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Information dispersion and equilibrium multiplicity

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Abstract

This paper studies the implications of correlation of private signals about the liquidation value of a risky asset in a variation of a standard noisy rational expectations model in which traders receive endowment shocks which are private information and have a common component. We find that a necessary condition to generate multiple linear partially revealing rational expectations equilibria is the existence of several sources of information dispersion. In this context equilibrium multiplicity tends to occur when information is more dispersed. A necessary condition to have strategic complementarity in information acquisition is to have multiple equilibria. When the equilibrium is unique there is strategic substitutability in information acquisition, corroborating the result obtained in Grossman and Stiglitz (1980).

JEL Classification: D82, D83, G14

Keywords: Multiplicity of equilibria, strategic complementarity, asymmetric information.

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1 Introduction

We study a standard static noisy rational expectations model, which has as particular cases the main extant models in the literature, and derive conditions for existence and uniqueness (or multiplicity) of equilibria. We find that a main driver of the characterization of equilibria is the level and number of sources of asymmetric information. This applies to multiplicity as well as to whether information acquisition decisions are strategic substitutes or complements.

The presence of multiple equilibria has proved important to show the possibility of strategic complementarity in information acquisition as well as possible coordination failures. In traditional rational expectations models with asymmetric information (Grossman and Stiglitz (1980), Hellwig (1980), Diamond and Verrecchia (1981), Admati (1985), see ch. 4 in Vives (2008) for an overview) there exists a unique linear partially revealing rational expectations equilibrium in financial markets. Moreover, stock prices are always more informative when more investors with private information (about the liquidation value) trade. This implies that private information is less valuable, and hence, traders have less incentives to get informed as the fraction of informed traders increases. In short, there is strategic susbtitutability in information acquisition.

There are several extensions of the models proposed by Grossman and Stiglitz (1980) and Diamond and Verrecchia (1981) showing the multiplicity of such equilibria and the possibility of strategic complementarity in information acquisition. Lundholm (1988) extends the rational expectations competitive model of Diamond and Verrecchia (1981), assuming that each investor receives both a public and a private signal. Lundholm proves the existence of a symmetric linear rational expectations equilibrium, with the possibility of multiplicity of such equilibria.[1] Barlevy and Veronesi (2007) show that when fundamentals and noise trading are correlated the existence of multiple equilibria and strategic complementarity in information acquisition may arise.[2]

Recently, Ganguli and Yang (2009) consider a variation of Diamond and Verrecchia (1981) where informed traders observe information about the liquidation value and the aggregate supply of the stock. They prove that a linear partially revealing rational expectations equilibrium may not exist. In case of existence (except for a set of parameters of measure zero), there are two of such equilibria, with opposing properties about the information content of the equilibrium price and, hence, in one equilibrium the market exhibits strategic complementarity

[1] Nevertheless, the main focus of this paper is the analysis of some curious comparative statics results. More precisely, Lundholm shows that when public and private signals' errors are positive correlated, the equilibrium price of the risky asset may move inversely with a signal. This contrasts with the intuitive result derived in rational expectations models with one risky asset in which the equilibrium price of the risky security increases in the signals observed by investors (see, for instance, Hellwig (1980) and Diamond and Verrecchia (1981)).

[2] Barlevy and Veronesi (2000) propose a model with risk-neutral traders who face a borrowing constraint, in the presence of noise traders and where the fundamentals follow a binomial distribution. They claim that as more traders acquire private information prices need not become more informative and, consequently, investors may want to acquire more private information. However, Chamley (2008a) proves that the previous paper has a mistake in the expression for the value of information.

in information acquisition, while in the other there is strategic substitutability in information acquisition. As in many models in market microstructure, in their paper the independence assumption of private signal noises among informed traders is made. In Hellwig and Veldkamp (2009), they found that this assumption is stringent in their framework since it ultimately determines the model's prediction. Hence, we would like to study the robustness of the results obtained in Ganguli and Yang (2009). Consequently, we develop an extension of Diamond and Verrecchia (1981), where there are a continuum of investors, the errors of private signals about the liquidation value may be correlated and the endowments of investors have a common factor.

We obtain that when private signal noises among informed traders are correlated the existence of a linear partially revealing rational expectations equilibrium is guaranteed, with the possibility of multiplicity of such equilibria. Therefore, this result shows that the problem related to the non-existence of such equilibria in Ganguli and Yang's framework is not robust to small perturbations in the correlation coefficient of private signal errors. Concerning the number of equilibria, we show that in our model at most there are three linear partially revealing rational expectations equilibria.

In relation to the possibility of strategic complementarity in information acquisition, we show that in case of uniqueness of linear partially revealing rational expectations equilibria, learning does not exhibit strategic complementarity. Therefore, we obtain that the corresponding result derived in Grossman and Stiglitz (1980) is robust. In case of two of such equilibria, at least in one equilibrium information acquisition decisions are strategic substitutes. In case of three of such equilibria, in one equilibrium information acquisition decisions are strategic complements, whereas the other equilibria display strategic substitutability. Finally, we develop some particular cases of the general model. We show that the general framework encompasses several of the main models presented in the literature. The results derived from this analysis suggest that equilibrium multiplicity tends to occur when information is more dispersed.

The remainder of this paper is organized as follows. Section 2 outlines the notation and the hypotheses of the model. Section 3 characterizes the symmetric linear equilibria in the general setup. Some particular cases are analyzed in Section 4. Concluding comments are presented in Section 5. Finally, all the proofs are included in the Appendix.

2 The Model

Consider a static asset market model with differential information, where a continuum of risk averse investors exchange a risky asset with liquidation value $v \sim N(v_e, \tau_v^{-1})$, and a riskless asset with unitary return.

Every investor $i \in [0, \mu]$ is (privately) informed about v. She has CARA preferences (denote with γ informed speculators' common degree of risk tolerance), is endowed with u_i shares of the risky asset, and maximizes the expected utility of her wealth: $W_i^I = vu_i + (v - p)x_i^I$. Thus, $U(W_i^I) = -\exp\{-W_i^I/\gamma\}$.

More in detail, every informed trader i:

- prior to trading receives a signal $s_i = v + \epsilon_i$, where $\epsilon_i \sim N(0, \tau_\epsilon^{-1})$, v and ϵ_i are independent for all i and $cov(\epsilon_i, \epsilon_j) = \rho\tau_\epsilon^{-1}$ for all i, j.
- Submits a demand schedule $X^I(s_i, u_i, p)$, contingent on the private signal s_i, on the endowment u_i and on the price p.

Every uninformed trader $j \in (\mu, 1]$ has also CARA preferences with the same degree of risk tolerance as informed investors, and maximizes the expected utility of her wealth: $W_j^U = u_j v + (v - p)x_j^U$. Thus, $U(W_j^U) = -\exp\{-W_j^U/\gamma\}$. More in detail, the uninformed trader j :

- Submits a demand schedule $X^U(u_j, p)$, contingent on the private endowment u_j, and on p.

Finally, assume that,

- $u_h = u + \eta_h$, for all $h \in [0, 1]$, where the error terms η_h are i.i.d., with $\eta_h \sim N(0, \tau_\eta^{-1})$, $u \sim N(u_e, \tau_u^{-1})$ and u and η_h are independent of all the other random variables in the model,
- given u the average endowment shock reveals u, that is $\int_0^1 u_h dh = u$ a.s.
- given v, the average signal satisfies

$$\frac{1}{\mu}\int_0^\mu s_i di = v + \frac{1}{\mu}\int_0^\mu \varepsilon_i di,$$

  where $\frac{1}{\mu}\int_0^\mu \varepsilon_i di$ is normally distributed with

$$E\left[\frac{1}{\mu}\int_0^\mu \varepsilon_i di\right] = 0 \text{ and } var\left[\frac{1}{\mu}\int_0^\mu \varepsilon_i di\right] = \rho\tau_\epsilon^{-1}.$$

3 Equilibrium

For the tractability of the analysis, we focus on symmetric linear rational expectations equilibria (*SLE*). Hence, we are interested in REE that have linear price functions as:

$$p = p(v + \frac{1}{\mu}\int_0^\mu \varepsilon_h dh, u) = A_0 + A_1\left(v + \frac{1}{\mu}\int_0^\mu \varepsilon_h dh\right) - A_2 u. \tag{1}$$

Next, we are interested in deriving under which conditions there exists a *SLE*. In order to perform this, we first express the coefficients of the price function as functions of a ratio, $\beta = \frac{A_1}{A_2}$, and then, we characterize this ratio as a root of a polynomial. If such a root exists, then we conclude that there exists a *SLE*.

Lemma 1.1: *Let* $\beta = \frac{A_1}{A_2}$. *In a SLE,*

$$\begin{aligned} A_0 &= v_e - A_1 v_e + A_2 u_e - \frac{1}{\gamma\left(\mu\tau_I + (1-\mu)\,\tau_U\right)} u_e, \\ A_1 &= 1 - \frac{\tau_v}{\left(\mu\tau_I + (1-\mu)\,\tau_U\right)} \text{ and} \\ A_2 &= \frac{1 + \beta\gamma\tau_u\tau_\varepsilon\left(\frac{\mu(1-\rho)}{\beta^2\rho(1-\rho)(\tau_u+\tau_\eta)+\tau_\varepsilon} + \frac{(1-\mu)}{\beta^2\rho(\tau_u+\tau_\eta)+\tau_\varepsilon}\right)}{\gamma\left(\mu\tau_I + (1-\mu)\,\tau_U\right)}, \end{aligned}$$

where

$$\begin{aligned} \tau_I &= var^{-1}(v|s_i, u_i, p) = \tau_v + \frac{\left(\beta^2\left(\tau_u + \tau_\eta\right)(1-\rho) + \tau_\varepsilon\right)\tau_\varepsilon}{\rho\,(1-\rho)\left(\tau_u + \tau_\eta\right)\beta^2 + \tau_\varepsilon} \text{ and} \\ \tau_U &= var^{-1}(v|u_j, p) = \tau_v + \frac{\left(\tau_u + \tau_\eta\right)\beta^2\tau_\varepsilon}{\rho\left(\tau_u + \tau_\eta\right)\beta^2 + \tau_\varepsilon}. \end{aligned}$$

In addition, β *is a root of the following polynomial:*

$$P(\beta) = c_5\beta^5 + c_4\beta^4 + c_3\beta^3 + c_2\beta^2 + c_1\beta + c_0,$$

where

$$\begin{aligned} c_5 &= \rho^2\,(1-\rho)\left(\tau_u + \tau_\eta\right)^2, \\ c_4 &= \rho\,(\rho - 1)\,\tau_\varepsilon\tau_\eta\left(\tau_u + \tau_\eta\right)\gamma, \\ c_3 &= \rho\,(2-\rho)\left(\tau_u + \tau_\eta\right)\tau_\varepsilon, \\ c_2 &= -\tau_\varepsilon^2\gamma\left(\tau_\eta + \rho\mu\tau_u\right), \\ c_1 &= \tau_\varepsilon^2, \text{ and} \\ c_0 &= -\mu\gamma\tau_\varepsilon^3. \end{aligned}$$

Remark 1: In a *SLE*, $0 \leq A_1 \leq 1$.

Corollary 1.2: *If* $\rho \neq 0$, *the existence of a SLE is guaranteed.*

As $P(\beta)$ is in general a polynomial of degree 5, one could think that we could have 5 *SLE*. However, the following corollary shows that at most the number of *SLE* is 3.

