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A way to play bankruptcy problems.

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Abstract

The commitment among agents has always been a difficult task, especially when they have to decide how to distribute the available amount of a scarce resource among all. On the one hand, there are a multiplicity of possible ways for assigning the available amount; and, on the other hand, each agent is going to propose that distribution which provides her the highest possible award. In this paper, with the purpose of making this agreement easier, firstly we use two different sets of *basic* properties, called *Commonly Accepted Equity Principles*, to delimit what agents can propose as reasonable allocations. Secondly, we extend the results obtained by Chun (1989) and Herrero (2003), obtaining new characterizations of old and well known bankruptcy rules. Finally, using the fact that bankruptcy problems can be analyzed from awards and losses, we define a mechanism which provides a new justification of the convex combinations of bankruptcy rules.

Keywords: Bankruptcy problems, Unanimous Concessions procedure, Diminishing Claims mechanism, Piniles' rule, Constrained Egalitarian rule
JEL classification: C71, D63, D71.

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1. Introduction.

Consider a group of agents who have deposited their savings in a bank. *How should the bank funds be distributed among these investors if bankruptcy?* Obviously, each agent is going to seek the highest amount of awards as possible, according to her savings. However, there are multiple possible allocations that agents can propose. So that, an agreement about the way of sharing the funds is not easily met. These kind of situations, where the resources are not sufficient to satisfy the agents' aggregate demand, are called bankruptcy problems. Moreover, according to each bankruptcy problem, the way of rationing the endowment among the agents, taking into account their claims, is prescribed by a bankruptcy rule.

Hereinafter, we analyze bankruptcy problems assuming that, as in the example above, agents act strategically, i.e., in a non-cooperative way. In this context, Chun (1989) and Herrero (2003) follow the bargaining model introduced by van Damme (1986), who prospects Nash equilibria of a non-cooperative game. Particularly, he defines a mechanism of successive concessions, where agents' strategies consist of the choice of a rule among a reasonable set of them. Applying this idea in bankruptcy problems, Chun (1989) proposed the *Diminishing Claims* procedure to solve bankruptcy situations where the allowed allocations are determined by the agreement of all agents on a set of "basic" requirements. Later, Herrero (2003) modifies the *Unanimous Concessions* mechanism, provided by Marco-Gil et al. (1995), for its application to bankruptcy problems. Finally, and more recently, Garcia-Jurado and Gónzalez-Villar (2006) propose an elementary game where each agent's strategy belong to a determined closed space of possible choices. With this game, they show that any acceptable rule can be obtained as the unique allocation of the corresponding Nash equilibria depending on its associated closed interval of strategies.

This paper strengths the set of *"Admissible"* rules to those that fulfil some reasonable or basic *"Commonly Accepted Equity Principles"*, P. Note that it can be easily found everyday situations in which agents have some legal restrictions on their proposals. See for instance, the law against dumping in competitive markets where, roughly speaking, firms cannot offer a product below its production costs; the different heritage laws; or, following the example which introduces this section, each investors cannot get more funds than her savings.

To this respect, we propose as basic properties the set P_1 composed by

Resource Monotonicity, *Super-Modularity* and *Midpoint Property*, and the set P_2, replacing on P_1 *Super-Modularity* by *Order Preservation*. We find out that, for every two-agents problem, in any Nash equilibrium of the game induced by the *Unanimous Concessions* procedure, the *Dual of Piniles'* and the *Dual Constrained Egalitarian* rules are retrieved for P_1 and P_2, respectively. Secondly, we generalize these results and show that the application of these procedures do not always provide desirable distributions. In this line, by applying the idea of duality, which we introduce in the next section, we recover the *Piniles'* and the *Constrained Egalitarian* rules for P_1 and P_2, respectively, in any Nash equilibrium of the two-agents game induced by the *Diminishing Claims* procedure.

