$ and player 4 have been equally treated at y can be justified. But consider now the situation of coalition $\{2, 4\}$. According to the per capita satisfaction it must be concluded that each player in the coalition receives a surplus of 1.6, i.e. more than the total payoff received by player 4. It seems incorrect to allocate a surplus of 1.6 to player 4 in coalition $\{2, 4\}$ at y. It seems more correct to consider that the total surplus of coalition $\{2, 4\}$ at y has been distributed as follows: player 2 gets 2.2 and player 4 gets 1.

These ideas motivate the definition of a new vector of satisfactions (and therefore a new lexicographic value) and the name of the new solution concept: Surplus Distributor Prenucleolus.

[3] In Section 4 we compute the SD-prenucleolus of this game, i.e. $(3, 2, 2, 1)$.

3.2 The Algorithm

Consider a game (N, v) and an allocation x. Our goal is to calculate a satisfaction vector $\{F(S,x)\}_{S \subsetneq N}$. We define the components of this vector recursively by defining an algorithm.

The algorithm has several steps (at most $2^n - 2$) and at each step we identify the collection of coalitions that has obtained the satisfaction. We denote by $\mathcal{H}$ this collection of coalitions. In the first step this collection $\mathcal{H}$ is empty. The algorithm ends when $\mathcal{H} = 2^N$.

For a collection $\mathcal{H}$ and a function $F : \mathcal{H} \to \mathbb{R}$ the function $F_{\mathcal{H}} : 2^N \to \mathbb{R}$ is defined. To this end, we introduce some notation.

For $\mathcal{H} \subset 2^N$ we denote

$$\sigma_{\mathcal{H}}(S) = \bigcup_{T \in \mathcal{H}, T \subset S} T$$

and also for a collection $\mathcal{H} \subset 2^N$ and a function $F : \mathcal{H} \to \mathbb{R}$ we denote by $f_{\mathcal{H},F}(i,S)$ the satisfaction of player i with respect to a coalition S and a collection $\mathcal{H}$ ($i \in \sigma_{\mathcal{H}}(S)$):

$$f_{\mathcal{H},F}(i,S) = \min_{T : T \in \mathcal{H}, i \in T \subset S} F(T)$$

Note that this definition can only be used in a situation when the function $F(S)$ is defined for all $S \in \mathcal{H}$.

Now we define a function $F_{\mathcal{H}} : 2^N \to \mathbb{R}$. We consider two cases (since it is evident that $\sigma_{\mathcal{H}}(S) \subset S$):

1. Relevant coalitions. $\sigma_{\mathcal{H}}(S) \neq S$. In this case the satisfaction of S is

$$F_{\mathcal{H}}(S) = \frac{x(S) - v(S) - \sum\limits_{i \in \sigma_{\mathcal{H}}(S)} f_{\mathcal{H},F}(i,S)}{|S| - |\sigma_{\mathcal{H}}(S)|}$$

Note that if the collection $\mathcal{H}$ is empty then the current satisfaction of the coalition S coincides with its *per capita satisfaction*:

$$F_{\emptyset}(S) = \frac{x(S) - v(S)}{|S|}$$

2. Non relevant coalitions. $\sigma_{\mathcal{H}}(S) = S$. In this case the current satisfaction of S is

$$F_{\mathcal{H}}(S) = x(S) - v(S) - \sum_{i \in S} f_{\mathcal{H},F}(i,S) + \max_{i \in S} f_{\mathcal{H},F}(i,S)$$

Therefore for any function $F : \mathcal{H} \to \mathbb{R}$ the value $f_{\mathcal{H},F}(i,S)$ can be calculated for every coalition S and player $i \in \sigma_{\mathcal{H}}(S)$. Also if a function $f_{\mathcal{H},F}(\cdot,\cdot)$ is defined for each $S \subsetneq N$ and $i \in \sigma_{\mathcal{H}}(S)$ then the function $F_{\mathcal{H}}$ can be defined.

The algorithm for the satisfaction vector is defined as follows:.

Algorithm 1 *Consider a game (N,v) and an allocation $x \in X(N,v)$.*

Step 1: *Set $k = 0$, $\mathcal{H}_0 = \emptyset$. Go to Step 2.*

Step 2: *Set*

$$\mathcal{H}_{k+1} = \mathcal{H}_k \cup \{S \notin \mathcal{H}_k : F_{\mathcal{H}_k}(S) = \min_{T \notin \mathcal{H}_k} F_{\mathcal{H}_k}(T)\}$$

Step 3: *Define for each $S \in \mathcal{H}_{k+1} \setminus \mathcal{H}_k$:*

$$F(S) = F_{\mathcal{H}_k}(S)$$

Step 4: *If $\mathcal{H}_{k+1} \neq 2^N \setminus \{N\}$ then let $k = k+1$ and go to Step 2, else go to Step 5.*

Step 5: *Stop. Return the vector*

$$\{F(S), S \subsetneq N\}$$

For simplicity we use the notation $F(S)$ instead of $F(S,x)$.

Note that according to this algorithm if the game introduced in this section and the allocation y (the per capita prenucleolus of the game) are considered it holds that $F(\{2,4\}, y) = 2.2 > 1.6$.

The outcome provided by the algorithm satisfies several interesting properties, which are pointed out in the following lemmas.

Lemma 2 *Let (N,v) be a TU game and x be an allocation. Let function F be the result of Algorithm 1 and let $\{\mathcal{H}_i\}_{i=1..k}$ be the associated collections of sets. Then*

1. the function F is defined for every $S \subsetneq N$

2. the function F is continuos.

Proof. 1. It holds that $\mathcal{H}_0 = \emptyset$, $\mathcal{H}_k = 2^N \setminus \{N\}$. In the i-th stage of the algorithm the function F is defined for all coalitions from $\mathcal{H}_i \setminus \mathcal{H}_{i-1}$. Therefore at the end this function is defined for all coalitions in

$$\bigcup_{i=1..k} (\mathcal{H}_i \setminus \mathcal{H}_{i-1}) = \mathcal{H}_k \setminus \mathcal{H}_0 = 2^N \setminus \{N\}.$$

2. This is immediately apparent. ■

Lemma 3 *Let (N, v) be a TU game and x be an allocation. Let function F be the result of Algorithm 1 and let $\{\mathcal{H}_i\}_{i=1..k}$ be the associated collections of sets. If $S \in \mathcal{H}_i$, $T \notin \mathcal{H}_i$ then $F(T) > F(S)$.*

The proof is in the Appendix. This lemma implies that for a relevant coalition ($\sigma_{\mathcal{H}}(S) \neq S$) it holds that

$$x(S) - v(S) = (|S| - |\sigma_{\mathcal{H}}(S)|)F_{\mathcal{H}}(S) + \sum_{i \in \sigma_{\mathcal{H}}(S)} f_{\mathcal{H},F}(i, S) = \sum_{i \in S} f_{\mathcal{H},F}(i, S)$$

which can be interpreted as a distribution of the total surplus of coalition S among its members. The following 3-person game is used to illustrate how this algorithm works. Let (N, v) be a game where $N = \{1, 2, 3\}$ and

$$v(S) = \begin{cases} 0 & \text{if } |S| = 1 \\ 4 & \text{if } S \in \{\{1,3\},\{1,2\}\} \\ -10 & \text{if } S = \{2,3\} \\ 6 & \text{if } S = N. \end{cases}$$

Consider the allocation $x = (5, 1, 0)$. Applying the algorithm the following is obtained:

Coalition	Satisfaction
$\{3\}$	0
$\{2\}\ \{1,2\}\ \{1,3\}$	1
$\{1\}$	5
$\{2,3\}$	11.

Coalition $\{2, 3\}$ is a non relevant coalition. The rest of the coalitions are relevant coalitions. Consider the satisfaction of coalition $\{1, 3\}$. This coalition has a subset (coalition $\{3\}$) that has already obtained its satisfaction.

This fact is incorporated into the computation of the satisfaction of coalition $\{1,3\}$ since $\sigma_{\mathcal{H}}(\{1,3\}) = \{3\}$. Therefore

$$F_{\mathcal{H}}(\{1,3\}, x) = \frac{x(\{1,3\}) - v(\{1,3\}) - \sum_{i \in \sigma_{\mathcal{H}}(\{1,3\})} f_{\mathcal{H},F}(i, \{1,3\})}{|\{1,3\}| - |\sigma_{\mathcal{H}}(\{1,3\})|} = \frac{5-4-0}{2-1}.$$

The total surplus of the coalition is divided as follows: player 1 gets 1 and player 3 gets 0.

The case of non relevant coalitions is different. If a coalition is non relevant for any player in the coalition there exists a subset of the coalition with a lower satisfaction and that subset determines the individual satisfaction of the player in the non relevant coalition. Note that

$$x(\{2,3\}) - v(\{2,3\}) = 11 > \sum_{i \in \sigma_{\mathcal{H}}(\{2,3\})} f_{\mathcal{H},F}(i, \{2,3\}) = 1 + 0.$$

4 The SD-prenucleolus

4.1 Definition

We define the new solution concept (the SD-prenucleolus) as a lexicographic value in the set of vectors of the new satisfactions. We denote the SD-prenucleolus of game (N, v) by $SD(N, v)$.

The definition of the SD-prenucleolus coincides with the definition of the classic prenucleolus, except that we use the vector of negative satisfactions $\{-F(S,x)\}$ instead of the vector of excesses. Therefore the SD-prenucleolus is a lexicographic value that selects from a set a vector that lexicographically dominates the other vectors of the set.

We now formulate it in detail.

We say that the satisfaction vector $F^x = \{F(S,x)\}_{S \subset N}$ *dominates* the satisfaction vector $F^y = \{F(S,y)\}_{S \subset N}$ if there is $k \geq 1$ such that

1. $\tilde{F}_i^x = \tilde{F}_i^y$ for all $i < k$
2. $\tilde{F}_k^x > \tilde{F}_k^y$,

where $\tilde{F}^x$ and $\tilde{F}^y$ are the vectors with the same components as the vectors F^x, F^y, but rearranged in a non decreasing order ($i > j \Rightarrow \tilde{F}_i^x \leq \tilde{F}_j^x$).

We say that the vector x belongs to the SD-prenucleolus if its satisfaction vector dominates (or weakly dominates) every other satisfaction vector.

Definition 4 *Let* (N, v) *be a TU game. Then* $x \in SD(N, v)$ *if and only if for any* $y \in X(N, v)$ *it holds that* $F^x \succeq_L F^y$.

Similarly to the prenucleolus, the SD-prenucleolus satisfies nonemptiness and single-valuedness on the class of all TU games.

Proposition 5 *Let* (N, v) *be a TU game. Then* $|SD(N, v)| = 1$.

Proof. The standard proof of the nonemptiness of the prenucleolus can be repeated in this case with no changes. The proof of single-valuedness is also very close to the standard one but has some differences. Assume that there is a pair of vectors $x, y \in X(N, v)$ such that both vectors $\{-F(S, x)\}_{S \subsetneq N}$,$\{-F(S, x)\}_{S \subsetneq N}$ dominates a vector $\{-F(S, z)\}_{S \subsetneq N}$ for every $z \in X(N, v)$.

Consider the allocation $t = \frac{x+y}{2}$ and the vector $\{-F(S, t)\}_{S \subsetneq N}$. Because $x \neq y$ the number k can be chosen such that for every $i < k$ it holds that $\mathcal{H}_i(x) = \mathcal{H}_i(y)$ and that $\mathcal{H}_k(x) \neq \mathcal{H}_k(y)$.