Corollary 1.3: *At most,there exist 3 SLE.*

Next, we analyze the possibility of strategic complementarity in information acquisition. Recall that strategic complementarity (substitutability) in information acquisition means that traders have more (less) incentives to get informed as the fraction of informed traders increases. Formally, let $R(\mu) = \frac{E\left(U(W_i^I)\right)}{E\left(U(W_j^U)\right)}$, where $E\left(U(W_i^I)\right)$ and $E\left(U(W_j^U)\right)$ denote the ex-ante expected indirect utility of an informed trader and an uninformed trader, respectively.

Definition. *A market exhibits strategic complementarity (substitutability) in information acquisition if* $R'(\mu) < 0$ *(*$R'(\mu) > 0$*).*

Corollary 1.4: $R(\mu) = \left(\frac{\tau_U}{\tau_I}\right)^{1/2}$.

This result tells us that, similar to Grossman and Stiglitz (1980), the value of information about the liquidation value of the risky asset is related to the square root of the ratio of the precision of uninformed traders to the precision of informed traders. From this corollary and the chain rule we obtain the following result:

Corollary 1.5: *a) In case of uniqueness of the SLE, the market does not exhibit strategic complementarity in information acquisition.*

b) In case of two SLE, at least in one equilibrium the market exhibits strategic substitutability in information acquisition.

c) In case of three SLE, the market exhibits strategic complementarity in information acquisition in the equilibrium whose value of β *is intermediate. In the remainder equilibria, the market exhibits strategic substitutability.*

Remark 2: We may conjecture that the equilibrium displaying strategic complementarity is unstable with respect to adaptive learning procedures while the ones displaying strategic substitutability are stable. This is the result obtained by Heinemann (2009) in the model of Ganguli and Yang (2009) as well as a related result and remark on a variation of the model by Chamley (2008b) -as referred to by Ganguli and Yang (2009).

4 Particular cases

The general framework presented above encompasses several of the main models presented in the literature of market microstructure.

Example 1: $\tau_\eta = 0$ *(a generalization of Grossman and Stiglitz (1980)).*

In this particular case, as in traditional rational expectations models, informed traders only have private information about the liquidation value of the risky asset. The next lemma shows that the result of the uniqueness of the partially revealing rational expectations equilibrium is robust.

Corollary 2.1: *If* $\tau_\eta = 0$*, a SLE exists and is unique.*

Using Corollary 1.5, Corollary 2.1 implies that the market does not exhibit strategic complementarity in information acquisition whenever informed traders hold private information only about the liquidation value of the asset, independently of the correlation among noise signal errors. The intuition of this result is similar to the one given in Grossman and Stiglitz (1980) model. The more informed traders there are, the equilibrium price becomes more informative, and hence, the lower the incremental value of private information.

Example 2: $\rho = 0$ *(as in Ganguli and Yang (2009)).*

In this case $P(\beta)$ is a polynomial of degree 2 in β and then we can explicitly obtain the roots of $P(\beta)$.

Corollary 2.2: *If* $1 < 4\tau_\varepsilon\tau_\eta\mu\gamma^2$ *this polynomial has no real roots and, hence, we conclude that a SLE does not exist. Otherwise, that is, if* $1 \geq 4\tau_\varepsilon\tau_\eta\mu\gamma^2$*, then we have the following roots:*[3]

$$\begin{aligned}\beta_1 &= \frac{1-\sqrt{1-4\tau_\varepsilon\tau_\eta\mu\gamma^2}}{2\tau_\eta\gamma} \text{ and} \\ \beta_2 &= \frac{1+\sqrt{1-4\tau_\varepsilon\tau_\eta\mu\gamma^2}}{2\tau_\eta\gamma}.\end{aligned}$$

From Corollary 1.2 we have that the result of non-existence of a *SLE* obtained when the error terms of private signals are uncorrelated is not robust.

Example 3: $\rho = 1$ *(informed traders receive the same signal).*

This case corresponds to a model where all the informed traders observe the same signal about the liquidation value.

The following result determines the number of *SLE* in this case:

Corollary 2.3:

- *If* $\mu \geq \frac{\tau_\eta}{8\tau_u+9\tau_\eta}$*, then* $P(\beta)$ *has a unique root, and hence, we conclude that there exists a unique SLE.*
- *If* $\mu < \frac{\tau_\eta}{8\tau_u+9\tau_\eta}$*, then* $\exists\, \overline{\tau}_\varepsilon$ *and* $\overline{\overline{\tau}}_\varepsilon$*, with* $\overline{\overline{\tau}}_\varepsilon < \overline{\tau}_\varepsilon$ *and* $\overline{\tau}_\varepsilon, \overline{\overline{\tau}}_\varepsilon \in \left(\frac{3(\tau_u+\tau_\eta)}{\gamma^2(\tau_\eta+\mu\tau_u)^2}, \infty\right)$, *such that*
  - *for all* τ_ε, $\tau_\varepsilon < \overline{\overline{\tau}}_\varepsilon$*, there exists a unique SLE.*
  - *when* $\tau_\varepsilon = \overline{\overline{\tau}}_\varepsilon$*, there exist two SLE,* β_1 *and* β_2*, such that* $0 < \beta_1 < \overline{\beta}_1$ *and* $\beta_2 = \overline{\beta}_2$.
  - *for all* τ_ε, $\overline{\overline{\tau}}_\varepsilon < \tau_\varepsilon < \overline{\tau}_\varepsilon$*, there exists three SLE,* β_1*,* β_2 *and* β_3*, such that* $0 < \beta_1 < \overline{\beta}_1 < \beta_2 < \overline{\beta}_2 < \beta_3$.

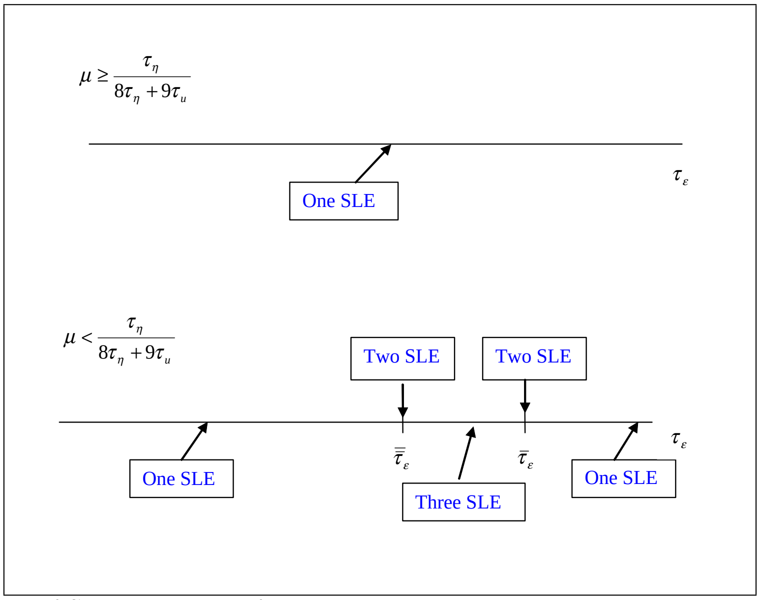
[3] It is important to point out that this case is similar to, the model proposed by Ganguli and Yang (2009) but not identical. In our model all the investors possess private signals about u, whereas in their model only the private informed agents. Obviously, the identity of both models is achieved when $\mu = 1$.

– *when $\tau_\varepsilon = \overline{\tau}_\varepsilon$, there exists two SLE, β_1 and β_2, such that $\beta_1 = \overline{\beta}_1$ and $\beta_2 > \overline{\beta}_2$.*

– *for all τ_ε, $\tau_\varepsilon > \overline{\tau}_\varepsilon$, there exists a unique SLE,*

where the expressions of $\overline{\beta}_1$ and $\overline{\beta}_2$ are stated in the Appendix.

The results derived in this corollary are represented in Figure 1:



Number of SLE when all informed investors receive the same signal about v.

Corollary 2.3 shows that when $\rho = 1$ the number of *SLE* depends on the parameter values. The first part of this result tells us that when the number of informed traders is large and all the investors in this economy are not very informed about the aggregate supply of the risky asset there is a unique *SLE*, independently of the precision of the private information about the liquidation value of the risky asset. To understand it, consider the extreme values: $\mu = 1$ and $\tau_\eta = 0$.

When $\mu = 1$ all the investors in the economy observe the signal about the liquidation value, and hence, there is a unique source of asymmetric/diverse information, the corresponding to the aggregate supply. This is the reason why we obtain unicity of the *SLE*.

When $\tau_\eta = 0$, the investors do not hold information about the aggregate supply. Hence, in this economy some investors only observe the private signal about v. Again, in this case there is a unique source of asymmetric/diverse information, and hence, we have unicity of *SLE*.

The second part of this corollary tells us that when the proportion of informed traders is not very high and the endowments are signals about u not too noisy, the number of equilibria depends on the precision of the private signal about the liquidation value. If τ_ε is very low, basically, we have an economy where there is a unique source of asymmetric/diverse information, the corresponding to the aggregate supply. If τ_ε is very high, we have an economy where the information about v is very precise and, consequently, they basically use the private signal of v when they determine their demands. This is the reason why there is a unique *SLE*. For intermediate values of τ_ε, the informed traders use the two private signals to determine their demand. In this case there are two relevant sources of asymmetric/diverse information and this generates the multiplicity of equilibria.

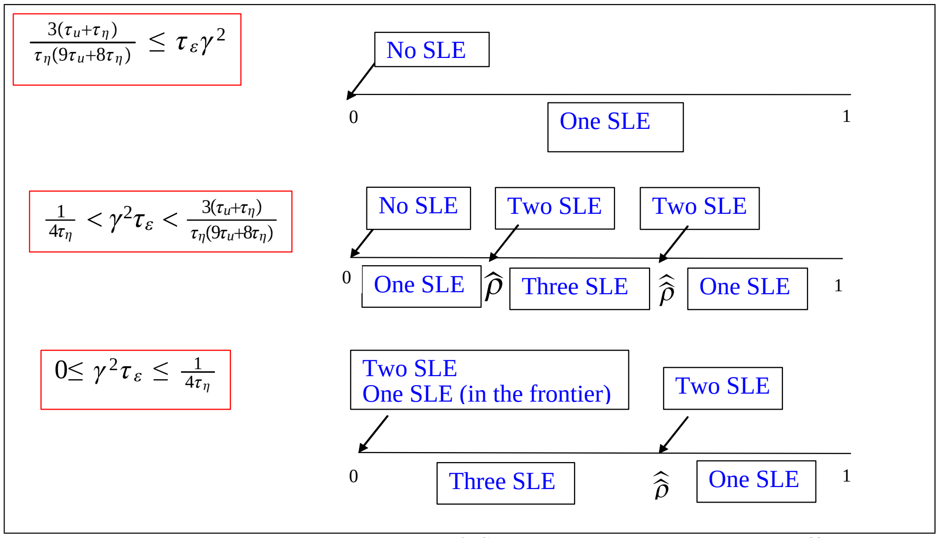
Example 4: $\mu = 1$ *(all the investors are informed about the liquidation value).*

Corollary 2.4: Suppose that $\mu = 1$.