Finally, using the fact that bankruptcy problems can be faced from a double point of view: awards and losses, we define a new mechanism, called *Double Concessions* procedure, which combines the philosophy of the *Diminishing Claims* and the *Unanimous Concessions* procedures. Then, we obtain that if the set of *"Admissible"* rules is defined by only two dual rules, our new procedure will correspond with the average of these two. This result has two consequences. On the one hand, we provide a new justification of a convex combination (the middle point) of two extreme and opposite ways of distributing the endowment. On the other hand, we obtain a new method for rationing the resources which is invariant to the point of view used (awards and losses).

The paper is organized as follows: Section 2 introduces the preliminaries. Sections 3 and 4 apply the *Unanimous Concessions* and the *Diminishing Claims* procedures for two different set of principles. Section 5 presents our new mechanism. Section 6 summarizes our conclusions. Finally, the Appendices gather technical proofs.

2. Prelimaries.

A **bankruptcy problem** is a vector $(E,c) \in \mathbb{R}_+ \times \mathbb{R}^n_+$, where E denotes the endowment and c is the vector of each agents' claim, c_i, for each $i \in N$, $N = \{1,...,i,...,n\}$, such that the agents' aggregate demand is higher than the endowment, $\sum\limits_{i \in N} c_i \geq E$.

For notational convenience, $\mathcal{B}$ will denote the set of all bankruptcy problems; C the sum of the agents' claims, $C = \sum\limits_{i \in N} c_i$; and L the total amount of losses to distribute among the agents, $L = C - E$.

Each bankruptcy problem can be faced from two points of views: awards and losses. Thus, we have two focal positions, depending on whether we worry about the awards we receive or the amount of our demand that is not satisfied. In this latter case, we consider the **dual bankruptcy problem**, which is the pair $(L, c) \in \mathbb{R}_+ \times \mathbb{R}^n_+$, such that L will denote the total amount of losses to distribute among the agents, $L = C - E$, and $\sum_{i \in N} c_i \geq L$.

In this context, a **rule** is a function, $\varphi : \mathcal{B} \to \mathbb{R}^n_+$, which associates for each $(E, c) \in \mathcal{B}$, a distribution of the endowment among the group of claimants, such that *(a)* $\sum_{i \in N} \varphi_i(E, c) = E$ and *(b)* $0 \leq \varphi_i(E, c) \leq c_i$.

Given a rule φ, its dual rule shares out losses in the same way that φ divides the endowment (Aumann and Maschler (1985)).

The **dual** of φ, denoted by φ^d, assigns for each $(E, c) \in \mathcal{B}$ and each $i \in N$, $\varphi^d_i(E, c) = c_i - \varphi_i(L, c)$.

It is straightforward to check that for each rule, φ, its dual rule is well defined, since given that $(E, c) \in \mathcal{B}$, $(L, c) \in \mathcal{B}$ and given that φ satisfies *efficiency*, *non-negativity* and *claim-boundedness*, the same will apply for φ^d. It is also clear that $(\varphi^d(E, c))^d = \varphi(E, c)$.

Additionally, if a rule recommends the same allocation when dividing awards and losses, it is called *Self-Dual*.

A rule φ is **Self-Dual**, if for each $(E, c) \in \mathcal{B}$ and each $i \in N$, $\varphi_i(E, c) = c_i - \varphi_i(L, c)$. [2]

Particularly, we focus on the following rules: the *Constrained Equal Awards*, *Piniles'* and the *Constrained Egalitarian* rules, and their dual rules.

The **Constrained Equal Awards** rule, *CEA*, (Maimonides 12th Century, among others) recommends, for each $(E, c) \in \mathcal{B}$, the vector $(\min \{c_i, \mu\})_{i \in N}$, where μ is chosen so that $\sum_{i \in N} \min \{c_i, \mu\} = E$.

Piniles' rule, *Pin*, (Piniles (1861)) provides, for each $(E, c) \in \mathcal{B}$, the vector $(CEA_i(E, c/2))_{i \in N}$, if $E \leq C/2$; and $(c_i/2 + CEA_i(E - C/2, c/2))_{i \in N}$, if $E \geq C/2$.