Assume that because of the linearity of functions f and F it can be concluded that for $i < k$ it is also true that $\mathcal{H}_i(x) = \mathcal{H}_i(t)$.

Consider the k-th stage of the algorithm for all three vectors (x, y, t). We can note that functions f and F are the same for these vectors. Denote $F^x_{\mathcal{H}_k(S)}$ for $S \in \mathcal{H}^x_k \backslash \mathcal{H}_{k-1}$ by G_k. Because of the coincidence of the satisfaction vectors for x and y it also holds that $G_k = F^y_{\mathcal{H}_k(T)}$ for $T \in \mathcal{H}^y_k \setminus \mathcal{H}_{k-1}$.

With no loss of generality it can be assumed that there exists $T \in H_{\{k\}}(x) \setminus H_{\{k\}}(y)$. By the linearity of the function F it can be concluded that

$$F^t_{\mathcal{H}_{k-1}}(T) = \frac{F^x_{\mathcal{H}_{k-1}}(T) + F^y_{\mathcal{H}_{k-1}}(T)}{2} = \frac{G_k + F^y_{\mathcal{H}_{k-1}}(T)}{2}$$

Because of $T \notin \mathcal{H}_k(y)$ we get $F^y_{\mathcal{H}_{k-1}}(T) > G_k$. Therefore

$$F^t_{\mathcal{H}_{k-1}}(T) = \frac{G_k + F^y_{\mathcal{H}_{k-1}}(T)}{2} > G_k$$

The same conclusions can be used for an arbitrary coalition U which belongs to $\mathcal{H}_k(y)$ but not to $\mathcal{H}_k(x)$. Therefore the collection of coalitions with satisfaction less than or equal to G_k for the vector t is equal to the intersection of such collections for vectors x and y. It means that the satisfaction vector for t dominates the satisfaction vectors for x and y and this contradicts the assumption. ■

4.2 Properties

The new solution shares other interesting properties with the classic prenucleolus and the per capita prenucleolus. It is not difficult to prove that the SD-prenucleolus satisfies desirability (and therefore the equal treatment property), anonymity, covariance and efficiency.

The SD-prenucleolus is a core selector, i.e. if a game is balanced its SD-prenucleolus is a core allocation. This is so because any core allocation has a non negative vector of satisfactions.

Unlike the prenucleolus, the SD-prenucleolus satisfies strong aggregate monotonicity. This property is also satisfied by the per capita prenucleolus.

Proposition 6 *The SD-prenucleolus satisfies the strong aggregate monotonicity property.*

The proof is in the Appendix.

We show that the SD-prenucleolus does not satisfy the null player property by showing that there is incompatibility between strong aggregate monotonicity, the null player property and core stability.

Proposition 7 *If a solution φ defined in the class of all TU games satisfies core stability and the null player property then φ does not satisfy the strong aggregate monotonicity property.*

Proof. Consider the following two games (N, v_1) and (N, v_2) where $N = \{1, 2, 3, 4\}$ and

$$v_1(S) = \begin{cases} 0 & \text{if } |S| = 1 \\ 0 & \text{if } |S| = 2 \text{ and } 4 \in S \\ 4 & \text{otherwise,} \end{cases}$$

$$v_2(S) = \begin{cases} 6 & \text{if } S = N \\ v_1(S) & \text{if } S \neq N. \end{cases}$$

In game (N, v_1) player 4 is a null player and therefore $\varphi_4(N, v_1) = 0$. In game (N, v_2) the core is $\{(2, 2, 2, 0)\}$ and therefore $\varphi_4(N, v_2) = 0$. It must be concluded that φ violates strong aggregate monotonicity. ■

Therefore, the SD-prenucleolus and the per capita prenucleolus do not satisfy the null player property on the class of all TU games.

Obviously, on the class of balanced games a solution that satisfies core stability must satisfy the null player property. But this is not necessarily true for the NPO property. For example, the per capita prenucleolus does not satisfy the NPO property. The result below reinforces the interest in the new solution.

Proposition 8 *The SD-prenucleolus satisfies the NPO property on the class of balanced games.*

The proof is in the Appendix.

In the class of balanced games the SD-prenucleolus is the only known single-valued core selector that satisfies the NPO property and strong aggregate monotonicity[4].

[4] The per capita prenucleolus violates the NPO property as the following example shows. Consider the games (N, v_1) and $(N \backslash \{4\}, v_2)$ where $N = \{1, 2, 3, 4\}$ and

$$v_1(S) = \begin{cases} 7 & \text{if } S \in \{\{1, 2, 3\}, N\} \\ 4 & \text{if } S \in \{\{1, 2\}, \{1, 2, 4\}\} \\ 0 & \text{otherwise,} \end{cases}$$

$$v_2(S) = \begin{cases} 7 & \text{if } S = N \backslash \{4\} = \{1, 2, 3\} \\ 4 & \text{if } S = \{1, 2\} \\ 0 & \text{otherwise.} \end{cases}$$

In game (N, v_1) player 4 is a null player and game $(N \backslash \{4\}, v_2)$ results after eliminating player 4 from game (N, v_1). The per capita prenucleolus of game (N, v_1) is $(2.25, 2.25, 1.5, 0)$ and the per capita prenucleolus of game $(N \backslash \{4\}, v_2)$ is $(3, 3, 1)$.

4.3 Kohlberg's characterization

We provide the equivalent of Kohlberg's theorem for the SD-prenucleolus. For this purpose we introduce the following notation. Given an allocation x and a real number α we define the following set of coalitions

$$\mathcal{B}_\alpha = \{S \subsetneq N : F(S,x) \leq \alpha\}.$$

The theorem is useful for checking whether an allocation is the SD-prenucleolus of a game or not. In fact, it is used to prove the main result of Section 6.

Theorem 9 *Let (N,v) be a TU game and x be an allocation. Then $x = SD(N,v)$ if and only if the collection of sets $\mathcal{B}_\alpha$ is empty or balanced*[5] *for every α.*

Proof. Assume that $x = SD(N,v)$ and that the theorem is not true. Let us choose the minimal α for which the collection of sets $\mathcal{B}_\alpha$ is nonempty and not balanced. It is immediate that the collection $\mathcal{B}_\alpha$ coincides with the collection $\mathcal{H}_k$ for some k.

The assumption of the minimality of the value α implies that for every $m < k$ the collection of sets $\mathcal{H}_m$ is balanced. The non-balancedness of the collection $\mathcal{H}_k$ implies that there exists a vector y such that

1. $\sum\limits_{i \in N} y_i = 0$
2. $\sum\limits_{i \in S} y_i \geq 0$ for each $S \in \mathcal{H}_k$
3. There is $S \in \mathcal{H}_k$ such that $\sum\limits_{i \in S} y_i > 0$

Moreover, by using the fact that the collection $\mathcal{H}_{k-1}$ is balanced we can conclude that $\sum\limits_{i \in T} y_i = 0$ for every $T \in \mathcal{H}_{k-1}$.

Let us consider the vector $x + \varepsilon y$ for "‘small"’ positive value ε. It holds that

1. for every $T \in \mathcal{H}_{k-1}$ the satisfaction with respect to vector $x + \varepsilon y$ is equal to the satisfaction with respect to vector x

[5] See Peleg and Sudholter (2007) for the definition of a balanced collection of sets.

2. for every $T \in \mathcal{H}_k \setminus \mathcal{H}_{k-1}$ the satisfaction with respect to vector $x + \varepsilon y$ is higher than or equal to the satisfaction with respect to vector x and there exists the coalition U such that this inequality is strong.

It is also immediately apparent that a value ε be chosen that is so small that the following collections $\mathcal{H}_m$ for $m > k$ will be not important. Therefore the vector $x + \varepsilon y$ dominates the vector x.

Therefore if a collection $\mathcal{H}_k$ is not balanced then the allocation x is not the SD-prenucleolus of the game. And if the allocation x is not the SD-prenucleolus of the game then there exists some collection $\mathcal{H}_k$ that is not balanced. ■

Using this theorem it can be asserted that the allocation $(5, 1, 0)$ is not the SD-prenucleolus of the second TU game in Section 3.

In general, the computation of the new solution is not an easy task. Like the prenucleolus, the calculation of the SD-prenucleolus of a game is an open challenge. In this sense, the characterization above is a first step that allows it to be checked whether an allocation is the SD-prenucleolus of the game. In Section 5 we introduce a formula for computing the SD-prenucleolus of veto balanced games.

5 Games with Veto Players

The class of games with veto players has been widely used to model economic situations where the presence of special players is needed in order to achieve some positive outcome. The list of papers that consider TU games with veto players is long. Our main purpose is to provide an easy way to compute the SD-prenucleolus of games with veto players.

Arin and Feltkamp (2012) introduce and characterize the Serial Rule for the class of veto balanced games. Let (N, v) be a game with veto players and let player 1 be a veto player. Define for each player i a value d_i as follows:

$$d_i = \max_{S \subseteq N \setminus \{i\}} v(S).$$

Then $d_1 = 0$. Let $d_{n+1} = v(N)$ and rename players according to the nondecreasing order of those values. That is, player 2 is the player with the lowest value besides player 1 and so on. The solution SR associates to each game with veto players, (N,v), the following payoff vector:

$$SR_l(N,v) = \sum_{i=l}^{n} \frac{d_{i+1} - d_i}{i} \text{ for all } l \in \{1,...,n\}.$$

Note that since $d_1 = 0$ the solution is efficient. If there is no veto player the solution is not efficient.

The example in Section 3 illustrates how the solution behaves. The 4-person game has a veto player, player 1. Recall the characteristic function of the game:

$$v(S) = \begin{cases} 1 & \text{if } S \in \{\{1,3,4\},\{1,2,4\}\} \\ 4 & \text{if } S = \{1,2,3\} \\ 8 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

Computing the vector of d-values we get:

$$(d_1, d_2, d_3, d_4, d_5) = (0,1,1,4,8).$$

Applying the formula

$$\begin{array}{llllll} SR_1 = & \frac{d_2-d_1}{1}+ & \frac{d_3-d_2}{2}+ & \frac{d_4-d_3}{3}+ & \frac{d_5-d_4}{4} & = 3 \\ SR_2 = & & \frac{d_3-d_2}{2}+ & \frac{d_4-d_3}{3}+ & \frac{d_5-d_4}{4} & = 2 \\ SR_3 = & & & \frac{d_4-d_3}{3}+ & \frac{d_5-d_4}{4} & = 2 \\ SR_4 = & & & & \frac{d_5-d_4}{4} & = 1. \end{array}$$

We prove that for monotonic[6] games with veto players the Serial Rule and the SD-prenucleolus coincide.

We present several lemmas that are used in the proof of the main theorem.

Lemma 10 *Let (N,v) be a monotonic veto game and let $x = SR(N,v)$. Let l be a non veto player and let S be a coalition such that $l \in S$ and $F(S,x) > x_l$. Then $f_{\mathcal{H},F}(l,S) = x_l$.*

[6] If a game with veto players is monotonic then is balanced since the allocation where a veto player receives $v(N)$ and the rest receive 0 is a core allocation.