- If $\tau_\varepsilon = 0$, then there is a unique *SLE* whenever $0 \leq \rho \leq 1$.
- If $\rho = 0$, then $\begin{cases} \text{There is no } SLE \text{ whenever } 1 < 4\gamma^2\tau_\eta\tau_\varepsilon \\ \text{There is a unique } SLE \text{ whenever } 1 = 4\gamma^2\tau_\eta\tau_\varepsilon \\ \text{There are two } SLE \text{ whenever } 1 > 4\gamma^2\tau_\eta\tau_\varepsilon \end{cases}$
- If $\rho = 1$, then there is a unique *SLE.*
- If $0 < \rho < 1$, we consider 3 cases:
  - If $\frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)} \leq \tau_\varepsilon\gamma^2$, then there is a unique *SLE* whenever $0 < \rho < 1$.
  - If $\frac{1}{4\tau_\eta} < \gamma^2\tau_\varepsilon < \frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)}$, then there exists a unique *SLE* whenever $\rho \in (0, \widehat{\rho})$, there are two *SLE* $\rho = \widehat{\rho}$, there are three *SLE* whenever $\rho \in \left(\widehat{\rho}, \widehat{\widehat{\rho}}\right)$, there are two *SLE* when $\rho = \widehat{\widehat{\rho}}$ and there exists a unique *SLE* whenever $\rho > \widehat{\widehat{\rho}}$.
  - If $0 < \gamma^2\tau_\varepsilon \leq \frac{1}{4\tau_\eta}$, there are three *SLE* whenever $\rho \in \left(0, \widehat{\widehat{\rho}}\right)$, there are two *SLE* when $\rho = \widehat{\widehat{\rho}}$ and there exists a unique *SLE* whenever $\rho > \widehat{\widehat{\rho}}$,

where $\widehat{\rho}$ and $\widehat{\widehat{\rho}}$ are implicitly determined in the Appendix.

The results obtained in Corollary 2.4 are represented in Figure 2:



Relationship between the number of *SLE* and the correlation coefficient among the errors of private signals about v when $\mu = 1$.

This corollary tells us that when the quality of private information held by investors is high enough (τ_η or τ_ε sufficiently high) there is a unique *SLE*. Notice that when $\tau_\varepsilon \to \infty$, there is a unique source of asymmetric/diverse information and hence, a unique *SLE* arises. A similar reasoning can be made when $\tau_\eta \to \infty$.

On the other extreme case, when the quality of private signals is very low, an increase in the correlation among the errors of private signals about v (and hence, a decrease in the diversity of such private information) reduces the number of equilibria. To understand this point recall that when $\rho = 1$, all investors have the same signal about v, therefore, there is a unique source of asymmetric/diverse information, and hence, there is a unique *SLE*. On the other hand, when ρ is small, there are two significant sources of asymmetric/diverse information and this generates multiple equilibria.

Note that for intermediate values of the precision of private signals, there is a non-monotonic relationship between the number of equilibria and ρ. However, it is important to point out that, for these values of the parameter space, when $\rho = 0$ there is no partially revealing rational expectations equilibria. A continuity property should explain why for low values of such correlation there is only a unique *SLE*.

5 Concluding remarks

This paper shows that the type of information observed by market participants affects their optimal behavior and hence influences the equilibrium price. Specifically, we show that the correlation among the errors of private signals about the liquidation value affects the number of partially revealing rational expectations equilibria and the possibility of strategic complementarity in information acquisition. This paper suggests that a necessary condition to generate multiple equilibria is the existence of several sources of asymmetric/diverse information. In this context equilibrium multiplicity tends to occur when information is more dispersed. Moreover, we show that in case of uniqueness of equilibria, the market does not exhibit strategic complementarity in information acquisition, thus corroborating the result obtained in Grossman and Stiglitz (1980).

6 Appendix

Proof of Lemma 1.1: Consider the informed trader i. Suppose that this agent conjectures the functional form of the price given by (1). The maximization problem of this investor is

$$\max_{x_i^I} E(-e^{-\frac{1}{\gamma}W_i^I}|s_i, u_i, p),$$

which is equivalent to

$$\max_{x_i^I} E(W_i^I|s_i, u_i, p) - \frac{1}{2\gamma} var(W_i^I|s_i, u_i, p).$$

Using the expression of the final wealth of this agent, we have

$$\begin{aligned} E(W_i^I|s_i, u_i, p) &= E(v|s_i, u_i, p)u_i + (E(v|s_i, u_i, p) - p)x_i^I \text{ and} \\ var(W_i^I|s_i, u_i, p) &= \left(x_i^I + u_i\right)^2 var(v|s_i, u_i, p). \end{aligned}$$

Substituting these expressions in the objective function and maximizing with respect to x_i^I, the first order condition implies that

$$x_i^I = \gamma \frac{E(v|s_i, u_i, p) - p - \frac{1}{\gamma} var(v|s_i, u_i, p)u_i}{var(v|s_i, u_i, p)}. \tag{2}$$

Applying standard normal theory, we have
$E(v|s_i, u_i, p) =$
$= v_e + \frac{\tau_\varepsilon^2 A_2^2 (s_i - v_e) + (1-\rho)\tau_\varepsilon \tau_\eta A_1 A_2 (u_i - u_e) + (1-\rho)(\tau_u + \tau_\eta)\tau_\varepsilon A_1 (p - (A_0 + A_1 v_e - A_2 u_e)))}{(\tau_\varepsilon + \rho\tau_v)(1-\rho)(\tau_u + \tau_\eta)A_1^2 + \tau_\varepsilon(\tau_v + \tau_\varepsilon)A_2^2}$
and

$$\tau_I^{-1} = var(v|s_i, u_i, p) = \frac{\tau_\varepsilon A_2^2 + \rho(1-\rho)(\tau_u + \tau_\eta)A_1^2}{(\tau_\varepsilon + \rho\tau_v)(1-\rho)(\tau_u + \tau_\eta)A_1^2 + \tau_\varepsilon(\tau_v + \tau_\varepsilon)A_2^2}.$$

Substituting these expressions into (2), we get

$$x_i^I = B_0 + B_1 u_i - B_2 p + B_3 s_i,$$

with

$$
\begin{aligned}
B_0 &= \gamma\left(\tau_v v_e + \frac{(1-\rho)\,\tau_\varepsilon A_1\left(-\left(\tau_u+\tau_\eta\right)A_0+\tau_u A_2 u_e\right)}{\rho\,(1-\rho)\left(\tau_u+\tau_\eta\right)A_1^2+\tau_\varepsilon A_2^2}\right),\\
B_1 &= -1+\frac{\gamma\,(1-\rho)\,\tau_\varepsilon\tau_\eta A_1 A_2}{\rho\,(1-\rho)\left(\tau_u+\tau_\eta\right)A_1^2+\tau_\varepsilon A_2^2},\\
B_2 &= \gamma\frac{(1-\rho)\left(\tau_u+\tau_\eta\right)\left(A_1\left(\tau_\varepsilon+\rho\tau_v\right)-\tau_\varepsilon\right)A_1+A_2^2\tau_\varepsilon\left(\tau_v+\tau_\varepsilon\right)}{\rho\,(1-\rho)\left(\tau_u+\tau_\eta\right)A_1^2+\tau_\varepsilon A_2^2} \text{ and}\\
B_3 &= \gamma\frac{\tau_\varepsilon^2 A_2^2}{\rho\,(1-\rho)\left(\tau_u+\tau_\eta\right)A_1^2+\tau_\varepsilon A_2^2}.
\end{aligned}
$$

Consider now the uninformed trader j. Suppose that this agent conjectures that the price has the functional form given in (1). The maximization problem of this investor is

$$\max_{x_j^U} E(-e^{-\frac{1}{\gamma}W_j^U}|u_j,p),$$

which is equivalent to the following maximization problem:

$$\max_{x_j^U} E(W_j^U|u_j,p)-\frac{1}{2\gamma}var(W_j^U|u_j,p).$$

Using the expression of the final wealth of this agent, we have

$$
\begin{aligned}
E(W_j^U|u_j,p) &= E(v|u_j,p)u_j+\left(E(v|u_j,p)-p\right)x_j^U \text{ and}\\
var(W_j^U|u_j,p) &= \left(x_j^U+u_j\right)^2 var(v|u_j,p).
\end{aligned}
$$

Substituting these expressions in the objective function and maximizing with respect to x_j^U, the first order condition implies that

$$x_j^U=\gamma\frac{E(v|u_j,p)-p-\frac{1}{\gamma}var(v|u_j,p)u_j}{var(v|u_j,p)}. \tag{3}$$

Again applying standard normal theory, we get

$$E(v|u_j,p)=v_e+\frac{\tau_\varepsilon\tau_\eta A_2A_1(u_j-u_e)+(\tau_u+\tau_\eta)\tau_\varepsilon A_1(p-(A_0+A_1v_e-A_2u_e))}{(\tau_\varepsilon+\rho\tau_v)(\tau_u+\tau_\eta)A_1^2+\tau_v\tau_\varepsilon A_2^2},$$

$$\tau_U^{-1}=var(v|u_j,p)=\frac{\left(A_2^2\tau_\varepsilon+A_1^2\rho\left(\tau_u+\tau_\eta\right)\right)}{\left(\tau_\varepsilon+\rho\tau_v\right)\left(\tau_u+\tau_\eta\right)A_1^2+\tau_v\tau_\varepsilon A_2^2}.$$

Plugging these expressions into (3), we have

$$x_j^U=C_0+C_1u_j-C_2p,$$

with

$$
\begin{aligned}
C_0 &= \gamma\left(\tau_v v_e+\frac{\tau_u\tau_\varepsilon u_e A_1A_2-\tau_\varepsilon\left(\tau_u+\tau_\eta\right)A_1A_0}{\rho\left(\tau_u+\tau_\eta\right)A_1^2+\tau_\varepsilon A_2^2}\right),\\
C_1 &= -1+\frac{\gamma\tau_\eta\tau_\varepsilon A_1A_2}{\rho\left(\tau_u+\tau_\eta\right)A_1^2+\tau_\varepsilon A_2^2} \text{ and}\\
C_2 &= \gamma\frac{\left(\tau_u+\tau_\eta\right)\left(\left(\tau_\varepsilon+\rho\tau_v\right)A_1-\tau_\varepsilon\right)A_1+\tau_v\tau_\varepsilon A_2^2}{\rho\left(\tau_u+\tau_\eta\right)A_1^2+\tau_\varepsilon A_2^2}.
\end{aligned}
$$

Using the optimal demands for all investors, the market clearing condition implies that

$$p=\frac{\mu B_0+(1-\mu)\,C_0+(\mu B_1+(1-\mu)\,C_1)\,u+\mu B_3\left(v+\frac{1}{\mu}\int_0^{\mu}\varepsilon_h dh\right)}{\mu B_2+(1-\mu)\,C_2}.$$

Equating coefficients according to (1)

$$A_0 = \frac{\mu B_0+(1-\mu)\,C_0}{\mu B_2+(1-\mu)\,C_2}, \qquad (4)$$

$$A_1 = \frac{\mu B_3}{\mu B_2+(1-\mu)\,C_2} \text{ and} \qquad (5)$$

$$A_2 = -\frac{\mu B_1+(1-\mu)\,C_1}{\mu B_2+(1-\mu)\,C_2}. \qquad (6)$$

Using the expressions of the coefficients B's and C's, we have

$$\frac{A_1}{A_2}=\frac{\mu\gamma\frac{\tau_\varepsilon^2 A_2^2}{\rho(1-\rho)(\tau_u+\tau_\eta)A_1^2+\tau_\varepsilon A_2^2}}{1-\tau_\varepsilon\tau_\eta\gamma\left(\mu\frac{(1-\rho)}{\rho(1-\rho)(\tau_u+\tau_\eta)A_1^2+\tau_\varepsilon A_2^2}+(1-\mu)\,\frac{1}{\rho(\tau_u+\tau_\eta)A_1^2+\tau_\varepsilon A_2^2}\right)A_1A_2}.$$

Let $\beta=\frac{A_1}{A_2}$. Operating the previous equality, we have that β is a root of $P(\beta)$, whose expression is given in the statement of this lemma.