The **Constrained Egalitarian** rule, *CE*, (Chun et al. (2001)) chooses, for each $(E, c) \in \mathcal{B}$, the vector $(CEA_i(E, c/2))_{i \in N}$, if $E \leq C/2$; and

[2] That is, $\varphi^d(E, c) = \varphi(E, c)$.

$(\max\{c_i/2, \min\{c_i, \delta\}\})_{i\in N}$, if $E \geq C/2$, where δ is chosen so that $\sum_{i\in N} CE_i(E,c) = E$.

Note that the *Constrained Equal Losses* rule*, CEL,* (Aumann and Maschler (1985)) is the dual of the *Constrained Equal Awards* rule (Herrero and Villar (2001)). Moreover, *DPin* and *DCE* will denote the *Dual of Piniles'* and the *Dual of Constrained Egalitarian* rules, respectively.

In this paper, we delimit the set of reasonable distributions with the definition different sets of basic properties, called *"Commonly Accepted Equity Principles"* sets, on which society agrees that the distribution of the resource must be made in base of. These extended problems, proposed by Jiménez-Gómez and Marco-Gil (2008), are called Bankruptcy Problem with Legitimate Principles, that is, these are problems where all the admissible rules must satisfy the *"Commonly Accepted Equity Principles"* set. Formally,

A **Bankruptcy Problem with Legitimate Principles** is a triplet (E, c, P_t) where $(E, c) \in \mathcal{B}$ and P_t is a fixed set of principles on which a particular society has agreed.

So that, the allowed rules for this problem must satisfy the set of equity principles, P_t. That is,

An **Admissible** rule, φ, is a rule satisfying all properties in P_t.

From now on, P denotes the set of all subsets of properties of rules. Each $P_t \in P$ represents a specific society which will always apply such principles for solving its problems; $\mathcal{B}_P$ denotes the set of all *Problems with Legitimate Principles,* Φ the set of all rules and $\Phi(P_t)$ the subset of rules satisfying P_t.

3. The Unanimous Concessions procedure: restuls.

In this section we present the procedure with which we apply the previous ideas, and we introduce some "basic" properties which define two *"Commonly Accepted Equity Principles"* sets, P_1 and P_2. Then, we analyze the two-agents problems, retrieving the *Dual of Piniles'* and the *Dual Constrained Egalitarian* rules. Finally, we study the general case, obtaining not so desirable results for P_2.

The *Unanimous Concessions* procedure (Herrero (2003)) says that, given that agents have chosen their preferred *Admissible* rules, if at the initial step there is no agreement, at the second step, each agent receives her minimal amount among all the proposed. Now, we redefine the residual bankruptcy problem, in which the endowment is the leftover resource, the claims are adjusted down by the amounts just given, and the same procedure is applied. The solution will be the limit of the procedure if it is feasible, and zero otherwise. Formally

Definition 1. ***Unanimous Concessions procedure***, u, *(Herrero (2003)):*
Let $(E, c, P_t) \in B_P$. At the first stage, each agent chooses a rule $\varphi^i \in \Phi(P_t)$. Let $\psi = (\varphi^i)$ be the profile of rules selected. The division proposed by the Unanimous Concessions procedure, $u\left[\varphi, (E, c, P_t)\right]$ is obtained as follows:

[Step 1] If all agents agree on $\varphi(E, c, P_t)$, then $u\left[\varphi, (E, c, P_t)\right] = \varphi\left(E, c, P_t\right)$. Otherwise, go to next step.

[Step 2] Let us define $s_i\left(E, c, P_t\right) = \min_{j\in N} \varphi_i^j\left(E, c, P_t\right)$,
$c^2 = c - s\left(E, c, P_t\right)$, and $E^2 = E - \sum_{i\in N} s_i\left(E, c, P_t\right)$.
Now, if all agents agree on $\varphi(E^2, c^2, P_t)$, then
$u\left[\varphi, (E, c, P_t)\right] = s\left(E, c, P_t\right) + \varphi\left(E^2, c^2, P_t\right)$. Otherwise, go to next step.