Proof. It is immediately apparent that the lemma is true for player n (the player with highest d-value) since $x_n = \min_{S \subset N} \frac{x(S)-v(S)}{|S|} = \frac{x(N \backslash \{n\})\_d_n}{n-1}$. The lemma also must be true for player n-1 since

$$x_{n-1} = \min_{S \subset N, S \notin \{N \backslash \{n\}, \{n\}\}} F(S,x) = F(N \backslash \{n-1\}, x) = \frac{\sum_{l=1}^{n-2} x_l - d_{n-2}}{n-2}.$$

Following similar arguments, it is not difficult to check that if the lemma holds for player k it must hold for player k-1. ■

Lemma 11 *Let (N,v) be a monotonic veto game. Let $x = SR(N,v)$ and let i be a non veto player. Then $F(N \backslash \{i\}, x) = x_i$.*

Proof. Let $T = \{l \in N \backslash \{i\} : x_i < x_l\}$ and let $P = \{l \in N \backslash \{i\} : x_i \geq x_l\}$. Note that since the game is monotonic $v(N \backslash \{i\}) = d_i$. Then by lemma 10

$$F(N \backslash \{i\}, x) = \frac{\sum_{l \in P} SR_l - d_i}{|P|} = SR_i(N,v).$$

This last equality is a consequence of the fact that for any k

$$\sum_{l=1}^{k-1} (SR_l - SR_k) = d_k.$$

■

Lemma 12 *Let (N,v) be a monotonic veto game and let $x = SR(N,v)$. Let S be a coalition without the veto player. Then $F(S,x) = \max_{i \in S} x_i$.*

Proof. Let $p = \max_{i \in S} x_i$. Let $T = \{i \in S : x_i = p\}$ and let $P = \{i \in S : x_i < p\}$. Then for $l \in P$ and applying lemma 10 it holds that

$$x_l = F(l,x) < F(S,x) = \frac{x(T)}{|T|} = p.$$

■

Lemma 13 *Let* (N, v) *be a monotonic veto game. Let* $x = SR(N, v)$ *and let* l *a non veto player. Let* S *be a coalition containing the veto players such that* $l \notin S$ *and* $x_l = \max_{i \notin S} x_i$. *Then* $F(\{l\}, x) = x_l \leq F(S, x)$.

Proof. Assume on the contrary that $x_l > F(S, x)$.

Let l be a non veto player such that $l \notin S$ and $x_l = \max_{i \notin S} x_i$. Let $T = \{i \in S : x_i \geq x_l\}$ and let $P = \{i \in S : x_i < x_l\}$. It is immediate that for $i \in P$ it holds that $F(\{i\}, x) < F(S; x)$. Therefore

$$F(S, x) = \frac{x(T) - v(S)}{|T|} \geq \frac{x(T) - v(N \setminus \{l\})}{|T|} = x_l.$$

The first equality results from applying lemma 10. The last inequality holds because of the monotonicity of (N, v) and the last equality is a consequence of lemma 11. ■

The main theorem of this section establishes the coincidence of the Serial Rule and the SD-prenucleolus on the class of veto monotonic games.

Theorem 14 *Let* (N, v) *be a monotonic veto game. Then* $SR(N, v) = SD(N, v)$.

Proof. The proof is based in the above lemmas. Consider the collection of coalitions S for which $F(S, SR(N, v) \leq k$. From lemma 12 if this collection contains a coalition without a veto player all players of this coalition appear also in the collection as singletons. By lemma 13 and 11 if there is a coalition S containing veto players and without non veto player l then coalitions $N \setminus \{l\}$ and $\{l\}$ are also present. Therefore for a non veto player i one of the two statements is true: either coalition $\{l\}$ is in the collection or all coalitions containing the veto players also contain player i. It is clear that such collection is always balanced. ■

It is clear that the solution denoted by SR satisfies monotonicity. That is, on the class of monotonic veto games the SD-prenucleolus satisfies coalitional monotonicity. (See Arin and Feltkamp (2012) to check that this result is not true for the prenucleolus and the per capita prenucleolus.)

The result of Theorem 14 is not necessarily true if the game is not monotonic. Consider the following 3-person balanced game. Let $N =$

$\{1,2,3\}$ and $v(\{1\}) = v(\{1,3\}) = v(\{1,2\}) = -3$ and $v(S) = 0$ otherwise. Then $SR(N,v) = (0,0,0) \neq SD(N,v) = (-2,1,1)$.

6 Conclusions

We introduce a new solution concept for TU games. This solution is a lexicographic value and can therefore be seen as a member of a family of solutions that includes the prenucleolus and the per capita prenucleolus[7]. The new solution is not a weighted prenucleolus and incorporates into its definition the idea that the surplus obtained by a coalition is divided among its members in a coherent way. This interpretation links the solution with the per capita prenucleolus and both solutions can be seen as members of a family of solutions that provides this distribution of the surplus[8]. Apart from the different way of interpreting the classic concept of excess/satisfaction, the attractiveness of the new solution relies on two interesting facts: The SD-prenucleolus is the only known solution that satisfies core stability, strong aggregate monotonicity and NPO property in the class of balanced games. The SD-prenucleolus is the only known solution defined in the class of all TU games that satisfies core stability, continuity and coalitional monotonicity in the class of veto balanced games. Another paper by Arin and Katsev (2013) adds to this list a new result: the SD-prenucleolus is monotonic in the class of convex games. Therefore, the SD-prenucleolus is the only continuous core concept monotonic in the class of convex games and in the class of veto balanced games.

7 Appendix

Proof of Lemma 3.

[7] Note that the SD-prenucleolus satisfies several properties that the per capita nucleolus does not satisfy (and the properties satisfied by the per capita nucleolus are satisfied by the SD-prenucleolus). From an axiomatic point of view in this family the SD-prenucleolus seems more attractive.

[8] The definition of the prenucleolus does not allow this interpretation.

Proof. Assume that the lemma is not true. Consider the minimal number k such that there exist coalitions S, T with $S \in \mathcal{H}_k$, $T \in \mathcal{H}_{k+1} \setminus \mathcal{H}_k$, and $F(T) \leq F(S)$. Consider the k-th stage of the algorithm where the collection $\mathcal{H}_{k-1}$ was fixed. It holds that $S \in \mathcal{H}_k$ and therefore

$$F_{\mathcal{H}_{k-1}}(S) = \min_{U \notin \mathcal{H}_{k-1}} F_{\mathcal{H}_{k-1}}(U)$$

It is also known that $T \in \mathcal{H}_{k+1} \setminus \mathcal{H}_k$, so $F(T) = F_{\mathcal{H}_k}(T)$.

Note that because of the assumption of the minimality of k

1. for every $i \in \sigma_{\mathcal{H}_{k-1}}(T)$ it holds that $f_{\mathcal{H}_k,F}(i,T) = f_{\mathcal{H}_{k-1},F}(i,T) < F(S)$
2. for every $i \in \sigma_{\mathcal{H}_k}(T) \setminus \sigma_{\mathcal{H}_{k-1}}(T)$ it holds that $f_{\mathcal{H}_k,F}(i,T) = F(S)$

Consider three cases:

A. $\sigma_{\mathcal{H}_k}(T) \neq T$. Then

$$F_{\mathcal{H}_k}(T) = \frac{x(T) - v(T) - \sum_{i \in \sigma_{\mathcal{H}_k}(T)} f_{\mathcal{H}_k,F}(i,T)}{|T| - |\sigma_{\mathcal{H}_k}(T)|} =$$

$$= \frac{x(T) - v(T) - \sum_{i \in \sigma_{\mathcal{H}_{k-1}}(T)} f_{\mathcal{H}_{k-1},F}(i,T) - F(S)(|\sigma_{\mathcal{H}_k}(T)| - |\sigma_{\mathcal{H}_{k-1}}(T)|)}{|T| - |\sigma_{\mathcal{H}_k}(T)|}$$

From the assumption that $F_{\mathcal{H}_k}(T) = F(T) \leq F(S)$

$$\frac{x(T) - v(T) - \sum_{i \in \sigma_{\mathcal{H}_{k-1}}(T)} f_{\mathcal{H}_{k-1},F}(i,T) - F(S)(|\sigma_{\mathcal{H}_k}(T)| - |\sigma_{\mathcal{H}_{k-1}}(T)|)}{|T| - |\sigma_{\mathcal{H}_k}(T)|} \leq F(S)$$

$$\Leftrightarrow x(T) - v(T) - \sum_{i \in \sigma_{\mathcal{H}_{k-1}}(T)} f_{\mathcal{H}_{k-1},F}(i,T) \leq F(S)(|T| - |\sigma_{\mathcal{H}_{k-1}}(T)|) \Leftrightarrow$$

$$\Leftrightarrow \frac{x(T) - v(T) - \sum_{i \in \sigma_{\mathcal{H}_{k-1}}(T)} f_{\mathcal{H}_{k-1},F}(i,T)}{|T| - |\sigma_{\mathcal{H}_{k-1}}(T)|} \leq F(S) \Leftrightarrow F_{\mathcal{H}_{k-1}}(T) \leq F(S)$$

But $F(S) = F_{\mathcal{H}_{k-1}}(S) = \min_{U \notin \mathcal{H}_{k-1}} F_{\mathcal{H}_{k-1}}(U)$.

Therefore $F_{\mathcal{H}_{k-1}}(T) = \min_{U \notin \mathcal{H}_{k-1}} F_{\mathcal{H}_{k-1}}(U)$ and from the Algorithm 1 it holds that $T \in \mathcal{H}_k$. This is in contradiction with the assumption that $T \in \mathcal{H}_{k+1} \setminus \mathcal{H}_k$.

B. $\sigma_{\mathcal{H}_{k-1}}(T) \neq T$, $\sigma_{\mathcal{H}_k}(T) = T$. Then

$$F_{\mathcal{H}_k}(T) = x(T) - v(T) - \sum_{i \in T} f_{\mathcal{H}_k,F}(i,T) + \max_{i \in T} f_{\mathcal{H},F}(i,T)$$

By using the fact that $\sigma_{\mathcal{H}_{k-1}}(T) \neq T$ it can be concluded that

$$F_{\mathcal{H}_k}(T) = x(T) - v(T) - \sum_{i \in T} f_{\mathcal{H},F}(i,T) + F(S) =$$

$$= x(T) - v(T) - \sum_{i \in \sigma_{\mathcal{H}_{k-1}}(T)} f_{\mathcal{H}_{k-1},F}(i,T) - F(S)(|T| - |\sigma_{\mathcal{H}_{k-1}}(T)|) + F(S)$$

From the assumption that $F_{\mathcal{H}_k}(T) = F(T) \leq F(S)$ it can be obtained that

$$x(T) - v(T) - \sum_{i \in \sigma_{\mathcal{H}_{k-1}}(T)} f_{\mathcal{H}_{k-1},F}(i,T) \leq F(S)(|T| - |\sigma_{\mathcal{H}_{k-1}}(T)|) \Leftrightarrow$$

$$\Leftrightarrow \frac{x(T) - v(T) - \sum_{i \in \sigma_{\mathcal{H}_{k-1}}(T)} f_{\mathcal{H}_{k-1},F}(i,T)}{|T| - |\sigma_{\mathcal{H}_{k-1}}(T)|} \leq F(S) \Leftrightarrow F_{\mathcal{H}_{k-1}}(T) \leq F(S)$$

and as in the previous case the contradiction is obtained.

C. $\sigma_{\mathcal{H}_{k-1}}(T) = \sigma_{\mathcal{H}_k}(T) = T$. Then

$$F_{\mathcal{H}_{k-1}}(T) = F_{\mathcal{H}_k}(T)$$

Because of the fact that $T \notin \mathcal{H}_k$ it can be concluded that

$$F_{\mathcal{H}_k}(T) = F_{\mathcal{H}_{k-1}}(T) > \min_{U \notin \mathcal{H}_{k-1}} F_{\mathcal{H}_{k-1}}(U) = F_{\mathcal{H}_{k-1}}(S).$$

■

Proof of Proposition 6.