Next, we derive the expressions of A_2, A_1 and A_0 as functions of β. Using the expressions of the coefficients B's and C's, from (6), direct computations yield the desired expression for A_2.

Concerning the expression of A_1, notice that substituting the expression of τ_I and τ_U in $1-\frac{\tau_v}{(\mu\tau_I+(1-\mu)\tau_U)}$ and operating, we get

$$1-\frac{\tau_v}{\mu\tau_I+(1-\mu)\,\tau_U}=\frac{\mu\frac{\left(A_1^2(\tau_u+\tau_\eta)(1-\rho)+\tau_\varepsilon A_2^2\right)\tau_\varepsilon}{\rho(1-\rho)(\tau_u+\tau_\eta)A_1^2+\tau_\varepsilon A_2^2}+(1-\mu)\,\frac{(\tau_u+\tau_\eta)A_1^2\tau_\varepsilon}{\rho(\tau_u+\tau_\eta)A_1^2+\tau_\varepsilon A_2^2}}{\tau_v+\mu\frac{\left(A_1^2(\tau_u+\tau_\eta)(1-\rho)+\tau_\varepsilon A_2^2\right)\tau_\varepsilon}{\rho(1-\rho)(\tau_u+\tau_\eta)A_1^2+\tau_\varepsilon A_2^2}+(1-\mu)\,\frac{(\tau_u+\tau_\eta)A_1^2\tau_\varepsilon}{\rho(\tau_u+\tau_\eta)A_1^2+\tau_\varepsilon A_2^2}}.$$

Using (5),

$$\mu\frac{\tau_\varepsilon^2A_2^2}{\rho(1-\rho)(\tau_u+\tau_\eta)A_1^2+\tau_\varepsilon A_2^2}=$$
$$A_1\left(\frac{\mu\left((1-\rho)(\tau_u+\tau_\eta)(\tau_\varepsilon(A_1-1)+\rho\tau_vA_1)A_1+A_2^2\tau_\varepsilon(\tau_v+\tau_\varepsilon)\right)}{\rho(1-\rho)(\tau_u+\tau_\eta)A_1^2+\tau_\varepsilon A_2^2}+\frac{(1-\mu)\left((\tau_u+\tau_\eta)((\tau_\varepsilon+\rho\tau_v)A_1-\tau_\varepsilon)A_1+\tau_v\tau_\varepsilon A_2^2\right)}{\rho(\tau_u+\tau_\eta)A_1^2+\tau_\varepsilon A_2^2}\right).$$

Substituting this expression in the RHS of the previous equality and operating, we get $1-\frac{\tau_v}{\mu\tau_I+(1-\mu)\tau_U}=A_1$.

Finally, isolating A_0 from (4), operating taking into account the expressions of A_1 and A_2, we get the formula for A_0 stated in this lemma. ■

Proof of Corollary 1.2: Suppose that $\mu\tau_\varepsilon\neq 0$.[4] If $\rho\neq 0$, then $P(0)<0$ and $\underset{\beta\to+\infty}{lim}P(\beta)=+\infty$. By continuity, we have that there exists a $\beta'>0$ such that $P(\beta')=0$, and this is equivalent to saying that a *SLE* exists. ■

[4]When $\mu\tau_\varepsilon=0$, we have an economy with no private information about the liquidation value of the risky asset. In this case, one can show that there is a unique partially revealing *SLE,* whose equilibrium price is given by p=$v_e-\tau_v^{-1}\gamma^{-1}u$.

Proof of Corollary 1.3: Notice that if ρ is either 0 or 1, then the result is trivial. Suppose that $\rho \neq 0, 1$. By virtue of Rolle's Theorem, to show the result it is enough to prove that there exists a unique β such that $P''(\beta) = 0$. Derivating we have

$$\begin{aligned} P''(\beta) &= 20c_5\beta^3 + 12c_4\beta^2 + 6c_3\beta + 2c_2 \text{ and} \\ P'''(\beta) &= 60c_5\beta^2 + 24c_4\beta + 6c_3. \end{aligned}$$

We distinguish three cases:

Case 1: $2\gamma^2(1-\rho)\tau_\varepsilon\tau_\eta^2 < 5\,(2-\rho)\,\rho\,(\tau_u + \tau_\eta)$.

In this case $P'''(\beta)$ hasn't real roots. The fact that $c_3 > 0$ implies that $P'''(\beta) > 0$ for all β, and consequently, $P''(\beta)$ is strictly increasing. Moreover, since $\underset{\beta\to-\infty}{lim} P''(\beta) = -\infty$ and $\underset{\beta\to+\infty}{lim} P''(\beta) = +\infty$, we conclude that in this case $P''(\beta)$ has a unique real root.

Case 2: $2\gamma^2(1-\rho)\tau_\varepsilon\tau_\eta^2 = 5\,(2-\rho)\,\rho\,(\tau_u + \tau_\eta)$.

In this case $P'''(\beta)$ has a unique real root, denoted by β'. The fact that $c_3 > 0$ implies that $P'''(\beta) > 0$ for all $\beta \neq \beta'$, and consequently, $P''(\beta)$ is strictly increasing for all $\beta \neq \beta'$. As before, since $\underset{\beta\to-\infty}{lim} P''(\beta) = -\infty$ and $\underset{\beta\to+\infty}{lim} P''(\beta) = +\infty$, we conclude that in this case $P''(\beta)$ has a unique real root.

Case 3: $2\gamma^2(1-\rho)\tau_\varepsilon\tau_\eta^2 > 5\,(2-\rho)\,\rho\,(\tau_u + \tau_\eta)$.

In this case $P'''(\beta)$ has two real roots, given by:

$$\beta_1 = \frac{2\tau_\varepsilon\tau_\eta\,(1-\rho)\,\gamma - \sqrt{2}\sqrt{(1-\rho)\,\tau_\varepsilon\left(\tau_\varepsilon 2\tau_\eta^2\gamma^2\,(1-\rho) - 5\rho\,(\tau_u+\tau_\eta)\,(2-\rho)\right)}}{10\,(1-\rho)\,\rho\,(\tau_u+\tau_\eta)}$$

and

$$\beta_2 = \frac{2\tau_\varepsilon\tau_\eta\,(1-\rho)\,\gamma + \sqrt{2}\sqrt{(1-\rho)\,\tau_\varepsilon\left(\tau_\varepsilon 2\tau_\eta^2\gamma^2\,(1-\rho) - 5\rho\,(\tau_u+\tau_\eta)\,(2-\rho)\right)}}{10\,(1-\rho)\,\rho\,(\tau_u+\tau_\eta)},$$

where β_1 is a local maximum of $P''(\beta)$. Using the fact that $\tau_\varepsilon > \frac{5(2-\rho)(\tau_u+\tau_\eta)\rho}{2\gamma^2(1-\rho)\tau_\eta^2}$, and after some tedious computations, we obtain that $P''(\beta_1) < 0$. Hence, we have that $P''(\beta)$ has a unique root. ∎

Next, we state a lemma that will be applied in the proof of Corollary 1.4.

Lemma A.1. Let $z \sim N(0, \Sigma)$ and $W = c + b\prime z + z\prime Az$, where $c \in R$, $b \in R^n$, and A is an n×n matrix. Then, if $\Sigma^{-1} + 2\rho A$ is positive definite, then

$E\left(e^{-\rho W}\right) = -\left|\Sigma\right|^{-1/2}\left|\Sigma^{-1} + 2\rho A\right|^{-1/2}\exp\left(-\rho\left(c - \frac{1}{2}\rho b'\left(\Sigma^{-1}+2\rho A\right)^{-1} b\right)\right)$

Proof. See Danthine and Moresi (1993).

Proof of Corollary 1.4: First, we want to derive the ex-ante expected utility function for an uninformed trader. Recall that the combination of CARA utility functions and the normality assumption implies that

$$E\left(U(W_j^U)|u_j, p\right) = -e^{\left(-\frac{1}{\gamma}\left(E(W_j^U|u_j,p) - \frac{1}{2\gamma}var(W_j^U|u_j,p)\right)\right)}.$$

Using the expression of the final wealth of this agent and (3), we have

$$E\left(U(W_j^U)|u_j,p\right)=-e^{\left(-\left(\frac{1}{\gamma}pu_j+\frac{1}{2}\frac{(E(v|u_j,p)-p)^2}{var(v|u_j,p)}\right)\right)}.$$

Let $z_1=p-E(p|u_j)$, where $E(p|u_j)=A_0+A_1v_e-A_2\left(u_e+\frac{\tau_\eta}{\tau_u+\tau_\eta}(u_j-u_e)\right)$. Using the expression of $E(v|u_j,p)$ and τ_U, the previous conditional expected utility can be written as:

$$E\left(U(W_j^U)|u_j\right)=-E(e^{\left(-\left(c+bz_1+Az_1^2\right)\right)}|u_j),$$

where

$$\begin{aligned}
A &= \frac{1}{2}\tau_U\left(\frac{1}{\tau_U}A_1\tau_\varepsilon\frac{\tau_u+\tau_\eta}{A_2^2\tau_\varepsilon+\rho A_1^2(\tau_u+\tau_\eta)}-1\right)^2,\\
b &= \frac{1}{\gamma}u_j+\tau_U\left(\frac{A_1}{\tau_U}\tau_\varepsilon\frac{\tau_u+\tau_\eta}{A_2^2\tau_\varepsilon+\rho A_1^2(\tau_u+\tau_\eta)}-1\right)(v_e-E(p|u_j)),\ \text{and}\\
c &= u_j\frac{1}{\gamma}E(p|u_j)+\frac{1}{2}\tau_U(v_e-E(p|u_j))^2.
\end{aligned}$$

Conditional on u_j, z_1 is normally distributed with zero mean and variance

$$\Sigma=var(z_1|u_j)=A_1^2\left(\frac{1}{\tau_v}+\rho\frac{1}{\tau_\varepsilon}\right)+A_2^2\frac{1}{(\tau_u+\tau_\eta)}=\frac{\left(A_2^2\tau_\varepsilon+A_1^2\rho(\tau_u+\tau_\eta)\right)\tau_U}{(\tau_u+\tau_\eta)\tau_v\tau_\varepsilon}.$$