[Step $m+1$] $s_i\left(E^m, c^m, P_t\right) = \min_{j\in N} \varphi_i^j\left(E^m, c^m, P_t\right)$,
$c^{m+1} = c^m - s\left(E^m, c^m, P_t\right)$, and $E^{m+1} = E^m - \sum_{i\in N} s_i\left(E^m, c^m, P_t\right)$.
Now, if all agents agree on $\varphi\left(E^{m+1}, c^{m+1}, P_t\right)$, then
$u\left[\varphi, (E, c, P_t)\right] = \sum_{k=1}^{m} s\left(E^k, c^k, P_t\right) + \varphi\left(E^{m+1}, c^{m+1}, P_t\right)$. Otherwise, go to next step.

[Limit case] Compute $\sum_{k=1}^{\infty} s\left(E^k, c^k, P_t\right)$. If it converges to an allocation, x, such that $\sum_{i\in N} x_i \leq E$, $u\left[\varphi, (E, c, P_t)\right] = x$. Otherwise, $u\left[\varphi, (E, c, P_t)\right] = 0$.

From now on, let $\Gamma_{P_t}^u$ denote the game induced by the *Unanimous Concessions* procedure when agents acts strategically, in which the set of players is N, the strategies for each agent are rules in $\Phi\left(E, c, P_t\right)$ and the payoffs are the sum of the amounts received by each agent in each step $m \in \mathbb{N}$. That is,

$$\Gamma_{P_t}^u = \left\{ N, \left\{\varphi^i \in \Phi(E, c, P_t)\right\}_{i=1}^n, \left\{ \sum_{k=1}^{m} s_i\left(E^k, c^k, P_t\right) \right\}_{i=1}^n \right\},$$

where m denotes the step where the agreement is reached, and ∞ otherwise.

Next, we introduce some properties of rules that have been understood by many authors as minimal requirements of fairness (see for instance Thomson (2003)), and with which we consider two possible choices of *"Commonly Accepted Equity Prinicples"* sets that a society could impose on the rules. Moreover, we present the notion of *duality* and *Self-Duality* between properties.

Resource Monotonicity (Curiel et al. (1987), Young (1988), among others) demands that if the endowment increases, then all individuals should get at least what they received initially.

Resource Monotonicity: for each $(E, c) \in \mathcal{B}$ and each $E' \in \mathbb{R}_+$ such that $C > E' > E$, then $\varphi_i(E', c) \geq \varphi_i(E, c)$, for each $i \in N$.

Order Preservation (Aumann and Maschler (1985)) requires respecting the ordering of the claims: if agent i's claim is at least as large as agent j's claim, he should receive and loss at least as much as agent j, does respectively.

Order Preservation: for each $(E, c) \in \mathcal{B}$, and each $i, j \in N$, such that $c_i \geq c_j$, then $\varphi_i(E, c) \geq \varphi_j(E, c)$, and $c_i - \varphi_i(E, c) \geq c_j - \varphi_j(E, c)$, that is $l_i \geq l_j$.

A *Super-Modular* rule (Dagan et al. (1997)) allocates each additional dollar in an "order preserving" manner. In other words, when the endowment increases, agents with higher claims receive a greater part of the increment than those with lower claims.

Super-Modularity: for each $(E, c) \in \mathcal{B}$, all $E' \in \mathbb{R}_+$ and each $i, j \in N$ such that $C > E' > E$ and $c_i \geq c_j$, then $\varphi_i(E', c) - \varphi_i(E, c) \geq \varphi_j(E', c) - \varphi_j(E, c)$.

Midpoint Property (Chun et al. (2001)) requires that if the estate is equal to the sum of the half-claims, then all agents should receive their half-claim.

Midpoint Property: for each $(E, c) \in \mathcal{B}$ and each $i \in N$, if $E = C/2$, then $\varphi_i(E, c) = c_i/2$.