Proof. Consider games (N,v) and (N,v_A) where $v_A(N) = v(N) + A|N|$ and $v_A(S) = v(S)$ for $S \neq N$. Let $x \in X(v)$ and $y \in X(v_A)$ such that for each $i \in N$ $y_i = x_i + A$.

It is sufficient to show that if the following holds for any $k \geq 0$

1. $F_{(N,v_A)}(S,y) = F_{(N,v)}(S,x) + A$ for each $S \in \mathcal{H}_k$
2. The collections $\mathcal{H}_k$ for (N,v_A,y) and (N,v,x) coincide

then the two facts hold also for $k+1$ (for $k=0$ it is evident). This is shown below. Note that for every $T \subsetneq N$, $i \in \sigma_{\mathcal{H}_k}(T)$ it holds that

$$f^{v_A}_{\mathcal{H}_k, F_{(N,v_A)}}(i,T) = \min_{U: U \in \mathcal{H}_k, i \in U \subset T} (F_{(N,v)}(U) + A) = f^{v}_{\mathcal{H}_k, F_{(N,v)}}(i,T) + A.$$

Consider a coalition $T \subsetneq N$ and two possible variants:

1. $\sigma_{\mathcal{H}_k}(T) \neq T$

$$F^{(N,v_A)}_{\mathcal{H}_k}(T,y) = \frac{y(T) - v_A(T) - \sum\limits_{i \in \sigma_{\mathcal{H}_k}(T)} f^{v_A}_{\mathcal{H}_k,F}(i,T)}{|T| - |\sigma_{\mathcal{H}_k}(T)|} =$$

$$= \frac{x(T) + A|T| - v(T) - \sum\limits_{i \in \sigma_{\mathcal{H}_k}(T)} f^{v}_{\mathcal{H}_k,F}(i,T) - A|\sigma_{\mathcal{H}_k}(T)|}{|T| - |\sigma_{\mathcal{H}_k}(T)|} =$$

$$= \frac{x(T) - v(T) - \sum\limits_{i \in \sigma_{\mathcal{H}_k}(T)} f^{v}_{\mathcal{H}_k,F}(i,T)}{|T| - |\sigma_{\mathcal{H}_k}(T)|} + A = F^{(N,v)}_{\mathcal{H}_k}(T,x) + A$$

2. $\sigma_{\mathcal{H}_k}(T) = T$

$$F^{(N,v_A)}_{\mathcal{H}_k}(T,y) = y(T) - v_A(T) - \sum_{i \in T} f^{v_A}_{\mathcal{H}_k,F}(i,T) + \max_{i \in T} f^{v_A}_{\mathcal{H},F}(i,T)$$

$$= x(T) + A|T| - v(T) - \sum_{i \in T} f^{v_A}_{\mathcal{H}_k,F}(i,T) - A|T| + \max_{i \in T} f_{\mathcal{H},F}(i,T) + A =$$

$$= x(T) - v(T) - \sum_{i \in T} f^{v_A}_{\mathcal{H}_k,F}(i,T) + \max_{i \in T} f_{\mathcal{H},F}(i,T) + A = F^{(N,v)}_{\mathcal{H}_k}(T,x) - A$$

In this way, for every coalition $T \subsetneq N$ it holds that

$$F^{(N,v_A)}_{\mathcal{H}_k}(T,y) = F^{(N,v)}_{\mathcal{H}_k}(T,x) + A$$

and therefore collections $\mathcal{H}_{k+1}$ in both games (N, v_A) and (N, v) coincide. ■

Proof of Proposition 8.

Proof. Consider a balanced game (N, v) where $i \in N$ is a null player. Let $x \in C(N, v)$ be a core allocation. To prove the NPO property of the SD-prenucleolus it is sufficient to show that for every $S \subset N \setminus \{i\}$

$$F(S, x) = F(S \cup \{i\}, x).$$

It is immediately apparent that $x_i = 0$ and coalition $\{i\}$ has the minimal satisfaction, which is 0. Therefore for coalition $P = \arg\min\limits_{S \subset N \setminus \{i\}} \frac{x(S)-v(S)}{|S|}$ it holds that $F(P, x) = F(P \cup \{i\}, x)$.

Consider that for coalitions that obtain their satisfaction before step k it holds that $F(S, x) = F(S \cup \{i\}, x)$. We will prove that for the step k of the algorithm and any coalition $S \in \mathcal{H}_k$, $S \subset N \setminus \{i\}$ it also holds that

$$F_{\mathcal{H}_k}(S, x) = F_{\mathcal{H}_k}(S \cup \{i\}, x). \tag{1}$$

Note that

$$\sigma_{\mathcal{H}_k}(S \cup \{i\}) = S \cup \{i\} \Leftrightarrow \sigma_{\mathcal{H}_k}(S) = S$$

Consider two cases (relevant and non relevant coalitions):

1. $\sigma_{\mathcal{H}_k}(S \cup \{i\}) \neq S \cup \{i\}$. Then

$$F_{\mathcal{H}_k}(S \cup \{i\}) = \frac{x(S \cup \{i\}) - v(S \cup \{i\}) - \sum\limits_{j \in \sigma_{\mathcal{H}_k}(S \cup \{i\})} f_{\mathcal{H}_k, F}(j, S)}{|S| + 1 - |\sigma_{\mathcal{H}_k}(S \cup \{i\})|} =$$

$$= \frac{x(S) - v(S) - \sum\limits_{j \in \sigma_{\mathcal{H}_k}(S)} f_{\mathcal{H}_k, F}(j, S)}{|S| + 1 - |\sigma_{\mathcal{H}_k}(S)| - 1} = F_{\mathcal{H}_k}(S)$$

2. $\sigma_{\mathcal{H}_k}(S \cup \{i\}) = S \cup \{i\}$. Then

$$F_{\mathcal{H}_k}(S \cup \{i\}) =$$

$$= x(S \cup \{i\}) - v(S \cup \{i\}) - \sum_{j \in S \cup \{i\}} f_{\mathcal{H}_k, F}(j, S) + \max_{j \in S \cup \{i\}} f_{\mathcal{H}_k, F}(j, S \cup \{i\}) =$$

$$= x(S) - v(S) - \sum_{j \in S} f_{\mathcal{H}_k, F}(j, S) + \max_{j \in S} f_{\mathcal{H}_k, F}(j, S \cup \{i\})$$

To show that $F_{\mathcal{H}_k}(S \cup \{i\}) = F_{\mathcal{H}_k}(S)$ it suffices to check that

$$\max_{j \in S} f_{\mathcal{H}_k, F}(j, S \cup \{i\}) = \max_{j \in S} f_{\mathcal{H}_k, F}(j, S).$$

But

$$f_{\mathcal{H}_k, F}(j, S \cup \{i\}) = \min_{T \in \mathcal{H}_k, j \in T \subset S \cup \{i\}} F_{\mathcal{H}_k}(T)$$

From the fact (1) for $k' < k$ it can be concluded that for every $T \in \mathcal{H}_k$ it holds that $F_{\mathcal{H}_k}(T) = F_{\mathcal{H}_k}(T \cup \{i\})$. Therefore

$$\min_{T \in \mathcal{H}_k, j \in T \subset S \cup \{i\}} F_{\mathcal{H}_k}(T) = \min_{T \in \mathcal{H}_k, j \in T \subset S} F_{\mathcal{H}_k}(T) \Rightarrow$$

$$\Rightarrow f_{\mathcal{H}_k,F}(j, S \cup \{i\}) = f_{\mathcal{H}_k,F}(j, S)$$

and the proposition has been proved. ■

References

[1] Arin J and Feltkamp V (2012) Coalitional games: Monotonicity and Core. European J Op Research 16, :208-213

[2] Arin J and Katsev I (2013) A monotonic core concept for convex games: The SD-prenucleolus. Mimeo

[3] Kleppe J (2010) Modeling Interactive Behavior, and Solution Concepts. Ph. D. thesis, Tilburg University

[4] Kolhberg E (1971) On the nucleolus of a characteristic function game. SIAM J. Appl. Math. **20**, 62-66

[5] Grotte J (1970) Computation of and observations on the nucleolus, the normalised nucleolus and the central games. Ph. D. thesis, Cornell University, Ithaca

[6] Meggido N (1974) On the monotonicity of the bargaining set, the kernel and the nucleolus of a game. SIAM J of Applied Mathematics 27:355-358

[7] Peleg B (1986) On the reduced game property and its converse. Int J of Game Theory 15:187-200

[8] Peleg B and Sudholter P (2007) Introduction to the theory of cooperative games. Berlin, Springer Verlag

[9] Schmeidler D (1969) The nucleolus of a characteristic function game. SIAM J on Applied Mathematics 17:1163-1170

[10] Sobolev A (1975) The characterization of optimality principles in cooperative games by functional equations. In: N Vorobiev (ed.) Mathematical Methods in the Social Sciences, pp: 95-151. Vilnius. Academy of Science of the Lithuanian SSR

[11] Young HP (1985) Monotonic solutions of cooperative games. Int J of Game Theory 14:65-72

Monotonic core concepts for convex games

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July 4, 2013

Abstract

The existence of a continuous core concept satisfying monotonicity in the class of convex games is an open question that we solve in the affirmative way. We prove that the SD-prenucleolus satisfies monotonicity in this class. The SD-prenucleolus is thus the only known continuous core concept that satisfies monotonicity for convex games, a class of games widely used to model economic situations.

Keywords: Convex games, prenucleolus, core, monotonicity
JEL classification: C71, C72.

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1 Introduction

How to divide the outcome obtained by agents that cooperate is one of the main issues analyzed in the literature of coalitional game theory. One approach to dealing with the problem consists of proposing rules or solutions that are used to solve the game. In this approach, the Shapley value (Shapley, 1953) and the prenucleolus (Schmeidler, 1969) stand out as the most well-known, widely analyzed single-valued solutions for coalitional games with transferable utility (TU games). One of the main reasons for the attractiveness of the Shapley value lies in the fact that it respects the principle of monotonicity, i.e. if a new TU game w is obtained from a given TU game v by increasing the worth of a coalition S then the members of S receive a payoff in game w that is no lower than in game v. On the other hand, the prenucleolus respects the core stability principle, i.e. the prenucleolus selects a core allocation whenever the game is balanced. A core allocation provides each coalition with at least the worth of the coalition, the amount that the members of the coalition can obtain by themselves. It seems very attractive to ask for a solution that fulfils both principles, since they share a kind of incentive compatibility principle that can be summarized in the following idea: the higher the worth of a coalition the better for its members. However, in the class of balanced games they are not compatible (Young, 1985) and therefore the Shapley value does not respect core stability and the prenucleolus fails to satisfy monotonicity.

This incompatibility does not exist if we restrict the analysis to the class of convex games where the Shapley value satisfies core stability. The nucleolus and the per capita nucleolus do not satisfy monotonicity in thus class.

These results motivate the question that this paper seeks to solve: Does there exists any continuous[1] core concept that satisfies monotonicity in the class of convex games? The answer is yes: the SD-prenucleolus.