Since $\Sigma^{-1}+2A>0$, we can apply the Lemma A.1 and operating, using the expression of τ_U, we get

$$E\left(U(W_j^U)|u_j\right)=-\frac{e^{\left(-\left(\widehat{c}+\widehat{b}u_j+\widehat{A}u_j^2\right)\right)}}{\left(\tau_U\frac{(\tau_u+\tau_\eta)\left(\tau_\varepsilon(A_1-1)^2+\rho A_1^2\tau_v\right)+A_2^2\tau_v\tau_\varepsilon}{(\tau_u+\tau_\eta)\tau_v\tau_\varepsilon}\right)^{1/2}},$$

where

$$\widehat{A}=-\frac{A_2^2\tau_\varepsilon+A_1^2\rho(\tau_u+\tau_\eta)-\tau_\varepsilon\tau_\eta A_2\gamma\left(\frac{\gamma A_2\tau_v\tau_\eta}{(\tau_u+\tau_\eta)}+2(A_1-1)\right)}{2\left((\tau_u+\tau_\eta)\left(\tau_\varepsilon(A_1-1)^2+\rho A_1^2\tau_v\right)+A_2^2\tau_v\tau_\varepsilon\right)\gamma^2},$$

$$\widehat{b}=\frac{\tau_\varepsilon\left(A_2u_e\frac{\tau_u}{(\tau_u+\tau_\eta)}-A_0\right)((\tau_u+\tau_\eta)(A_1-1)+\gamma A_2\tau_v\tau_\eta)+v_e\tau_v\left(A_1^2\rho(\tau_u+\tau_\eta)-\tau_\varepsilon A_2(\tau_\eta\gamma(A_1-1)-A_2)\right)}{\left((\tau_u+\tau_\eta)\left(\tau_\varepsilon(A_1-1)^2+\rho A_1^2\tau_v\right)+A_2^2\tau_v\tau_\varepsilon\right)\gamma}$$

and

$$\widehat{c}=\frac{(\tau_u+\tau_\eta)\tau_v\tau_\varepsilon\left(v_e-A_0-A_1v_e+A_2\frac{\tau_u}{\tau_u+\tau_\eta}u_e\right)^2}{2\left((\tau_u+\tau_\eta)\left(\tau_\varepsilon(A_1-1)^2+\rho A_1^2\tau_v\right)+A_2^2\tau_v\tau_\varepsilon\right)}.$$

Hence, the ex-ante expected utility function for the uninformed trader is given by

$$E\left(U(W_j^U)\right) = -\frac{E\left(e^{\left(-\left(\widehat{c}+\widehat{b}u_j+\widehat{A}u_j^2\right)\right)}\right)}{\left(\tau_U \frac{(\tau_u+\tau_\eta)\left(\tau_\varepsilon(A_1-1)^2+\rho A_1^2\tau_v\right)+A_2^2\tau_v\tau_\varepsilon}{(\tau_u+\tau_\eta)\tau_v\tau_\varepsilon}\right)^{1/2}}. \qquad (7)$$

Performing similar computations as before, we obtain that the ex-ante expected utility function for an informed trader is given by

$$E(U(W_i^I)) = -\frac{E\left(e^{\left(-\left(\widehat{c}+\widehat{b}u_i+\widehat{A}u_i^2\right)\right)}\right)}{\left(\tau_I \frac{(\tau_u+\tau_\eta)\left(\tau_\varepsilon(A_1-1)^2+\rho A_1^2\tau_v\right)+A_2^2\tau_v\tau_\varepsilon}{\tau_\varepsilon\tau_v(\tau_u+\tau_\eta)}\right)^{1/2}}.$$

Using the fact that u_i and u_j are identically distributed, from the previous equality and (7), it follows that $R(\mu) = \left(\frac{\tau_U}{\tau_I}\right)^{1/2}$. ■

Proof of Corollary 1.5: Using Corollary 1.4, the expressions of τ_U, τ_I and the chain rule, we have

$$R'(\mu) = \frac{1}{2}\left(\frac{\tau_U}{\tau_I}\right)^{1/2} 2\frac{\left(\tau_\varepsilon(\tau_\varepsilon+\tau_v\rho(2-\rho))+\beta^2 2\rho(\tau_u+\tau_\eta)(1-\rho)(\tau_\varepsilon+\rho\tau_v)\right)(\tau_u+\tau_\eta)\beta\tau_\varepsilon^3}{(\beta^2(\tau_u+\tau_\eta)(1-\rho)(\tau_\varepsilon+\rho\tau_v)+\tau_\varepsilon(\tau_v+\tau_\varepsilon))^2(\tau_\varepsilon+\beta^2\rho(\tau_u+\tau_\eta))^2}\beta'(\mu).$$

Since in equilibrium $\beta > 0$, $R'(\mu) < 0$ if and only if $\beta'(\mu) < 0$. From Lemma 1.1 and the Implicit Function Theorem, we have $\beta'(\mu) = -\frac{\left(-\tau_\varepsilon^2\gamma\left(\tau_\varepsilon+\rho\beta^2\tau_u\right)\right)}{P'(\beta)}$. Hence, the sign of $\beta'(\mu)$ is the sign of $P'(\beta)$.

a) In case of uniqueness of the *SLE*, $P'(\beta)$ evaluated at the unique zero of $P(\beta)$ is strictly positive, which allows us to conclude that $R'(\mu) > 0$.

b) In case of two *SLE,* the shape of $P(\beta)$ allows us to conclude that there exists at least one *SLE* with a value of β such that $P'(\beta)$ is strictly positive, and hence, $R'(\mu) > 0$ in this equilibrium.

c) In case of three *SLE*, denote by β_1, β_2 and β_3 the corresponding equilibrium values of β, where $\beta_1<\beta_2<\beta_3$. In this case, $P'(\beta_1) > 0$, $P'(\beta_2) < 0$ and $P'(\beta_3) < 0$. This implies that $R'(\mu) > 0$ in the *SLE* with extreme values of β, whereas $R'(\mu) < 0$ in the *SLE* whose value of β is intermediate. ■

Proof of Corollary 2.1: Direct computations yield $P(\beta) = g(\beta)\left(\tau_\varepsilon + \rho\beta^2\tau_u\right)$, where $g(\beta) = (1-\rho)\rho\tau_u\beta^3 + \tau_\varepsilon\beta - \tau_\varepsilon^2\gamma\mu$. Hence, $P(\beta) = 0$ is equivalent to $g(\beta) = 0$. Since $g(\beta)$ is strictly increasing and the fact that $g(0) < 0$ and $\lim\limits_{\beta\to\infty} g(\beta) = +\infty$, we conclude that $g(\beta)$, and also $P(\beta)$, has a unique root. Therefore, in this case there is a unique *SLE.* ■

Proof of Corollary 2.2: When $\rho = 0$, we have

$$P(\beta) = \tau_\varepsilon^2\left(-\tau_\eta\gamma\beta^2 + \beta - \mu\gamma\tau_\varepsilon\right).$$

Notice that $P(\beta)$ is a polynomial of degree 2 in β. Therefore, we can explicitly compute the roots of this polynomial. If $1 < 4\tau_\varepsilon\tau_\eta\mu\gamma^2$, this polynomial has no real roots and, hence, we conclude that a *SLE* does not exist. Otherwise, that is, if $1 \geq 4\tau_\varepsilon\tau_\eta\mu\gamma^2$, then β_1 and β_2 are the roots $P(\beta)$, whose expressions are given in the statement of this Corollary. ■

Proof of Corollary 2.3: When $\rho = 1$, $P(\beta)$ is a polynomial of degree 3 in β given by

$$P(\beta) = \beta^3 (\tau_u + \tau_\eta) - \beta^2 \tau_\varepsilon \gamma (\tau_\eta + \mu\tau_u) + \beta\tau_\varepsilon - \mu\gamma\tau_\varepsilon^2.$$

Derivating, we have

$$P'(\beta) = 3\beta^2 (\tau_u + \tau_\eta) - 2\beta\tau_\varepsilon\gamma (\tau_\eta + \mu\tau_u) + \tau_\varepsilon.$$

We distinguish 3 cases:

Case 1: $(\tau_\eta + \mu\tau_u)^2 \gamma^2 \tau_\varepsilon < 3 (\tau_u + \tau_\eta)$.

In this case $P'(\beta) > 0$ for all β. Hence, $P(\beta)$ is strictly increasing in β. Moreover, $P(0) < 0$ and $\underset{\beta\to+\infty}{lim} P(\beta) = +\infty$. Therefore, we conclude that there is a unique root of $P(\beta)$, and consequently, a unique *SLE*.

Case 2: $(\tau_\eta + \mu\tau_u)^2 \gamma^2 \tau_\varepsilon = 3 (\tau_u + \tau_\eta)$.

In this case $P'(\beta)$ has a unique root, denoted by β^*, and $P'(\beta) > 0$ whenever $\beta \neq \beta^*$. Again, $P(0) < 0$ and $\underset{\beta\to+\infty}{lim} P(\beta) = +\infty$. Therefore, we conclude that there is a unique root of $P(\beta)$, and consequently, a unique *SLE*.

Case 3: $(\tau_\eta + \mu\tau_u)^2 \gamma^2 \tau_\varepsilon > 3 (\tau_u + \tau_\eta)$.

In this case $P'(\beta)$ has two real roots

$$\overline{\beta}_1 = \frac{\tau_\varepsilon\gamma(\tau_\eta + \mu\tau_u) - \sqrt{\tau_\varepsilon(\tau_\varepsilon\gamma^2 (\tau_\eta + \mu\tau_u)^2 - 3(\tau_u + \tau_\eta))}}{3(\tau_u + \tau_\eta)} \text{ and}$$
$$\overline{\beta}_2 = \frac{\tau_\varepsilon\gamma(\tau_\eta + \mu\tau_u) + \sqrt{\tau_\varepsilon(\tau_\varepsilon\gamma^2 (\tau_\eta + \mu\tau_u)^2 - 3(\tau_u + \tau_\eta))}}{3(\tau_u + \tau_\eta)},$$

where $\overline{\beta}_1$ is a local maximum and $\overline{\beta}_2$ is a local minimum of $P(\beta)$. Direct computations yield:

$$P(\overline{\beta}_1) = -\frac{\tau_\varepsilon^{3/2} f_1(\tau_\varepsilon)}{27(\tau_u + \tau_\eta)^2} \text{ and } P(\overline{\beta}_2) = -\frac{\tau_\varepsilon^{3/2} f_2(\tau_\varepsilon)}{27(\tau_u + \tau_\eta)^2},$$

where

$$f_1(\tau_\varepsilon) = \tau_\varepsilon^{1/2}\gamma \left(9(\tau_u + \tau_\eta)(2\mu\tau_u - \tau_\eta + 3\mu\tau_\eta) + \tau_\varepsilon 2\gamma^2 (\tau_\eta + \mu\tau_u)^3\right)$$
$$-2\left(\tau_\varepsilon\gamma^2 (\tau_\eta + \mu\tau_u)^2 - 3(\tau_u + \tau_\eta)\right)^{\frac{3}{2}}$$

and

$$f_2(\tau_\varepsilon) = \tau_\varepsilon^{1/2}\gamma \left(9(\tau_u + \tau_\eta)(2\mu\tau_u - \tau_\eta + 3\mu\tau_\eta) + \tau_\varepsilon 2\gamma^2 (\tau_\eta + \mu\tau_u)^3\right)$$
$$+2\left((\tau_\varepsilon\gamma^2 (\tau_\eta + \mu\tau_u)^2 - 3(\tau_u + \tau_\eta)\right)^{3/2}.$$

Moreover, we have $f_1(\tau_\varepsilon) < f_2(\tau_\varepsilon)$ whenever $\tau_\varepsilon > \frac{3(\tau_u+\tau_\eta)}{(\tau_\eta+\mu\tau_u)^2\gamma^2}$, $\frac{d}{d\tau_\varepsilon}f_1(\tau_\varepsilon) > 0$ and $\frac{d}{d\tau_\varepsilon}f_2(\tau_\varepsilon) > 0$ whenever $\tau_\varepsilon > \frac{3(\tau_u+\tau_\eta)}{(\tau_\eta+\mu\tau_u)^2\gamma^2}$, and

$$f_1\left(\frac{3(\tau_u+\tau_\eta)}{(\tau_\eta+\mu\tau_u)^2\gamma^2}\right) = f_2\left(\frac{3(\tau_u+\tau_\eta)}{(\tau_\eta+\mu\tau_u)^2\gamma^2}\right) = \frac{(3(\tau_u+\tau_\eta))^{3/2}}{(\tau_\eta+\mu\tau_u)}(8\mu\tau_u - \tau_\eta + 9\mu\tau_\eta).$$

Thus, we distinguish 2 subcases: 3.1) $8\mu\tau_u - \tau_\eta + 9\mu\tau_\eta \geq 0$, and 3.2) $8\mu\tau_u - \tau_\eta + 9\mu\tau_\eta < 0$.