The dual relation defined between rules has been carried to the concept of property. In this sense, given two properties, we say that they are dual of each other if whenever a rule satisfies one of them, its dual satisfies the other.

Two properties, $\mathcal{P}$ and $\mathcal{P}^d$, are **dual** if whenever a rule, φ, satisfies $\mathcal{P}$, its dual, φ^d, satisfies $\mathcal{P}^d$.

It is worth noting that all the principles we have introduced are invariant to the perspective from which the problem is thought, that is, they do not change when dividing "what is available" or "what is missing", so , they are *Self-Dual*. Formally:

A property, $\mathcal{P}$, is **Self-Dual** when it coincides with its dual.

Specifically, we consider the two following sets,

$$P_1 = \{\textit{Resource Monotonicity, Super-Modularity, Midpoint Property}\}$$
$$P_2 = \{\textit{Resource Monotonicity, Midpoint Property, Order Preservation}\}$$

Note that since *Super-Modularity* implies *Order Preservation* (Thomson (2003)), we obtain $P_1 \subseteq P_2$.

Given P_1, next propositions tell us that, on the hand, if some agent announces the *DPin* rule, then the *Unanimous Concessions* procedure converge to this rule, and, on the other hand that, the *DPin* rule is a weakly dominant strategy for the agent with the highest claimant. Then, as a direct consequence of these two results, we show that in all noncooperative Nash equilibria, each agent will receive the awards recommended by the *DPin* rule.

Proposition 2. *For each* $(E, c, P_1) \in \mathcal{B}_P$*, and each* $i \in N$*, if* $\varphi^i(E, c) \in \Phi\left(B_{P_1}\right)$*, and for some* $j \in N$*,* $\varphi^j(E, c) = DPin(E, c)$*, then* $u\left[\varphi, (E, c, P_1)\right] = DPin(E, c)$*.*

Proof. See Appendix 2.

Proposition 3. *In the game* $\Gamma^u_{P_1}$*, the DPin rule is a weakly dominant strategy for the agent with the highest claim.*

Proof. See Appendix 3.

Theorem 4. *In any Nash equilibrium induced by the game* $\Gamma^u_{P_1}$*, each agent receives the amount given by the DPin rule.*

Proof. See Appendix 4.

Finally, we obtain similar results with P_2. That is, if some agent announces the *DCE* rule, then the *Unanimous Concessions* procedure converge to this rule. Moreover, the *DCE* and the *CE* rules are a weakly dominant strategy for the agents with the highest and the smallest claimant, respectively. However, we show that, in general, the Nash equilibrium induced by the application of the *Unanimous Concessions* procedure not only does not provide one of the *Admissible* rules, but also the allocation proposed by it fails one of *"Commonly Accepted Equity Principles"* on which the process is based.

Proposition 5. *For each* $(E, c, P_2) \in \mathcal{B}_P$*, with* $|N| = 2$*, and each* $i \in \{1, 2\}$*, if* $\varphi^i(E, c) \in \Phi(B_{P_2})$ *, and for some* $j \in \{1, 2\}$ *,* $\varphi^j(E, c) = DCE(E, c)$*, then* $u[\varphi, (E, c, P_2)] = DCE(E, c)$*.*

Proof. See Appendix 5.

Proposition 6. *In the game* $\Gamma^u_{P_2}$*, the DCE rule is a weakly dominant strategy for the agent with the highest claim.*

Proof. See Appendix 6.

Proposition 7. *In the game* $\Gamma^u_{P_2}$*, the CE rule is a weakly dominant strategy for the agent with the smallest claim.*

Proof. See Appendix 6.

Theorem 8. *For two-agents problems, in any Nash equilibrium induced by the game* $\Gamma^u_{P_2}$*, each agent receives the amount given by the DCE rule.*

Proof. See Appendix 7.

Theorem 9. *There is a problem,* $(E, c, P_2) \in \mathcal{B}_P$*, for which if* $\varphi^i(E, c) \in \Phi(B_{P_2})$ *, and for some* $j \in N$*,* $\varphi^j(E, c) = DCE(E, c)$*, then* $u[\varphi, (E, c, P_2)] \neq DCE(E, c)$*.*

Proof. See Appendix 8.