This solution, introduced by Arin and Katsev (2011) also satisfies ag-

[1]With the requirement of continuity we avoid core concepts defined as follows; Let ϕ be a core concept that coincides with the Shapley value if the game is convex and with the nucleolus otherwise. The solution ϕ satisfies core stability, monotonicity for convex games but not continuity,

gregate monotonicity and monotonicity for games with veto players. Thus, the SD-prenucleolus is the only known concept that satisfies monotonicity for convex games while respecting the principle of core stability, that is, it selects a core allocation whenever the game is balanced.

The rest of the paper is organized as follows:

Section 2 introduces TU games, solutions and properties. Section 3 provides a detailed introduction to the definition of SD-prenucleolus of a game, the notion of "relevant coalition" and the concept of *SD-reduced game property.* This section is based on Arin and Katsev (2011). In Section 4 we analyze the monotonicity of the SD-prenucleolus when considering SD-relevant games and we prove that convex games are SD-relevant games.

2 Preliminaries: TU games

A *cooperative n-person game in characteristic function form* is a pair (N, v), where N is a finite set of n elements and $v : 2^N \to \mathbb{R}$ is a[2] real-valued function in the family 2^N of all subsets of N with $v(\emptyset) = 0$. Elements of N are called *players* and the real valued function v the characteristic function of the game. Any subset S of N is called a *coalition.* Singletons are coalitions that contain only one player. A game is *monotonic* if whenever $T \subset S$ then $v(T) \leq v(S)$. The number of players in S is denoted by $|S|$. Given $S \subset N$ we denote by $N \backslash S$ the set of players of N that are not in S. A distribution of $v(N)$ among the players, an allocation, is a real-valued vector $x \in \mathbb{R}^N$ where x_i is the payoff assigned by x to player i. A distribution satisfying $\sum\limits_{i \in N} x_i = v(N)$ is called an *efficient allocation* and the set of efficient allocations is denoted by $X(v)$. We denote $\sum\limits_{i \in S} x_i$ by $x(S)$. The core of a game is the set of allocations that cannot be blocked by any coalition, i.e.

$$C(N, v) = \{x \in X(N, v) : x(S) \geq v(S) \text{ for all } S \subset N\}.$$

It has been shown that a game with a non-empty core is balanced[3] and therefore games with non-empty core are called balanced games. *Player i is*

[2] See next section for a formal definition of these concepts.

[3] See Peleg and Südholter (2007).

a veto player if $v(S) = 0$ for all S where player i is not present. A balanced game with at least one veto player is called a veto balanced game. We denote by Γ_{VB} the class of balanced games and by Γ_{VB} the class of veto balanced games.

We say that a game (N, v) is convex if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subset N$. We denote by Γ_C the class of convex games.

A solution φ in a class of games Γ_0 is a correspondence that associates a set $\varphi(N, v)$ in $\mathbb{R}^N$with each game (N, v) in Γ_0 such that $x(N) \leq v(N)$ for all $x \in \varphi(N, v)$. This solution is *efficient* if this inequality holds with equality. The solution is *single-valued* if the set contains a *single* element for each game in the class.

We say that the vector x *weakly lexicographically dominates* the vector y (denoted by $x \preceq_L y$) if either $\widetilde{x} = \widetilde{y}$ or there exists k such that $\widetilde{x}_i = \widetilde{y}_i$ for all $i \in \{1, 2, ..., k-1\}$ and $\widetilde{x}_k > \widetilde{y}_k$ where $\widetilde{x}$ and $\widetilde{y}$ are the vectors with the same components as the vectors $\widetilde{x}$, $\widetilde{y}$, but rearranged in a non decreasing order ($i > j \Rightarrow \widetilde{x}_i \leq \widetilde{x}_j$).

Given $x \in \mathbb{R}^N$ the *satisfaction of a coalition S with respect to x* in a game v is defined as $e(S, x) := x(S) - v(S)$. Let $\theta(x)$ be the vector of all satisfactions at x arranged in non decreasing order. Schmeidler (1969) introduced the *prenucleolus* of a game v, denoted by $PN(v)$, as the unique allocation that lexicographically maximizes the vector of non decreasingly ordered satisfactions over the set of allocations. In formula:

$$PN(N, v) = \{x \in X(N, v) \,|\theta(x) \preceq_L \theta(y) \text{ for all } y \in X(N, v)\}.$$

For any game v the prenucleolus is a single-valued solution, is contained in the prekernel and lies in the core provided that the core is non-empty.

The per capita prenucleolus (Groote, 1970) is defined analogously by using the concept of per capita satisfaction instead of excess. Given S and x the per capita satisfaction of S at x is

$$e^{pc}(S, x) := \frac{x(S) - v(S)}{|S|}$$

Other weighted prenucleoli can be defined in a similar way whenever a weighted excess function is defined. The same solution concepts can be analogously defined using the notion of satisfaction instead of excess. Given

$x \in \mathbb{R}^N$ the *excess of a coalition S with respect to x* in a game (N, v) is defined as $f(S, x) := x(S) - v(S)$. In this paper we use the notion of satisfaction in defining the new solution.

For two-person games the nucleolus and the per capita nucleolus coincide with the *standard solution*. Let $(\{i, , j\}, v)$ be a two person game. Then the standard solution of the game is

$$(v(\{i\}) + \frac{v(\{i,j\}) - v(\{i\}) - v(\{j\})}{2}, v(\{j\}) + \frac{v(\{i,j\}) - v(\{i\}) - v(\{j\})}{2}).$$

Some convenient and well-known properties of a solution concept φ on Γ_0 are the following.

- φ satisfies **core stability** if it selects core allocations whenever the game is balanced.

The following properties are defined for single-valued solutions.

- φ satisfies **coalitional monotonicity**: if for all $v, w \in \Gamma_0$, if for all $S \neq T$, $v(S) = w(S)$ and $v(T) < w(T)$, then for all $i \in T$, $\varphi_i(v) \leq \varphi_i(w)$.

- φ satisfies **aggregate monotonicity**: if for all $v, w \in \Gamma_0$, if for all $S \neq N$, $v(S) = w(S)$ and $v(N) < w(N)$, then for all $i, j \in N$, $\varphi_i(w) - \varphi_i(v) \geq 0$.

- φ satisfies **strong aggregate monotonicity**: if for all $v, w \in \Gamma_0$, if for all $S \neq N$, $v(S) = w(S)$ and $v(N) < w(N)$, then for all $i, j \in N$, $\varphi_i(w) - \varphi_i(v) = \varphi_j(w) - \varphi_j(v) \geq 0$.

Young (1985) proves that no solution satisfies coalitional monotonicity and core stability. However there are solutions, including the per capita prenucleolus and the SD-prenucleolus, that satisfy core stability and strong aggregate monotonicity. Clearly, strong aggregate monotonicity implies aggregate monotonicity. The prenucleolus does not satisfy aggregate-monotonicity in the class of convex games (Hokari, 2000). The per capita prenucleolus does not satisfy monotonicity in the class of convex games (see Arin, 2013).

The following notation is widely used in this work. We denote by (N, u_S) the game:

$$u_S(T) = \begin{cases} 1 & \text{if } T = S \\ 0 & \text{otherwise.} \end{cases}$$

3 The SD-prenucleolus

3.1 Definition

In 2011, Arin and Katsev introduce the SD-prenucleolus, a solution concept for TU games. In this section we recall some definitions and results that are needed in the present paper.

The definition of the SD-prenucleolus is based in the concept of satisfaction of a coalition given an allocation. Given a game (N, v) and an allocation x we calculate a satisfaction vector $\{F(S, x)\}_{S \subsetneq N}$. The components of this vector are obtained recursively by defining an algorithm.

The algorithm has several steps (at most $2^n - 2$) and at each step we identify the collection of coalitions that has obtained the satisfaction. This collection of coalitions is denoted by $\mathcal{H}$. In the first step this collection $\mathcal{H}$ is empty. The algorithm ends when $\mathcal{H} = 2^N$.

For a collection $\mathcal{H}$ and a function $F : \mathcal{H} \to \mathbb{R}$ the function $F_{\mathcal{H}} : 2^N \to \mathbb{R}$ is defined. To that end, we introduce some notation. Denote by $\mathcal{H} \subset 2^N$

$$\sigma_{\mathcal{H}}(S) = \bigcup_{T \in \mathcal{H}, T \subset S} T$$

and also for a collection $\mathcal{H} \subset 2^N$ and a function $F : \mathcal{H} \to \mathbb{R}$ we denote by $f_{\mathcal{H},F}(i, S)$ the satisfaction of player i with respect to a coalition S and a collection $\mathcal{H}$ $(i \in \sigma_{\mathcal{H}}(S))$:

$$f_{\mathcal{H},F}(i, S) = \min_{T : T \in \mathcal{H}, i \in T \subset S} F(T)$$

Note that this definition can only be used in a situation when the function $F(S)$ is defined for all $S \in \mathcal{H}$.

Now we define a function $F_{\mathcal{H}} : 2^N \to \mathbb{R}$. We consider two cases (since it is evident that $\sigma_{\mathcal{H}}(S) \subset S$):

1. *Relevant coalitions.* $\sigma_{\mathcal{H}}(S) \neq S$. In this case the satisfaction of S is

$$F_{\mathcal{H}}(S) = \frac{x(S) - v(S) - \sum_{i \in \sigma_{\mathcal{H}}(S)} f_{\mathcal{H},F}(i,S)}{|S| - |\sigma_{\mathcal{H}}(S)|}$$

Note that if the collection $\mathcal{H}$ is empty then the current satisfaction of the coalition S coincides with its *per capita satisfaction*:

$$F_{\emptyset}(S) = \frac{x(S) - v(S)}{|S|}$$

2. *Non relevant coalitions.* $\sigma_{\mathcal{H}}(S) = S$. In this case the current satisfaction of S is

$$F_{\mathcal{H}}(S) = x(S) - v(S) - \sum_{i \in S} f_{\mathcal{H},F}(i,S) + \max_{i \in S} f_{\mathcal{H},F}(i,S)$$

Therefore for any function $F : \mathcal{H} \to \mathbb{R}$ the value $f_{\mathcal{H},F}(i,S)$ can be calculated for every coalition S and player $i \in \sigma_{\mathcal{H}}(S)$. Also if a function $f_{\mathcal{H},F}(\cdot,\cdot)$ is defined for each $S \subsetneq N$ and $i \in \sigma_{\mathcal{H}}(S)$ then the function $F_{\mathcal{H}}$ can be defined.

The algorithm for the satisfaction vector is defined as follows:.

Algorithm 1 *Consider a game (N,v) and an allocation $x \in X(N,v)$.*

Step 1: *Set $k = 0$, $\mathcal{H}_0 = \emptyset$. Go to Step 2.*

Step 2: *Set*

$$\mathcal{H}_{k+1} = \mathcal{H}_k \cup \{S \notin \mathcal{H}_k : F_{\mathcal{H}_k}(S) = \min_{T \notin \mathcal{H}_k} F_{\mathcal{H}_k}(T)\}$$

Step 3: *Define for each $S \in \mathcal{H}_{k+1} \setminus \mathcal{H}_k$:*

$$F(S) = F_{\mathcal{H}_k}(S)$$

Step 4: *If $\mathcal{H}_{k+1} \neq 2^N \setminus \{N\}$ then let $k = k+1$ and go to Step 2, else go to Step 5.*

Step 5: *Stop. Return the vector*

$$\{F(S), S \subsetneq N\}$$

For the sake of simplicity we use the notation $F(S)$ instead of $F(S,x)$.

The lemma below proves that the surplus of a relevant coalition ($x(S) - v)S)$) is fully divided among the members of the coalition. This is not the case with non relevant coalitions, where the surplus of the coalition is higher than the sum of the surpluses of the members of the coalition.