Subcase 3.1: $8\mu\tau_u - \tau_\eta + 9\mu\tau_\eta \geq 0$

If $\mu \geq \frac{\tau_\eta}{8\tau_u+9\tau_\eta}$, then $f_1(\tau_\varepsilon) > 0$ and $f_2(\tau_\varepsilon) > 0$ whenever $\tau_\varepsilon > \frac{3(\tau_u+\tau_\eta)}{(\tau_\eta+\mu\tau_u)^2\gamma^2}$, which implies $P(\overline{\beta}_1) < 0$ and $P(\overline{\beta}_2) < 0$. The fact that $P(\overline{\beta}_1) < 0$ allows us to conclude that there is a unique *SLE*.

Subcase 3.2: $8\mu\tau_u - \tau_\eta + 9\mu\tau_\eta < 0$

If $\mu < \frac{\tau_\eta}{8\tau_u+9\tau_\eta}$, then $f_1(\frac{3(\tau_u+\tau_\eta)}{(\tau_\eta+\mu\tau_u)^2\gamma^2}) = f_2(\frac{3(\tau_u+\tau_\eta)}{(\tau_\eta+\mu\tau_u)^2\gamma^2}) < 0$. The fact that $f_1(\tau_\varepsilon)$ and $f_2(\tau_\varepsilon)$ are strictly increasing in τ_ε whenever $\tau_\varepsilon > \frac{3(\tau_u+\tau_\eta)}{(\tau_\eta+\mu\tau_u)^2\gamma^2}$, $f_1(\tau_\varepsilon) < f_2(\tau_\varepsilon)$, and that $\lim_{\tau_\varepsilon\to\infty} f_1(\tau_\varepsilon) = \lim_{\tau_\varepsilon\to\infty} f_2(\tau_\varepsilon) = \infty$, implies that $\exists\ \overline{\tau}_\varepsilon$ and $\overline{\overline{\tau}}_\varepsilon$, with $\overline{\overline{\tau}}_\varepsilon < \overline{\tau}_\varepsilon$ and $\overline{\tau}_\varepsilon, \overline{\overline{\tau}}_\varepsilon \in \left(\frac{3(\tau_u+\tau_\eta)}{(\tau_\eta+\mu\tau_u)^2\gamma^2}, \infty\right)$, such that

- for all τ_ε, $\tau_\varepsilon < \overline{\overline{\tau}}_\varepsilon$, $f_2(\tau_\varepsilon) < 0$, and hence, $P(\overline{\beta}_2) > 0$. As $\overline{\beta}_2$ is the local minimum of $P(\beta)$, this inequality implies that in this case there exists a unique *SLE*.
- when $\tau_\varepsilon = \overline{\overline{\tau}}_\varepsilon$, $f_2(\tau_\varepsilon) = 0$ and $f_1(\tau_\varepsilon) < 0$, and hence, $P(\overline{\beta}_1) > 0$ and $P(\overline{\beta}_2) = 0$. This implies that there exist two *SLE*, β_1 and β_2, such that $0 < \beta_1 < \overline{\beta}_1$ and $\beta_2 = \overline{\beta}_2$.
- for all τ_ε, $\overline{\overline{\tau}}_\varepsilon < \tau_\varepsilon < \overline{\tau}_\varepsilon$, $f_2(\tau_\varepsilon) > 0$ and $f_1(\tau_\varepsilon) < 0$, or equivalently, $P(\overline{\beta}_2) < 0$ and $P(\overline{\beta}_1) > 0$. This implies that in this case there exist three *SLE*, β_1, β_2 and β_3, such that $\beta_1 < \overline{\beta}_1 < \beta_2 < \overline{\beta}_2 < \beta_3$.
- when $\tau_\varepsilon = \overline{\tau}_\varepsilon$, $f_2(\tau_\varepsilon) > 0$ and $f_1(\tau_\varepsilon) = 0$, and hence, $P(\overline{\beta}_1) = 0$ and $P(\overline{\beta}_2) < 0$. This implies that there exists two *SLE*, β_1 and β_2, $\beta_1 = \overline{\beta}_1$ and $\beta_2 > \overline{\beta}_2$.
- for all τ_ε, $\tau_\varepsilon > \overline{\tau}_\varepsilon$, $f_2(\tau_\varepsilon) > 0$ and $f_1(\tau_\varepsilon) > 0$, and hence, $P(\overline{\beta}_2) < 0$ and $P(\overline{\beta}_1) < 0$. This implies that in this case there exists a unique *SLE*. ■

In the proof of Corollary 2.4 we use the following result:

Lemma A.2: Let $g(\rho) = -27(\tau_u+\tau_\eta)^2\rho^2 + \rho\tau_\eta\left(9\tau_u + 9\tau_\eta + 2\gamma^2\tau_\eta^2\tau_\varepsilon\right) - 2\gamma^2\tau_\eta^3\tau_\varepsilon$.

If $\frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)} \leq \tau_\varepsilon\gamma^2$, then $g(\rho) < 0$ for all $0 < \rho < \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}$.

If $\tau_\varepsilon\gamma^2 < \frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)}$, then $\begin{cases} g(\rho) < 0 \text{ for all } \rho < \rho_1, \text{ and} \\ g(\rho) \geq 0 \text{ for all } \rho_1 \leq \rho < \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}, \end{cases}$, where ρ_1 is the smallest root of $g(\rho)$.

Proof of Lemma A.2: Notice that $g(0) = -2\tau_\varepsilon \tau_\eta^3 \gamma^2 < 0$ and $g(\frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}) = \frac{3(\tau_u+\tau_\eta)\gamma^2\tau_\eta^3\tau_\varepsilon\left(3(\tau_u+\tau_\eta)-\tau_\varepsilon\tau_\eta\gamma^2(9\tau_u+8\tau_\eta)\right)}{\left(3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon\right)^2}$. Moreover,

$$\left(\tau_\eta\left(9\tau_u+9\tau_\eta+2\gamma^2\tau_\eta^2\tau_\varepsilon\right)\right)^2 - 4\left(-27\left(\tau_u+\tau_\eta\right)^2\right)\left(-2\gamma^2\tau_\eta^3\tau_\varepsilon\right) = \tau_\eta^2 q(\tau_\varepsilon\gamma^2),$$

where $q(x) = 4\tau_\eta^4 x^2 - 36\tau_\eta\left(\tau_u+\tau_\eta\right)\left(6\tau_u+5\tau_\eta\right)x + 81\left(\tau_u+\tau_\eta\right)^2$. Operating, we have that the roots of $q(x)$ are given by:

$$\begin{aligned} x_1 &= \frac{18\left(\tau_u+\tau_\eta\right)}{4\tau_\eta^3}\left(\left(6\tau_u+5\tau_\eta\right) - 2\sqrt{3}\sqrt{\left(3\tau_u+2\tau_\eta\right)\left(\tau_u+\tau_\eta\right)}\right) \text{ and} \\ x_2 &= \frac{18\left(\tau_u+\tau_\eta\right)}{4\tau_\eta^3}\left(\left(6\tau_u+5\tau_\eta\right) + 2\sqrt{3}\sqrt{\left(3\tau_u+2\tau_\eta\right)\left(\tau_u+\tau_\eta\right)}\right). \end{aligned}$$

We distinguish three cases: 1) $\tau_\varepsilon\gamma^2 \leq x_1$, 2) $x_1 < \tau_\varepsilon\gamma^2 < x_2$, and 3) $\tau_\varepsilon\gamma^2 \geq x_2$.

- **Case 1:** $\tau_\varepsilon\gamma^2 \leq x_1$.

In this case $q(\tau_\varepsilon\gamma^2) \geq 0$, and hence, we know that $g(\rho)$ has real roots given by

$$\begin{aligned} \rho_1 &= \frac{\tau_\eta\left(9\left(\tau_u+\tau_\eta\right)+2\gamma^2\tau_\eta^2\tau_\varepsilon - \sqrt{q(\tau_\varepsilon\gamma^2)}\right)}{54\left(\tau_u+\tau_\eta\right)^2} \text{ and} \\ \rho_2 &= \frac{\tau_\eta\left(9\left(\tau_u+\tau_\eta\right)+2\gamma^2\tau_\eta^2\tau_\varepsilon + \sqrt{q(\tau_\varepsilon\gamma^2)}\right)}{54\left(\tau_u+\tau_\eta\right)^2}. \end{aligned}$$

Next, we compare $\frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}$ with ρ_1 and ρ_2. Notice that

$$\rho_1 - \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3\left(\tau_u+\tau_\eta\right)+\gamma^2\tau_\eta^2\tau_\varepsilon} = \frac{\tau_\eta\left(r(\tau_\varepsilon\gamma^2) - \left(3\left(\tau_u+\tau_\eta\right)+\gamma^2\tau_\eta^2\tau_\varepsilon\right)\sqrt{q(\tau_\varepsilon\gamma^2)}\right)}{54\left(3\left(\tau_u+\tau_\eta\right)+\gamma^2\tau_\eta^2\tau_\varepsilon\right)\left(\tau_u+\tau_\eta\right)^2},$$

where $r(x) = 27\left(\tau_u+\tau_\eta\right)^2 - 3x\tau_\eta\left(\tau_u+\tau_\eta\right)\left(18\tau_u+13\tau_\eta\right) + 2x^2\tau_\eta^4$. It is easy to see that $r(x)$ is strictly decreasing whenever $x \in [0, x_1]$ and $r(x_1) > 0$. Hence, we have that $r(\tau_\varepsilon\gamma^2) > 0$ whenever $\tau_\varepsilon\gamma^2 \leq x_1$. Direct computations yield that $\rho_1 \geq \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}$ if and only if $\tau_\varepsilon\gamma^2 \geq \frac{3\tau_\eta+3\tau_u}{\tau_\eta(9\tau_u+8\tau_\eta)}$. On the other hand, we obtain that

$$\rho_2 - \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3\left(\tau_u+\tau_\eta\right)+\gamma^2\tau_\eta^2\tau_\varepsilon} = \frac{\tau_\eta\left(r(\tau_\varepsilon\gamma^2) + \left(3\tau_u+3\tau_\eta+\gamma^2\tau_\eta^2\tau_\varepsilon\right)\sqrt{q(\tau_\varepsilon\gamma^2)}\right)}{54\left(3\left(\tau_u+\tau_\eta\right)+\gamma^2\tau_\eta^2\tau_\varepsilon\right)\left(\tau_u+\tau_\eta\right)^2} > 0,$$

since $r(\tau_\varepsilon\gamma^2) \geq 0$ in this case. Thus, it holds $\rho_2 > \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}$. Therefore, we have that if $\tau_\varepsilon\gamma^2 \geq \frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)}$, then $\rho_1 \geq \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}$, and consequently, $g(\rho) < 0$ for all $0 < \rho < \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}$. Otherwise, that is, when $\tau_\varepsilon\gamma^2 < \frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)}$, $\rho_1 < \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon} < \rho_2$ and, hence, we have that $g(\rho) < 0$ for all $\rho < \rho_1$, and $g(\rho) \geq 0$ for all $\rho_1 \leq \rho < \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}$.