Theorem 10. *The Nash equilibrium induced by the game* $\Gamma^u_{P_2}$ *does not fulfil Resource Monotonicity.*

Proof. See Appendix 8.

4. Diminishing Claims procedure: results.

By using the idea of duality and the fact all the properties proposed are *Self-Dual*, the previous results can be analyzed from the viewpoint of sharing losses, i.e., we focus on the maximum awards that each agent can ensure.

In this sense, the *Diminishing Claims* procedure (Chun (1989)), denoted by d, says that, given that agents have chosen their preferred rules, if at the initial step there is no agreement, at the second step, we redefine the residual bankruptcy problem, in which the endowment does not change, and each agent's claim is truncated by the highest amount among all the proposed at step 1. Then, the procedure is again applied until an agreement is reached. If this is not the case, the solution will be the limit of the procedure if is feasible, and zero otherwise. Formally,

Definition 11. ***Diminishing Claims procedure***, *d, (Chun (1989)):*

Let $(E, c, P_t) \in \mathcal{B}_P$. At the first stage, each agent chooses a rule $\varphi^i \in \Phi(P_t)$. Let $\psi = (\varphi^i)$ be the profile of selected rules. The division proposed by the Diminishing Claims procedure, $d[\varphi, (E, c, P_t)]$ is obtained as follows:

[Step 1] If all agents agree on $\varphi(E, c, P_t)$, then $d[\varphi, (E, c, P_t)] = \varphi(E, c, P_t)$. Otherwise, go to next step.

[Step 2] Let us define $ce_i(E, c, P_t) = \max_{j \in N} \varphi_i^j(E, c, P_t)$,
$c^2 = ce(E, c, P_t)$, and $E^2 = E$.
Now, if all agents agree on $\varphi(E^2, c^2, P_t)$, then
$d[\varphi, (E, c, P_t)] = \varphi(E^2, c^2, P_t)$. Otherwise, go to next step.

[Step $m+1$] Let us define $ce_i(E^m, c^m, P_t) = \max_{j \in N} \varphi_i^j(E^m, c^m, P_t)$,
$c^{m+1} = ce(E^m, c^m, P_t)$, and $E^{m+1} = E$.
Now, if all agents agree on $\varphi(E^{m+1}, c^{m+1}, P_t)$, then
$d[\varphi, (E, c, P_t)] = \varphi(E^{m+1}, c^{m+1}, P_t)$. Otherwise, go to next step.

[Limit case] Compute $\lim_{k \to \infty} \varphi(E^k, c^k, P_t)$. If it converges to an allocation, x, such that $\sum_{i \in N} x_i \leq E$, $d[\varphi, (E, c, P_t)] = x$. Otherwise, $d[\varphi, (E, c, P_t)] = 0$.

From now on, let $\Gamma_{P_t}^d$ denote the game induced by the *Diminishing Claims* procedure when agents act strategically, in which the set of players is N, the strategies for each agent are rules in $\Phi(E, c, P_t)$ and the payoffs are the amount recommending to each agent by the accorded rule. That is,

$$\Gamma^d_{P_t} = \left\{N, \left\{\varphi^i \in \Phi(E,c,P_t)\right\}_{i=1}^n, \left\{\varphi_i\left(E^m, c^m, P_t\right)\right\}_{i=1}^n\right\}.$$

It can be easily checked that the *Diminishing Claims* and the *Unanimous Concessions* procedures are dual, since the maximum amount that each agent can receive in the former mechanism can be interpreted as the minimal losses in which each agent can incur applying the latter mechanism.

Therefore, next results come straightforwardly by duality.