Lemma 2 *For every game (N,v) and allocation $x \in X(v)$ it holds that*

1. *For every relevant coalition $S \subset N$*

$$\sum_{i\in S} f(i,S) = x(S) - v(S).$$

2. *For every non relevant coalition $S \subset N$*

$$\sum_{i\in S} f(i,S) < x(S) - v(S).$$

3. *For every coalition $S \subset T \subset N$*

$$f(i,S) \geq f(i,T) \quad \text{for any } i \in S.$$

Proof. 1. By the definition of satisfaction of a relevant coalition it holds that

$$F_{\mathcal{H}}(S) = \frac{x(S) - v(S) - \sum_{i\in\sigma_{\mathcal{H}}(S)} f_{\mathcal{H},F}(i,S)}{|S| - |\sigma_{\mathcal{H}}(S)|}.$$

Therefore

$$x(S) - v(S) = F_{\mathcal{H}}(S)(|S| - |\sigma_{\mathcal{H}}(S)|) + \sum_{i\in\sigma_{\mathcal{H}}(S)} f_{\mathcal{H},F}(i,S) =$$

$$= \sum_{i\in S\setminus\sigma_{\mathcal{H}}(S)} f(i,S) + \sum_{i\in\sigma_{\mathcal{H}}(S)} f(i,S) = \sum_{i\in S} f(i,S)$$

2. By the definition of satisfaction of a non relevant coalition it holds that

$$F_{\mathcal{H}}(S) = x(S) - v(S) - \sum_{i\in S} f_{\mathcal{H},F}(i,S) + \max_{i\in S} f_{\mathcal{H},F}(i,S).$$

By Lemma 3 from Arin and Katsev (2011) $F_{\mathcal{H}}(S) > \max_{i \in S} f_{\mathcal{H},F}(i,S)$ and therefore

$$\sum_{i \in S} f(i,S) < x(S) - v(S).$$

3. It is immediately apparent. ■

We illustrate the notion of *relevant coalition* by using the following 3-person game. Let (N, v) be a game where $N = \{1,2,3\}$ and

$$v(S) = \begin{cases} 0 & \text{if } |S| = 1 \\ 4 & \text{if } S \in \{\{1,3\}, \{1,2\}\} \\ -10 & \text{if } S = \{2,3\} \\ 6 & \text{if } S = N. \end{cases}$$

Consider the allocation $x = (5,1,0)$. Applying the algorithm the following is obtained:

Coalition	Satisfaction
$\{3\}$	0
$\{2\}$ $\{1,2\}$ $\{1,3\}$	1
$\{1\}$	5
$\{2,3\}$	11.

Coalition $\{2,3\}$ is a non relevant coalition. The rest of the coalitions are relevant coalitions. Consider the satisfaction of coalition $\{1,3\}$. This coalition has a subset (coalition $\{3\}$) that has already obtained its satisfaction. This fact is incorporated into the computation of the satisfaction of coalition $\{1,3\}$ since $\sigma_{\mathcal{H}}(\{1,3\}) = \{3\}$. Therefore

$$F_{\mathcal{H}}(\{1,3\}, x) = \frac{x(\{1,3\}) - v(\{1,3\}) - \sum_{i \in \sigma_{\mathcal{H}}(\{1,3\})} f_{\mathcal{H},F}(i, \{1,3\})}{|\{1,3\}| - |\sigma_{\mathcal{H}}(\{1,3\})|} = \frac{5-4-0}{2-1}.$$

The total surplus of the coalition is divided as follows: player 1 gets 1 and player 3 gets 0.

The case of non relevant coalitions is different. If a coalition is non relevant for any player in the coalition there exists a subset of the coalition with a lower satisfaction and that subset determines the individual satisfaction of the player in the non relevant coalition. Note that

$$x(\{2,3\}) - v(\{2,3\}) = 11 > \sum_{i \in \sigma_{\mathcal{H}}(\{2,3\})} f_{\mathcal{H},F}(i, \{2,3\}) = 1 + 0.$$

We define the SD-prenucleolus as a lexicographic value in the set of vectors of satisfactions. We denote the SD-prenucleolus of game (N, v) by $SD(N, v)$.

We say that the vector x belongs to the SD-prenucleolus if its satisfaction vector dominates (or weakly dominates) every other satisfaction vector.

Definition 3 *(Arin and Katsev, 2011) Let (N, v) be a TU game. Then $x \in SD(N, v)$ if and only if for any $y \in X(N, v)$ it holds that $F^x \succeq_L F^y$.*

The SD-prenucleolus satisfies nonemptiness and single-valuedness in the class of all TU games.

In the proof of the main theorem we need to use the fact that in the class of all TU games the SD-prenucleolus satisfies the SD-reduced game property. Arin and Katsev (2011) introduce the SD-reduced game.

Definition 4 *Let (N, v) be a TU game, $S \subset N$ and $x \in X(N)$. A game (S, v_S^x) is the SD-reduced game with respect to S and x if*

1. $v_S^x(S) = v(N) - x(N \setminus S)$

2. for every $T \subsetneq S$

$$F^{(S,v_S^x)}(T, x_S) = \min_{U \in N \setminus S} F^{(N,v)}(U \cup T, x).$$

For any game (N, v) and any allocation x the SD-reduced game exists and is unique.

We say that a solution ϕ satisfies the *SD-reduced game property* on Γ, *SD-RGP*, if for every game $v \in \Gamma$, for all nonempty coalitions S and for all $x \in \phi(v)$, $x^S \in \phi((S, v_S^x))$.

The SD-prenucleolus satisfies the SD-reduced game property. This type of property[4] plays a determinant role in the characterization of lexicographic

[4]The definition of the reduced game depends on the definition of the satisfaction of a coalition given an allocation.

values such as the prenucleolus and the per capita prenucleolus. The reduced games associated to the prenucleolus and the per capita prenucleolus can be reformulated in an explicit way.

This property plays a determinant role in the proof of the main theorem (the monotonicity of the SD-prenucleolus in the class of convex games).

3.2 Antipartitions

The notion of antipartition (Arin and Inarra, 1998) also plays a central role in the main results of this paper.

A collection of sets $\mathcal{C}=\left\{\dot{S} : S \subset N\right\}$ is called *antipartition* if the collection of sets $\{N\backslash S : S \in \mathcal{C}\}$ is a partition of N. An antipartition is a balanced collection of sets[5]. In order to balance an antipartition $\mathcal{Q}$ each coalition receives the same weight, i.e. $\frac{1}{|\mathcal{C}|-1}$.

For any game (N, v) the *satisfaction of an antipartition* $\mathcal{C}$ with weights $(\frac{1}{|\mathcal{C}|-1})_{S\in\mathcal{C}}$ is defined by

$$F(\mathcal{C}) := \frac{v(N) - \sum_{S\in\mathcal{C}} \frac{1}{|\mathcal{C}|-1} v(S)}{|N|}.$$

Let (N, v) be a TU game and x be an allocation. We denote by $\mathcal{B}(x)$ the set of coalitions with minimal satisfaction at x.

Lemma 5 *Given a game $(N, v\ )$ and an allocation x, if the collection of sets with minimal satisfaction $\mathcal{B}(x)$ contains an antipartition $\mathcal{C}$ then $F(S, x) = F(\mathcal{C})$ for all S belonging to $\mathcal{B}(x)$.*

Proof. Let x be an allocation and let $\mathcal{C}$ be an antipartition in $\mathcal{B}(x)$. Note that for $S \in \mathcal{C}$ it results that $F(S, x) = \frac{x(S)-v(S)}{|S|} = \alpha$.

Since $\mathcal{C}$ is balanced

$$\sum_{S\in\mathcal{C}} \lambda_s x(S) = \sum_{i\in N} x_i = v(N).$$

[5] See Peleg and Sudholter (2007) for the definition of a balanced collection of sets.

From the definition above the following emerges:

$$|N|\, F(\mathcal{C}) = v(N) - \sum_{S\in\mathcal{C}} \frac{1}{|\mathcal{C}|-1} v(S).$$

From the balancedness of $\mathcal{C}$ it holds that

$$v(N) - \sum_{S\in\mathcal{C}} \frac{1}{|\mathcal{C}|-1} v(S) = \sum_{S\in\mathcal{C}} \frac{1}{|\mathcal{C}|-1} \left(x(S) - v(S)\right) =$$

$$\sum_{S\in\mathcal{C}} \frac{1}{|\mathcal{C}|-1} |S|\, F(S,x) = \alpha \sum_{S\in\mathcal{C}} \frac{1}{|\mathcal{C}|-1} |S| = \alpha\, |N|\,.$$

Last equality is a direct consequence of the fact that each player is present in all coalitions of the antipartition but one. ■

Note that if the set of coalitions with minimal satisfaction with respect to the SD-prenucleolus of the game contains an antipartition then the satisfaction of these coalitions only depends on the characteristic function of the game.

4 Monotonicity of the SD-prenucleolus

4.1 SD-relevant games

In this section we introduce the class of SD-relevant TU games. This class includes the class of convex games.

A TU game is SD-relevant if given the SD-prenucleolus of the game all its coalitions are relevant. Formally,

Definition 6 *We say that a TU game (N, v) is SD-relevant if any S, $S \subset N$, is relevant with respect to $SD(N, v)$.*

The game (N, v) where $N = \{1, 2, 3, 4\}$ and

$$v(S) = \begin{cases} 0 & \text{if } |S| = 1 \text{ or } S \in \{\{1,2\},\{3,4\}\} \\ 2 & \text{if } |S| = 2 \\ 2 & \text{if } |S| = 3 \\ 4 & \text{if } S = N \end{cases}$$

is not SD-relevant since $SD(N,v)=(1,1,,1,1)$ and clearly, coalition $\{1,2,3\}$ is non relevant at $(1,1,1,1)$.

In what follows we provide an alternative way of computing the SD-reduced games of an SD-relevant game which bears strong similarities to the well known Davis Maschler reduced game.

If a game is SD-relevant then any SD-reduced game with respect to the SD-prenucleolus of the game is also SD-relevant. Therefore in the class of SD-relevant games all the SD-reduced games with respect to the SD-prenucleolus belong to this class.

Lemma 7 *Let (N,v) be an SD-relevant TU game. Let (S,v^{SD}) be an SD-reduced game with respect to the SD-prenucleolus of (N,v). Then (S,v^{SD}) is an SD-relevant TU game.*

Proof. Let P and M two subsets of S such that $F(M,SD(S,v^{SD}))\geq F(P,SD(S,v^{SD}))$ and $M\cup P\neq N$. We seek to prove that

$$F(M\cup P,SD(S,v^{SD}))\leq \max\left\{F(M,SD(S,v^{SD})),F(P,SD(S,v^{SD}))\right\}.$$

Let $F(M,SD(S,v^{SD}))=F(M\cup Q,SD(N,v^{SD}))$ for some $Q\subseteq N\backslash S$ and let $F(P,SD(S,v^{SD}))=F(P\cup T,SD(N,v^{SD}))$ for some $T\subseteq N\backslash S$. Clearly, $(M\cup Q)\cup(P\cup T)\neq N$.