- **Case 2:** $x_1 < \tau_\varepsilon\gamma^2 < x_2$

In this case $q(\tau_\varepsilon\gamma^2) < 0$, and hence, we know that $g(\rho)$ doesn't have real roots. As $g(0) < 0$, we conclude that $g(\rho) < 0$ for all ρ.

- **Case 3:** $\tau_\varepsilon\gamma^2 \geq x_2$

In this case $q(\tau_\varepsilon\gamma^2) \geq 0$, and hence, we know that $g(\rho)$ has real roots, ρ_1 and ρ_2. Next, we compare $\frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}$ with ρ_1 and ρ_2. Recall that

$$\rho_1 - \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon} = \frac{\tau_\eta\left(r(\tau_\varepsilon\gamma^2) - \left(3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon\right)\sqrt{q(\tau_\varepsilon\gamma^2)}\right)}{54\left(3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon\right)(\tau_u+\tau_\eta)^2}.$$

It is easy to see that $r(x)$ is strictly increasing whenever $x > x_2$ and $r(x_2) > 0$. Hence, we have that $r(\tau_\varepsilon\gamma^2) > 0$ whenever $\tau_\varepsilon\gamma^2 \geq x_2$. Performing the same computations as in Case 1 $\rho_1 \geq \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}$ if and only if $\tau_\varepsilon\gamma^2 \geq \frac{3\tau_\eta+3\tau_u}{\tau_\eta(9\tau_u+8\tau_\eta)}$. This inequality is satisfied whenever $\tau_\varepsilon\gamma^2 \geq x_2$. Therefore, we conclude that in this case $g(\rho) < 0$ for all $\rho < \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}$. ■

Proof of Corollary 2.4: In this case $P(\beta;\rho) = \left(\tau_\varepsilon + \beta^2\rho(\tau_u+\tau_\eta)\right) f(\beta,\rho)$, where

$$f(\beta;\rho) = \beta^3\rho(\tau_u+\tau_\eta)(1-\rho) - \beta^2\tau_\varepsilon\tau_\eta\gamma(1-\rho) + \beta\tau_\varepsilon - \gamma\tau_\varepsilon^2.$$

We focus on the case $\tau_\varepsilon > 0$. In this case, the roots of $P(\beta;\rho)$ and the roots of $f(\beta;\rho)$ coincide.

$$\rho = 0$$

Evaluating, we have $f(\beta;0) = \tau_\varepsilon\left(-\gamma\tau_\varepsilon + \beta - \beta^2\gamma\tau_\eta\right)$. Hence, we have if $-4\gamma^2\tau_\eta\tau_\varepsilon + 1 < 0$, $P(\beta,0)$ hasn't real roots, and consequently, there isn't any *SLE*. If $-4\gamma^2\tau_\eta\tau_\varepsilon + 1 \geq 0$, then the real roots are given by:

$$\beta_1 = \frac{1}{\gamma\tau_\eta}\left(-\frac{1}{2}\sqrt{-4\gamma^2\tau_\eta\tau_\varepsilon+1} + \frac{1}{2}\right) \text{ and } \beta_2 = \frac{1}{\gamma\tau_\eta}\left(\frac{1}{2}\sqrt{-4\gamma^2\tau_\eta\tau_\varepsilon+1} + \frac{1}{2}\right).$$

$$\rho = 1$$

Evaluating, we have $f(\beta;1) = \tau_\varepsilon(-\gamma\tau_\varepsilon + \beta)$. Therefore, there is a unique real root: $\beta = \gamma\tau_\varepsilon$.

$$0 < \rho < 1$$

Derivating we have

$$f'(\beta;\rho) = \beta^2 3\rho\,(\tau_u + \tau_\eta)\,(1-\rho) - \beta 2\tau_\varepsilon\tau_\eta\gamma\,(1-\rho) + \tau_\varepsilon.$$

Notice that
$$(2\tau_\varepsilon\tau_\eta\gamma\,(1-\rho))^2 - 4\,(3\rho\,(\tau_u+\tau_\eta)\,(1-\rho))\,\tau_\varepsilon = 4\tau_\varepsilon\,(1-\rho)\left(\gamma^2\tau_\eta^2\tau_\varepsilon - \rho\left(3\,(\tau_u+\tau_\eta) + \gamma^2\tau_\eta^2\tau_\varepsilon\right)\right).$$

Hence, we distinguish 3 cases: 1) $\rho > \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}$, 2) $\rho = \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}$ and 3) $\rho < \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}$.

CASE 1: $\rho > \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}$.

In this case $f'(\beta;\rho) > 0$ for all β. Hence, $f(\beta;\rho)$ is strictly increasing in β. Moreover, $f(0;\rho) = -\gamma\tau_\varepsilon^2 < 0$ and $\lim_{\beta\to\infty} f(\beta;\rho) = +\infty$. Therefore, we conclude that there is a unique root of $f(\beta;\rho)$, and consequently, a unique *SLE*.

CASE 2: $\rho = \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}$.

In this case $f'(\beta;\rho)$ has a unique root, $\beta = \frac{\tau_\varepsilon\tau_\eta\gamma}{3\rho(\tau_u+\tau_\eta)}$ and $f'(\beta;\rho) > 0$ whenever $\beta \neq \frac{\tau_\varepsilon\tau_\eta\gamma}{3\rho(\tau_u+\tau_\eta)}$. Again, $f(0;\rho) = -\gamma\tau_\varepsilon^2 < 0$ and $\lim_{\beta\to\infty} f(\beta;\rho) = +\infty$. Therefore, we conclude that there is a unique root of $f(\beta;\rho)$, and consequently, a unique *SLE*.

CASE 3: $\rho < \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}$.

In this case $f'(\beta;\rho)$ has two real roots given by:

$$\widehat{\beta}_1 = \frac{\tau_\varepsilon\tau_\eta\gamma\,(1-\rho) - \sqrt{\tau_\varepsilon\,(1-\rho)\left(-3\rho\,(\tau_u+\tau_\eta) + \tau_\varepsilon\tau_\eta^2\gamma^2\,(1-\rho)\right)}}{3\rho\,(\tau_u+\tau_\eta)\,(1-\rho)} \text{ and}$$
$$\widehat{\beta}_2 = \frac{\tau_\varepsilon\tau_\eta\gamma\,(1-\rho) + \sqrt{\tau_\varepsilon\,(1-\rho)\left(-3\rho\,(\tau_u+\tau_\eta) + \tau_\varepsilon\tau_\eta^2\gamma^2\,(1-\rho)\right)}}{3\rho\,(\tau_u+\tau_\eta)\,(1-\rho)}.$$

Notice that $\widehat{\beta}_1$ is a local maximum of $f(\beta;\rho)$ and $\widehat{\beta}_2$ is a local minimum of $f(\beta;\rho)$. In particular, we have

- if $f(\widehat{\beta}_1,\rho) < 0$, there is a unique root of $f(\beta;\rho)$;
- if $f(\widehat{\beta}_1,\rho) = 0$, there are two roots of $f(\beta;\rho)$;
- if $f(\widehat{\beta}_1,\rho) > 0$ and $f(\widehat{\beta}_2,\rho) < 0$, there are three roots of $f(\beta;\rho)$;
- if $f(\widehat{\beta}_1,\rho) > 0$ and $f(\widehat{\beta}_2,\rho) = 0$, there are two roots of $f(\beta;\rho)$;
- if $f(\widehat{\beta}_2,\rho) > 0$, there is a unique root of $f(\beta;\rho)$.

Direct computations yield

$$f(\widehat{\beta}_1;\rho) = \frac{\tau_\varepsilon^2\gamma\left(\rho-1\right)^2 g(\rho) + 2\left(\tau_\varepsilon\left(\rho-1\right)\left(\rho\left(3\left(\tau_u+\tau_\eta\right)+\gamma^2\tau_\eta^2\tau_\varepsilon\right)-\gamma^2\tau_\eta^2\tau_\varepsilon\right)\right)^{3/2}}{27\left(\rho-1\right)^2\left(\tau_u+\tau_\eta\right)^2\rho^2} \text{ and}$$

$$f(\widehat{\beta}_2;\rho) = \frac{\tau_\varepsilon^2\gamma\left(\rho-1\right)^2 g(\rho) - 2\left(\tau_\varepsilon\left(\rho-1\right)\left(\rho\left(3\left(\tau_u+\tau_\eta\right)+\gamma^2\tau_\eta^2\tau_\varepsilon\right)-\gamma^2\tau_\eta^2\tau_\varepsilon\right)\right)^{3/2}}{27\left(\rho-1\right)^2\left(\tau_u+\tau_\eta\right)^2\rho^2},$$

where $g(\rho) = -27\left(\tau_u+\tau_\eta\right)^2\rho^2 + \rho\tau_\eta\left(9\tau_u+9\tau_\eta+2\gamma^2\tau_\eta^2\tau_\varepsilon\right) - 2\gamma^2\tau_\eta^3\tau_\varepsilon$.

Next, taking into account Lemma A.2, we consider two subcases: 3.1) $\frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)} \le \tau_\varepsilon\gamma^2$, and 3.2) $\tau_\varepsilon\gamma^2 < \frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)}$.

CASE 3.1: $\frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)} \le \tau_\varepsilon\gamma^2$

Lemma A.2 provides that in this case $g(\rho) < 0$ for all $\rho < \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}$. Next, we show that in this case $f(\widehat{\beta}_1,\rho) < 0$, and hence, we conclude that there exists a unique *SLE*. Direct computations yield that $f(\widehat{\beta}_1;\rho) < 0$ is equivalent to showing $h(\tau_\varepsilon\gamma^2) < 0$ whenever $\frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)} \le \tau_\varepsilon\gamma^2$, with

$$h(x) = -4\tau_\eta^3\left(\rho-1\right)^2 x^2 + x\left(1-\rho\right)\left(-\rho^2 27\left(\tau_u+\tau_\eta\right)^2 + 18\tau_\eta\left(\tau_u+\tau_\eta\right)\rho + \tau_\eta^2\right) - 4\left(\tau_u+\tau_\eta\right)\rho.$$

Operating, we get

$$h\left(\frac{3\left(\tau_u+\tau_\eta\right)}{\tau_\eta\left(9\tau_u+8\tau_\eta\right)}\right) = \frac{\left(9\rho\left(\tau_u+\tau_\eta\right)-\tau_\eta\right)^2\left(\rho\left(9\tau_u+8\tau_\eta\right)-12\tau_\eta-9\tau_u\right)\left(\tau_u+\tau_\eta\right)}{\left(9\tau_u+8\tau_\eta\right)^2\tau_\eta},$$

$h'\left(\frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)}\right) = \frac{\left(\rho\left(51\tau_u\tau_\eta+27\tau_u^2+24\tau_\eta^2\right)-16\tau_\eta^2-15\tau_u\tau_\eta\right)(9(\tau_u+\tau_\eta)\rho-\tau_\eta)(\rho-1)}{(9\tau_u+8\tau_\eta)}$ and $h''(x) < 0$ for all x.