Corollary 12. *In any Nash equilibrium induced by the game* $\Gamma^d_{P_1}$ *each agent receives the amount given by the Pin rule.*

Corollary 13. *For two-agents problems, in any Nash equilibrium induced by the game* $\Gamma^d_{P_2}$ *each agent receives the amount given by the CE rule.*

Corollary 14. *There is a problem* $(E,c,P_2) \in \mathcal{B}_P$*, for which if* $\varphi^i(E,c) \in \Phi\left(P_2\right)$ *and for some* $j \in N, \varphi^j(E,c) = CE\left(E,c\right),$ *then* $u\left[\varphi,(E,c,P_2)\right] \neq DCE(E,c)$.

Corollary 15. *The Nash equilibrium induced by the game* $\Gamma^d_{P_2}$ *does not fulfil Resource Monotonicity.*

5. The solution is "in the middle".

In this section, we define a new mechanism which combines the philosophy of the *Diminishing Claims* and the *Unanimous Concessions* procedures, using the fact that they are dual each other.

This new method, named the *Double Concessions* procedure, says that, given that agents have chosen their preferred rules, if at the initial step there is no agreement, at the second step, each agent receives the smallest amount among all the proposed at step 1. Now, we redefine the residual problem, in which the endowment is the leftover resources, and the claims are truncated by the maximum amount recommended by all the suggested rules and adjusted down by the amounts just given. Then, the procedure is again applied until an agreement is reached. If this is not the case, the solution will be the limit of the procedure if it is feasible, and zero otherwise. Formally:

Definition 16. ***Double Concessions procedure***, *du:*

Let $(E, c, P_t) \in \mathcal{B}_P$. *At the first stage, each agent chooses a rule* $\varphi^i \in \Phi(E, c, P_t)$. *The proposal of the Double Concessions procedure,* $du[\varphi, (E, c, P_t)]$ *is obtained as follows:*

[Step 1] If all agents agree on $\varphi(E, c, P_t)$, *then* $du[\varphi, (E, c, P_t)] = \varphi(E, c, P_t)$. *Otherwise, go to next step.*

[Step 2] Let us define

$s_i(E, c, P_t) = \min_{j \in N} \varphi_i^j (E, c, P_t)$,

$ce_i(E, c, P_t) = \max_{j \in N} \varphi_i^j (E, c, P_t)$,

$c^2 = ce(E, c, P_t) - s(E, c, P_t)$, *and*

$E^2 = E - \sum_{i \in N} s_i(E, c, P_t)$.

Now, if all agents agree on $\varphi(E^2, c^2, P_t)$, *then* $du[\varphi, (E, c, P_t)] = s(E, c, P_t) + \varphi(E^2, c^2, P_t)$. *Otherwise, go to next step.*

[Step $m+1$*] Let us define*

$s_i(E^m, c^m, P_t) = \min_{j \in N} \varphi_i^j (E^m, c^m, P_t)$,

$ce_i(E^m, c^m, P_t) = \max_{j \in N} \varphi_i^j (E^m, c^m, P_t)$,

$c^{m+1} = ce(E^m, c^m, P_t) - s(E^m, c^m, P_t)$, *and*

$E^{m+1} = E^m - \sum_{i \in N} s_i(E^m, c^m, P_t)$.

Now, if all agents agree on $\varphi(E^{m+1}, c^{m+1}, P_t)$, *then*

$du[\varphi, (E, c, P_t)] = \sum_{k=1}^{m} s\left(E^k, c^k, P_t\right) + \varphi(E^{m+1}, c^{m+1}, P_t)$. *Otherwise, go to next step.*

[Limit case] Compute $\lim_{m \to \infty} \sum_{k=1}^{m} s\left(E^k, c^k, P_t\right)$. *If it converges to an allocation,* x, *such that* $\sum_{i \in N} x_i \leq E$, $du[\varphi, (E, c, P_t)] = x$. *Otherwise,* $du[\varphi, (E, c, P_t)] = 0$.