Since all coalitions in the game (N,v) are relevant It holds that

$$F((M\cup Q)\cup(P\cup T),SD(N,v^{SD}))\leq$$

$$\max\left\{F(M\cup Q,SD(N,v^{SD})),F(P\cup T,SD(N,v^{SD}))\right\}=$$
$$F(M\cup Q,SD(N,v^{SD})).$$

Note that $(M\cup Q)\cup(P\cup T)=(M\cup P)\cup(Q\cup T)$ and therefore

$$F(M\cup P,SD(S,v^{SD}))\leq F((M\cup Q)\cup(P\cup T),SD(N,v^{SD}))\leq$$

$$\leq F(M\cup Q,SD(N,v^{SD})).$$

Therefore the SD-reduced game of an SD-relevant game is SD-relevant.
■

Arin and Inarra (1998) prove that, given a convex game, the collection of coalitions with minimal satisfaction with respect to the prenucleolus of the game contains either a partition or an antipartition. In the case of the SD-prenucleolus of an SD-relevant game only antipartitions should be considered, as the following theorem shows.

Lemma 8 *Let (N, v) be an SD-relevant TU game. Then the collection of sets with minimal satisfactions with respect to $SD(N, v)$ contains an antipartition.*

Proof. Let $x = SD(N, v)$ and let $\mathcal{B}(x)$ be the set of coalitions with minimal satisfaction with respect to x. Let S be a maximal coalition in $\mathcal{B}(x)$, that is, there is no coalition T in $\mathcal{B}(x)$ such that $S \subset T$. Since $\mathcal{B}(x)$ is balanced for each $i \in S$ there exists a coalition, T^i, such that $i \notin T^i$and $T^i \in \mathcal{B}(x)$. Since (N, v) is SD-relevant the maximality of S implies that $N \backslash S \subset T^i$. Let $\{T^i : i \in S\}$ be the set of maximal coalitions for each i in S ((perhaps the case in which for two players i, j it holds that $T^i = T^j$). Then $\{T^i : i \in S\} \cup \{S\}$ is an antipartition. It is immediately apparent that $(N \backslash T^i) \cap (N \backslash S)$ is empty. If for any $i, j \in S$ it holds that $(N \backslash T^i) \cap (N \backslash T^j)$ is nonempty it is clear that $T^i \cup T^j \neq N$ which contradicts the maximality of T^I and T^j since the fact that (N, v) is SD-relevant implies that $T^i \cup T^j$ is an element of the set $\mathcal{B}(x)$. ■

The above results allow for a different interpretation of the SD-reduced game of an SD-relevant game. The SD-reduced games with respect to the SD-prenucleolus can be easily computed according to the result established by the following lemma.

Lemma 9 *Let (N, v) be an SD-relevant TU game, $S \subset N$ and $x = SD(N, v)$. Consider the SD-reduced game (S, v_S^x). Then*

$$v_S^x(T) = v(T \cup (N \setminus S)) - \sum_{i \in N \setminus S} z_i(T \cup (N \setminus S)) = \sum_{i \in T} z_i(T \cup (N \setminus S)).$$

where $z_i(T \cup (N \setminus S)) = x_i - f_i(T \cup (N \setminus S))$

Proof. By Lemma 7 (S, v_S^x) is SD-relevant. We denote by f_S^x the analog of function f for the game (S, v_S^x).

By definition of $f_S^x(i,T)$ it holds that

$$f_S^x(i,T) = \min_{i\in U\subset T} F^{(S,v_S^x)}(U) = \min_{i\in U\subset T}\min_{R\subset N\setminus S} F(U\cup R) =$$

$$= \min_{i\in M\subset T\cup(N\setminus S)} F(M) = f(i, T\cup(N\setminus S)).$$

Therefore

$$v_S^x(T) = x(T) - \sum_{i\in T} f_S^x(i,T) = x(T) - \sum_{i\in T} f(i, T\cup(N\setminus S)) =$$

$$= \sum_{i\in T} z_i(T\cup(N\setminus S)) = v(T\cup(N\setminus S)) - \sum_{i\in N\setminus S} z_i(T\cup(N\setminus S)).$$

Here we use the fact that the coalition $T\cup(N\setminus S)$ is relevant in the game (N,v). ■

The corollary below presents a simple formula for computing some SD-reduced games. This result is used in the proof of the main theorem.

Corollary 10 *Let (N,v) be an SD-relevant TU game, $x = SD(N,v)$ and $S\in\mathcal{B}(x)$. Consider the SD-reduced game $(N\backslash S, v^x)$ and assume that coalition $T\subset N\backslash S$ is relevant in game $(N\backslash S, v^x)$ at x. Then*

$$v_S^x(T) = v(T\cup S) - \sum_{i\in S} z_i(T\cup S) = v(T\cup S) - v(S).$$

Proof. Since $S\in\mathcal{B}(x)$ it holds that $f(i,S) = \frac{x(S)-v(S)}{|S|}$. Since (N,v) is SD-relevant, for any T such that $S\subset T$ it holds that $f(i,T) = f(i,S)$. Therefore,

$$\sum_{i\in S} z_i(T\cup S) = \sum_{i\in S} x_i(T\cup S) - \sum_{i\in S} f(i, T\cup S) =$$

$$\sum_{i\in S}(x_i(S) - f(i,S) = x(S) - |S|\,\frac{x(S)-v(S)}{|S|} == v(S).$$

■

The notion of SD-equivalent games is also needed in the proof of the main results of the paper. We say that two TU games are SD-equivalents if the two games have in the set of coalitions of minimal satisfaction with respect to the SD-prenucleolus the same antipartition.

Definition 11 *Let* $x = SD(N, v)$ *and let* $y = SD(N, w)$*. We say that TU games* (N, v) *and* (N, w) *are SD-equivalent if there exists an antipartition* Q *such that* $Q \subseteq \mathcal{B}(x)$ *and* $Q \subseteq \mathcal{B}(y)$*.*

Next lemma allows us to consider only SD-relevant and SD-equivalent game while analyzing the monotonicity of the SD-prenucleolus in the class of SD-relevant games.

Lemma 12 *For some* $S \subset N$ *and any* $\gamma \in [0, \alpha]$*, let* $(N, v + \gamma u_S)$ *be an SD-relevant TU game. Then, there exists* β*,* $0 < \beta \leq \alpha$ *such that:*

1 (N, v) *and* $(N, v + \beta u_S)$ *are SD-equivalent TU games.*

2 $(N, v + \beta u_S)$ *and* $(N, v + \alpha u_S)$ *are SD-equivalent TU games.*

Proof. Let $y = SD(N, v + \alpha u_S) = SD(N, w)$ and $x = SD(N, v)$. Let $\mathcal{Q}$ be an antipartition contained in $\mathcal{B}(x)$. If $\mathcal{Q}$ is contained in $\mathcal{B}(y)$ then it is evident that $\alpha = \beta$. If $\mathcal{Q}$ is not contained in $\mathcal{B}(y)$ then it must be the case that $S \in \mathcal{B}(y)$ and $F(\mathcal{Q}, v) > F(S, y, w)$ and any antipartition in $\mathcal{B}(y)$ must include S. Let $\mathcal{M}$ be an antipartition in $\mathcal{B}(y)$. If $\mathcal{M}$ is an antipartition in $\mathcal{B}(y)$ the proof is completed and $\alpha = \beta$. If not, it is clear that $F(\mathcal{M}, v) > F(\mathcal{Q}, v) > F(\mathcal{Q}, w) > F(\mathcal{M}, w)$. Therefore by decreasing α we can find a new game $(N, q) = (N, v + \beta u_S)$ such that

$$F(\mathcal{M}, q) = F(\mathcal{Q}, v) = F(\mathcal{Q}, q) > F(\mathcal{M}, w)$$

Therefore with the game (N, q) the statement of the lemma is proved for this last case. ■

The proof of the main theorem uses the following facts:

1 We only consider SD-equivalent games.

2 The set of coalitions with minimal satisfaction with respect to the SD-prenucelolus contains an antipartition. The satisfaction of these coalitions only depends on the characteristic function of the game.

3 The SD-reduced games with respect to the SD-prenucleolus are SD-relevant and can be easily computed.

Now we are in a position to present the main theorem of this section.

Theorem 13 *For some $S \subset N$ and any $\gamma \in [0, \alpha]$, let $(N, v + \gamma u_S)$ be an SD-relevant TU game. Then $SD_i(N, v + \alpha u_S) \geq SD_i(N, v)$ for any $l \in S$.*

Proof. The fact can be proved by induction for $|N|$. If $|N| \leq 2$ then the monotonicity holds since the SD-prenucleolus of the game is the standard solution of the game. Assume that it holds for all games with no more than k players. Now we show that it also holds for each game with $k+1$ players.

Consider the game (N, v) with $|N| = k+1$ and a game $(N, w) \equiv (N, v + \alpha u_S)$ for $S \subset N$ and $\alpha > 0$. We will show that for each $i \in S$ it holds that $SD_i(N, v) \leq SD_i(N, w)$. Assume that (N, w) and (N, v) are SD-equivalent.

From Lemma 8 for the two games there is an antipartition, Q, in the set of coalitions with minimal satisfaction. Consider a coalition T of this antipartition Q. We seek to compare the SD-prenucleolus of the two SD-reduced games $(N \backslash T, v^{SD(v)})$ and $(N \backslash T, w^{SD(w)})$. We distinguish 3 cases:

1. $S \notin Q$ and T is not a subset of S. By applying Corollary 10, the two SD-reduced games must coincide. Therefore players in $S \cap N \backslash T$ receive the same payoff in both games. Since the SD-prenucleolus satisfies the SD-reduced game property it must be the case that in games (N, v) and (N, w) players in $S \cap N \backslash T$ also must receive the same payoff.

2. $S \in Q$. Note that this implies that S must be in the same antipartition with T since otherwise the two games cannot be SD-equivalent.

If $|N \backslash T| = 1$ then it is clear that $SD(T, v) = SD(T, w)$ and consequently $SD_i(N, v) = SDiN, w)$ for $i \in N \backslash T$. Therefore we only consider the case $|N \backslash T| > 1$.

Let $F(S, SD(N, v), v) = F_1$ and $F(S, SD(N, w), w) = F_2$. Since S is in the same antipartition in both games by applying Lemma 5 it is quite immediately apparent that $F_2 = F_1 - \alpha \frac{1}{|N|(|Q|-1)}$ and

$$SD(S, w) = w(S) + |S| F_2 = v(S) + \alpha + |S| (F_1 - \alpha \frac{1}{|N| (|Q| - 1)}) =$$

$$SD(S, v) + \alpha(1 - \frac{|S|}{|N| (|Q| - 1)}) > SD(S, v).$$

In this case the characteristic function w^S for relevant coalitions in the reduced game (S, w^{SD}) with respect to the SD-prenucleolus of the game (N, w)

results

$$w^{SD}(U) = \begin{cases} v^{SD}(S) + \alpha(1 - \frac{|S|}{|N|(|Q|-1)}) & U = N \backslash T \\ v^{SD}(U) & \text{otherwise} \end{cases}$$

This means that (by strong aggregate monotonicity of the SD-prenucleolus) for each $i \in S \cap (N \backslash T)$ it holds that

$$SD_i(N \backslash T, w^{SD(w)}) > SD_i(N \backslash T, v^{SD(V)}) \Leftrightarrow SD_i(N, w) > SD_i(N, v).$$

3 $S \notin Q$ and $T \subset S$.