We distinguish 3 subcases: 3.1.1) $\rho \ge \frac{\tau_\eta}{\tau_u+\tau_\eta}$, 3.1.2) $\frac{\tau_\eta}{9(\tau_u+\tau_\eta)} < \rho < \frac{\tau_\eta}{\tau_u+\tau_\eta}$, and 3.1.3) $\rho \le \frac{\tau_\eta}{9(\tau_u+\tau_\eta)}$.

Subcase 3.1.1: If $\rho \ge \frac{\tau_\eta}{\tau_u+\tau_\eta}$, then $h'\left(\frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)}\right) < 0$. As $h\prime(x)$ is strictly decreasing, we conclude that $h'(\tau_\varepsilon\gamma^2) < 0$ whenever $\frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)} \le \tau_\varepsilon\gamma^2$, and therefore, $h(x)$ is strictly decreasing whenever $\frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)} \le \tau_\varepsilon\gamma^2$. The fact that $h\left(\frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)}\right) < 0$ implies that $h(\tau_\varepsilon\gamma^2) < 0$ whenever $\frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)} \le \tau_\varepsilon\gamma^2$.

Subcase 3.1.2: If $\frac{\tau_\eta}{9(\tau_u+\tau_\eta)} < \rho < \frac{\tau_\eta}{\tau_u+\tau_\eta}$, $h(x)$ does not have real roots. As, $h\left(\frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)}\right) < 0$, we have that in this case $h(\tau_\varepsilon\gamma^2) < 0$ whenever $\frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)} \le \tau_\varepsilon\gamma^2$.

Subcase 3.1.3: If $\rho \le \frac{\tau_\eta}{9(\tau_u+\tau_\eta)}$, then $h'\left(\frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)}\right) \le 0$. Doing a similar reasoning as in Case 1, we obtain that $h(\tau_\varepsilon\gamma^2) < 0$ whenever $\frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)} \le \tau_\varepsilon\gamma^2$.[5]

[5] In the particular case $\tau_\varepsilon\gamma^2 = \frac{3\left(\tau_u+\tau_\eta\right)}{\tau_\eta\left(9\tau_u+8\tau_\eta\right)}$, we are considering $\rho < \frac{\tau_\eta}{9\left(\tau_u+\tau_\eta\right)}$, and hence, $h\left(\frac{3\left(\tau_u+\tau_\eta\right)}{\tau_\eta\left(9\tau_u+8\tau_\eta\right)}\right) < 0$.

CASE 3.2: $\tau_\varepsilon\gamma^2 < \frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)}$

We divide this case in two subcases: 3.2.1) $0 < \rho < \rho_1$, and 3.2.2) $\rho_1 \leq \rho < \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{\left(3\tau_u+3\tau_\eta+\gamma^2\tau_\eta^2\tau_\varepsilon\right)}$.

Case 3.2.1: $0 < \rho < \rho_1$.

Lemma A.2 tells us that in this case $g(\rho) < 0$. Hence, $f(\widehat{\beta}_2;\rho) < 0$, where $\widehat{\beta}_2$ is the local minimum of $f(\beta;\rho)$. Now, we want to study the sign of $f(\widehat{\beta}_1;\rho)$. Direct computations yield that $f(\widehat{\beta}_1,\rho) \geq 0$ if and only if $p(\rho) \geq 0$, with

$p(\rho) = \rho^3 27\tau_\varepsilon\gamma^2\left(\tau_u+\tau_\eta\right)^2 - \rho^2\tau_\varepsilon\gamma^2\left(72\tau_u\tau_\eta + 27\tau_u^2 + 45\tau_\eta^2 + 4\gamma^2\tau_\eta^3\tau_\varepsilon\right) + \rho\left(8\gamma^4\tau_\eta^3\tau_\varepsilon^2 + \tau_\varepsilon\tau_\eta\gamma^2\left(18\tau_u+17\tau_\eta\right) - 4\tau_\eta - 4\tau_u\right) + \tau_\varepsilon\tau_\eta^2\gamma^2\left(1-4\gamma^2\tau_\eta\tau_\varepsilon\right).$

Direct computations yield $p''(\rho) < 0$ whenever $0 < \rho < \rho_1$ and $\tau_\varepsilon\gamma^2 < \frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)}$. Therefore, $p(\rho)$ is a concave function in this interval. Moreover, using the fact that ρ_1 is a root of $g(\rho)$, we have that

$p(\rho_1) = \frac{\left(2\left(\tau_\varepsilon(\rho_1-1)\left(3\rho_1\tau_u+3\rho_1\tau_\eta-\gamma^2\tau_\eta^2\tau_\varepsilon+\rho_1\gamma^2\tau_\eta^2\tau_\varepsilon\right)\right)^{3/2}\right)^2}{27\tau_\varepsilon^3\rho_1^2(1-\rho_1)^3(\tau_u+\tau_\eta)^2} > 0$. We have two possibilities: a) $1-4\gamma^2\tau_\eta\tau_\varepsilon \geq 0$, and b) $1-4\gamma^2\tau_\eta\tau_\varepsilon < 0$.

Subcase a: $1-4\gamma^2\tau_\eta\tau_\varepsilon \geq 0$. Direct computations yield $p(0) \geq 0$. Combining the fact that $p(\rho)$ is strictly concave in the interval $(0,\rho_1)$, the previous inequality and $p(\rho_1) > 0$, we can conclude that $p(\rho) > 0$ for all $\rho \in (0,\rho_1)$. Hence, $f(\widehat{\beta}_1;\rho) > 0$ for all $\rho \in (0,\rho_1)$, which implies that $f(\beta;\rho)$ has three positive roots and, consequently, there are three *SLE* in this case.

Subcase b: $1-4\gamma^2\tau_\eta\tau_\varepsilon < 0$. In this case $p(0) < 0$. Combining the fact that $p(\rho)$ is strictly concave in the interval $(0,\rho_1)$, the previous inequality and $p(\rho_1) > 0$, we can conclude that there exists a value $\widehat{\rho} \in (0,\rho_1)$ such that $p(\rho) < 0$ whenever $\rho \in (0,\widehat{\rho})$, $p(\widehat{\rho}) = 0$ and $p(\rho) > 0$ whenever $\rho \in (\widehat{\rho},\rho_1)$. Hence, we have $f(\widehat{\beta}_1;\rho) < 0$ whenever $\rho \in (0,\widehat{\rho})$, $f(\widehat{\beta}_1;\widehat{\rho}) = 0$ and $f(\widehat{\beta}_1;\widehat{\rho}) > 0$ whenever $\rho \in (\widehat{\rho},\rho_1)$. Therefore, we can conclude that there exists a unique *SLE* whenever $\rho \in (0,\widehat{\rho})$, there are two *SLE* $\rho = \widehat{\rho}$ and there are three *SLE* whenever $\rho \in (\widehat{\rho},\rho_1)$.

Case 3.2.2: $\rho_1 \leq \rho < \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}$.

Lemma A.2 tells us that in this case $g(\rho) \geq 0$. Hence, $f(\widehat{\beta}_1;\rho) > 0$, where recall that $\widehat{\beta}_1$ is the local maximum of $f(\beta;\rho)$. Now, we want to study the sign of $f(\widehat{\beta}_2;\rho)$. Using the expression of $f(\widehat{\beta}_2;\rho)$, we have that $f(\widehat{\beta}_2;\rho) < 0$ if and only if $p(\rho) > 0$. Recall that $p(\rho_1) > 0$. Moreover, $p(\frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}) = -\frac{\left(\tau_\varepsilon\tau_\eta\gamma^2(9\tau_u+8\tau_\eta)-3\tau_\eta-3\tau_u\right)^2(\tau_u+\tau_\eta)\gamma^2\tau_\eta^2\tau_\varepsilon}{\left(3\tau_u+3\tau_\eta+\gamma^2\tau_\eta^2\tau_\varepsilon\right)^3} < 0$. Direct computations yield that $p''(\rho) < 0$ whenever $\rho_1 \leq \rho < \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3\tau_u+3\tau_\eta+\gamma^2\tau_\eta^2\tau_\varepsilon}$ and $\tau_\varepsilon\gamma^2 < \frac{3(\tau_u+\tau_\eta)}{\tau_\eta(9\tau_u+8\tau_\eta)}$. Therefore, $p(\rho)$ is a strictly concave function in $\left[\rho_1, \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}\right)$. Hence, we obtain that there exists a value $\widehat{\widehat{\rho}} \in \left[\rho_1, \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}\right)$ such that $p(\rho) > 0$ whenever $\rho \in \left[\rho_1, \widehat{\widehat{\rho}}\right)$, $p\left(\widehat{\widehat{\rho}}\right) = 0$ and $p(\rho) > 0$ whenever $\rho \in \left(\widehat{\widehat{\rho}}, \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}\right)$.

Hence, we have $f(\widehat{\beta}_2;\rho) < 0$ whenever $\rho \in \left[\rho_1, \widehat{\widehat{\rho}}\right)$, $f(\widehat{\beta}_2;\widehat{\widehat{\rho}}) = 0$ and $f(\widehat{\beta}_2;\widehat{\widehat{\rho}}) > 0$ whenever $\rho \in \left(\widehat{\widehat{\rho}}, \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}\right)$. Therefore, we can conclude that there are three *SLE* whenever $\rho \in \left[\rho_1, \widehat{\widehat{\rho}}\right)$, there are two *SLE* $\rho = \widehat{\widehat{\rho}}$ and there is one *SLE* whenever $\rho \in \left(\widehat{\widehat{\rho}}, \frac{\gamma^2\tau_\eta^2\tau_\varepsilon}{3(\tau_u+\tau_\eta)+\gamma^2\tau_\eta^2\tau_\varepsilon}\right)$. ■

References

[1] Admati, A. R. (1985): "A noisy rational expectations equilibrium for multi-asset securities markets." Econometrica 53, 629-657.

[2] Barlevy, G. and P. Veronesi (2000):"Information acquisition in financial markets", Review of Economic Studies 67, 79-90.

[3] Barlevy, G. and P. Veronesi (2007): "Information acquisition in financial markets: a correction", Chicago Fed Working Paper 2007-06.

[4] Chamley, C. (2008a): "On acquisition of information in financial markets", Review of Economic Studies 75, 1081-1084.

[5] Chamley, C. (2008b), Comment on equilibria on Ganguli and Yang (2006, 2007), Mimeo, Boston University.

[6] Danthine, J. P. and S. Moresi (1993): "Volatility, information, and noise trading", European Economic Review 37, 961–982.

[7] Diamond, D. W. and R. E. Verrecchia (1981): "Information aggregation in a noisy rational expectations economy." Journal of Financial Economics 9, 221-235.

[8] Ganguli, J. V. and L. Yang (2009): "Complementarities, multiplicity, and supply information", Journal of the European Economic Association 7, 90-115.

[9] Grossman, S. and J. Stiglitz (1980): "On the impossibility of informationally efficient markets", American Economic Review 70, 393-408.

[10] Heinemann, M. (2009): "Stability under learning of equilibria in financial markets with supply information", Working Paper 122, University of Lüneburg.

[11] Hellwig, M. F. (1980): "On the efficiency of competitive stock markets where traders have diverse information", Journal of Economic Theory 22, 477-498.

[12] Hellwig, C. and L. Veldkamp (2009): "Knowing what others know: Coordination motives in information acquisition", Review of Economic Studies, 76, 223-251.

[13] Lundholm, R. J. (1988): "Price-signal relations in the presence of correlated public and private information", Journal of Accounting Research 26, 107-118.

[14] Vives, X. (2008), Information and learning in markets, Princeton: Princeton University Press.