In this case by applying Corollary 10,

$$w^{SD}(U) = \begin{cases} v^{SD}(S) + \alpha & U = S \backslash T \\ v^{SD}(U) & \text{otherwise.} \end{cases}$$

In this case the TU game $(N \backslash T, w^{SD(w)})$ can be written as $(N \backslash T, v^{SD(v)} + au_{S \backslash T}\dot{)}$. From Lemma 7 the SD-reduced games $(N \backslash T, v^{SD(v)}\dot{)}$ and $(N \backslash T, v^{SD(v)} + au_{S \backslash T}\dot{)}$ are SD-relevant. Note also that for any $\gamma \in [0, \alpha]$ it also holds that $(N \backslash T, v^{SD(v)} + \gamma u_{S \backslash T}\dot{)}$ is an SD-relevant game[6]. We distinguish two cases;

3a) $(N \backslash T, v^{SD(v)}\dot{)}$ and $(N \backslash T, v^{SD(v)} + au_{S \backslash T}\dot{)}$ are SD-equivalent. The analysis can be repeated for these two TU games. If the analysis ends in case 1 or 2 the proof is complete. Otherwise the analysis is repeated for the resulting two new SD-reduced games. In this last case the new TU games have fewer players. Since the result is true when the number of players is 2 it can be asserted that at some stage the procedure will end in case 1 or 2.

3b) $(N \backslash T, v^{SD(v)}\dot{)}$ and $(N \backslash T, v^{SD(v)} + au_{S \backslash T}\dot{)}$ are not SD-equivalent. By Lemma 12 there exists β, $\beta < \alpha$, such that $(N \backslash T, v^{SD(v)}\dot{)}$ and $(N \backslash T, v^{SD(v)} + \beta u_{S \backslash T}\dot{)}$ are SD-equivalent and $(N \backslash T, v^{SD(v)} + \beta u_{S \backslash T}\dot{)}$ and $(N \backslash T, v^{SD(v)} + \alpha u_{S \backslash T})$ are SD-equivalent and SD-relevant. The analysis can be repeated for these two pairs of TU games. If the analysis ends in case 1 or 2 the proof is complete. Otherwise the analysis is repeated for the resulting two new SD-reduced games. In this last case the new TU games have fewer players.

[6] This is so because $(N \backslash T, v^{SD(v)} + au_{S \backslash T}\dot{)}$ is the SD-reduced game of $(N, v + \gamma u_S\dot{)}$ with respect to the SD-prenucleolus of $(N, v + \gamma u_S\dot{)}$. Recall that $(N, v + \gamma u_S\dot{)}$ is by assumption SD-relevant.

Since the result is true when the number of players is 2 it can be asserted that at some stage the procedure will end in case 1 or 2. It is thus proved that for any player i in $S \cap (N \backslash T)$ it holds that $SD_i(N, w) \geq SD_i(N, v)$. Since this is true for any coalition T in the antipartition Q it must be concluded that for any player i in S it holds that $SD_i(N, w) \geq SD_i(N, v)$.

Assume that (N, w) and (N, v) are not SD-equivalent. By Lemma 12 there exists β, $\beta < \alpha$, such that (N, v) and $(N, v + \beta u_S)$ are SD-equivalent and (N, w) and $(N, v + \beta u_S)$ are SD-equivalent. Therefore the above arguments can be used to conclude that for any player i in S it holds that $SD_i(N, v) \leq SD_i(N, v + \beta u_S)$.Similarly, it must be concluded that for any player i in S it holds that $SD_i(N, w) \geq SD_i(N, v + \beta u_S)$. ■

4.2 Convex games

In the class of convex games[7] (Shapley, 1971) core stability and coalitional monotonicity are compatible. In thus class, the Shapley value satisfies the two properties. In general, the Shapley value is not a core concept. Therefore the issue of whether a core concept satisfying monotonicity in the class of convex games exists has been an open question. The following theorem answers the question in the affirmative.

Theorem 14 *In the class of convex games the SD-prenucleolus satisfies coalitional monotonicity.*

The proof of this theorem results immediately from the facts that convex games are SD-relevant games (see lemma below) and the fact that if (N, v) and $(N, v + \alpha u_S)$ are convex then $(N, v + \gamma u_S)$ is convex for any $\gamma \in [0, \alpha]$.

Lemma 15 *Let (N, v) be a convex game and x be an allocation. Then all coalitions are relevant with respect to x. Therefore convex games are SD-relevant games.*

Proof. The lemma is obviously true for coalitions with minimal satisfaction, coalitions in $\mathcal{B}(x)$. We seek to prove that given any two relevant coalitions, S and T, $S \cup T$ is relevant.

[7] Convex games have been widely used to model many different economic situations.

Assume that S and T are relevant coalitions, $S \cup T \neq N$ and $S \cup T$ is non relevant. By convexity

$$x(S \cup T) - v(S \cup T) + x(S \cap T) - v(S \cap T) \leq x(S) - v(S) + x(T) - v(T).$$

Since S and T are relevant:

$$x(S) - v(S) = \sum_{i \in S} f_{\mathcal{H},F}(i, S)$$

$$x(T) - v(T) = \sum_{i \in T} f_{\mathcal{H},F}(i, T).$$

Since $S \cup T$ is non relevant:

$$x(S \cup T) - v(S \cup T) = \sum_{i \in S \cup T} f_{\mathcal{H},F}(i, S \cup T) + (F(x, S \cup T) - \max_{i \in S} f_{\mathcal{H},F}(i, S \cup T)).$$

We consider two cases.

a) There is no relevant coalition $Q \subset S \cup T$ such that $Q \nsubseteq S$, $Q \nsubseteq T$ and $F(Q, x) < \max(F(S, x), F(T, x))$.

In this case it holds that

$$\sum_{i \in S \cup T} f_{\mathcal{H},F}(i, S \cup T) =$$

$$= \sum_{i \in S \setminus T} f_{\mathcal{H},F}(i, S) + \sum_{i \in T \setminus S} f_{\mathcal{H},F}(i, T) + \sum_{i \in T \cap S} \min(f_{\mathcal{H},F}(i, T), f_{\mathcal{H},F}(i, S)) + \alpha$$

where $\alpha > 0$ since $S \cup T$ is non relevant. Therefore

$$\alpha + x(S \cap T) - v(S \cap T) \leq \sum_{i \in T \cap S} \max(f_{\mathcal{H},F}(i, T), f_{\mathcal{H},F}(i, S))$$

or (since $\alpha > 0$)

$$x(S \cap T) - v(S \cap T) < \sum_{i \in T \cap S} \max(f_{\mathcal{H},F}(i, T), f_{\mathcal{H},F}(i, S))$$

or (assuming $S \cap T$ is relevant[8])

$$\sum_{i \in T \cap S} f_{\mathcal{H},F}(i, T \cap S) < \sum_{i \in T \cap S} \max(f_{\mathcal{H},F}(i, T), f_{\mathcal{H},F}(i, S))$$

[8] If it is non relevant the proof is identical: a strictly positive number α just needs to be added on the right-hand side of the inequality. Since α is positive the arguments do not change.

Since $(S \cap T) \subset S$ for any $i \in S \cap T$ it holds that

$$f_{\mathcal{H},F}(i,S) \leq f_{\mathcal{H},F}(i,T \cap S)$$

and similarly, since $(S \cap T) \subset T$ for any $i \in S \cap T$ it holds that

$$f_{\mathcal{H},F}(i,T) \leq f_{\mathcal{H},F}(i,T \cap S).$$

Consequently, for any $i \in S \cap T$ it holds that

$$f_{\mathcal{H},F}(i,T \cap S) \geq \max(f_{\mathcal{H},F}(i,T), f_{\mathcal{H},F}(i,S))$$

which contradicts the fact that

$$x(S \cap T) - v(S \cap T) = \sum_{i \in T \cap S} f_{\mathcal{H},F}(i,T \cap S) < \sum_{i \in T \cap S} \max(f_{\mathcal{H},F}(i,T), f_{\mathcal{H},F}(i,S)).$$

b) There is a relevant coalition $Q \subset S \cup T$ such that $Q \nsubseteq S$, $Q \nsubseteq T$ and $F(x,Q) < \max(F(S,x), F(T,x))$. Among the relevant coalitions satisfying these conditions Q has the minimal satisfaction.

Consider the following coalitions S^1 and T^1 defined as follows:

$$S^1 = \begin{cases} S & \text{if } F(Q,x) \geq F(S,x) \\ S \cup Q & \text{if } F(Q,x) < F(S,x) \end{cases} \text{ and}$$

$$T^1 = \begin{cases} T & \text{if } F(Q,x) \geq F(T,x) \\ T \cup Q & \text{if } F(Q,x) < F(T,x) \end{cases}.$$

We consider two cases:

b1) Coalitions S^1 and T^1 are relevant.

Using coalitions S^1 and T^1 , repeat the arguments used for coalitions S and T. Note that since Q has been chosen with minimal satisfaction then for these two coalitions ($S^1$and T^1) case b) does not occur and it is concluded that $S^1 \cup T^1 = S \cup T$ is relevant.

b2) Assume, without loss of generality, that S^1 is non relevant. Note that $S^1 \subset S \cup T$ and the set of players is finite. Repeat the proof with coalitions S and Q. This ends up either in a contradiction (cases a) and b1)) or in case b2) with two coalitions S and P (or Q and P) such that $S \cup P$ (or $Q \cup P$) is non relevant. Repeat the proof again for coalitions S and P (or Q and P). If the proof ends in case a) or b1) the contradiction is found. If not, repeat the proof with another two coalitions. Note that at the end two coalitions need to be found for which case b2) does not occur since the number of players is finite and the size of the coalitions is reduced at each step whenever the proof ends in case b2). ■

5 Concluding remarks

This paper follows up the research started by Arin and Katsev in 2011. Considering the results included in the two papers the SD-prenucleolus stands out as the only known core concept that satisfies monotonicity in the class of convex games and in the class of veto balanced games. Convex games[9] and games with veto players have been widely used to model many different economic situations. In both classes the compatibility between core stability and monotonicity was known. However the existence of a continuous core concept satisfying monotonicity in those two classes was an open question that has been answered in the positive way: the SD-prenucleolus is a continuous core concept that satisfies aggregate-monotonicity, monotonicity for of convex games and for of veto balanced games. That is, the SD-prenucleolus respects monotonicity and core stability in two important classes of games where the two principles are compatible.

References

[1] Arin J (2013) Monotonic core solutions: Beyond Young´s theorem. Int J of Game Theory 42:325–337

[2] Arin J and Katsev I (2011) The SD-prenucleolus for TU games. https://addi.ehu.es/bitstream/10810/6230/1/IL201156.pdf

[9]This research opens up several questions concerning the study of the SD-prenucleolus for subclasses of convex games such as bankruptcy games and airport games.

[3] Arin J and Inarra E (1998) A characterization of the nucleolus for convex ga,es. Games Econ Behavior 23:12-24

[4] Grotte J (1970) Computation of and observations on the nucleolus, the normalised nucleolus and the central games. Ph. D. thesis, Cornell University, Ithaca

[5] Hokari T (2000) The nucleolus is not aggregate-monotonic on the domain of convex games. Int J of Game Theory 29:133-137

[6] Maschler M, Peleg, B and Shapley LS (1972) The kernel and the bargaining set for convex games. Int J of Game Theory 15: 73-93

[7] Peleg B and Sudholter P (2007) Introduction to the theory of cooperative games. Berlin, Springer Verlag

[8] Schmeidler D (1969) The nucleolus of a characteristic function game. SIAM J on Applied Mathematics 17:1163-1170

[9] Shapley LS (1971). "Cores and convex games". Int. J. of Game Theory **1**:11-26

[10] Young HP (1985) Monotonic solutions of cooperative games. Int J of Game Theory 14:65-72