From now on, let $\Gamma_{P_t}^{du}$ denote the game induced by the *Double Concessions* procedure when agents acts strategically, in which the set of players is N, the strategies for each agent are rules in $\Phi(E, c, P_t)$ and the payoffs are the sum of the amounts received by each agent in each step $m \in \mathbb{N}$. That is,

$$\Gamma_{P_t}^{du} = \left\{ N, \left\{\varphi^i \in \Phi(E,c,P_t)\right\}_{i=1}^n, \left\{ \sum_{k=1}^{m} s_i\left(E^k, c^k, P_t\right) \right\}_{i=1}^n \right\},$$

where m denotes the step where the agreement is reached, and ∞ otherwise.

Next theorem takes as starting point situations where discrepancy for sharing the estate is considered by means of the existence of two fixed focal rules representing two prominent proposals. Such approach was introduced by Gadea-Blanco et al. (2010) in a more general framework from a cooperative point of view, under the name of *Bifocal distribution problems.* Particularly, we consider two rules, f and g, called **Focal** rules, which mark out the area of all the admissible paths of awards, φ_i, satisfying properties for P_t, that is, for each $(E,c,P_t) \in \mathcal{B}_{\mathcal{P}}$, and each $i \in N$ $\min\{f_i(E,c,P_t), g_i(E,c,P_t)\} \leq \varphi_i\,(E,c,P_t) \leq \max\{f_i(E,c,P_t), g_i(E,c,P_t)\}$. Then, the next result shows that the *Double Concessions* procedure for P_t will coincide with the average of the two *Focal* rules if they are *dual* to each other.

Theorem 17. *In any Nash equilibrium induced by the game* $\Gamma_{P_t}^{du}$*, such that for each* $i \in N$*, any Admissible rule,* φ*, fulfils that:* $\min\{f_i(E,c,P_t), g_i(E,c,P_t)\} \leq \varphi_i\,(E,c,P_t) \leq \max\{f_i(E,c,P_t), g_i(E,c,P_t)\}$*, and* $f(E,c,P_t) = c - g(L,c,P_t)$*, then,*

$$du\left[\varphi, (E,c,P_t)\right] = \frac{f(E,c,P_t) + g(E,c,P_t)}{2}.$$

Proof.- See Appendix 9.

A direct consequence of the above results is that our mechanism provides the same allocation of the endowment when distributing awards or losses, i.e., is *Self-Dual.* Moreover, we recover the midpoint between the two rules which represent extreme and opposite ways of sharing awards among claimants according to the imposed requirements. So, in other words, it could be said that the allocation so obtained neither favour nor hurts to any agent in particular. Following Thomson and Yeh (2008),

> *'When two rules express opposite points of views on how to solve a bankruptcy problem, it is natural to compromise between them by averaging.*

5.1. Applications.

Now, following Arin (2007), Dutta and Ray (1989), Cowell (2000) and Lambert (2001), and with the aim of determining the two *Focal* rules which represent the discrepancy for sharing the resources, we considered the *Lorenz (equity) criterion* (Lorenz (1905)). That is, we combine the two focus, awards and losses, that 'naturally' arises in bankruptcy problems, with the two most egalitarian rules according to these points of views, i.e., the *Lorenz-Gains Maximal* and the *Lorenz-Losses Maximal* for each (E, c, P_t). Therefore, now we consider the two following sets,

$P_1 =\{$*Resource Monotonicity, Super-Modularity, Midpoint Property, Lorenz criterion*$\}$

$P_2 =\{$*Resource Monotonicity, Midpoint Property, Order Preservation, Lorenz criterion*$\}$

Lorenz comparisons of bankruptcy rules from the awards point of view can be found in Bosmans and Lauwers (2011) and Thomson (2007). These results together with duality define the *Focal* rules that mark out the region of the *Admissible* rules for P_1, and P_2. So, *Bankruptcy Problems with Legitimate Principles* for each of these principles sets are well-defined, being their elements triplets, such that, for each $(E, c) \in \mathcal{B}$,

(E, c, P_1) with *Focal* rules Pin and $DPin$,
(E, c, P_2) with *Focal* rules CE and DCE.

Consequently, next results are straightforwardly obtained by applying Theorem 17.

Corollary 18. *In any Nash equilibrium induced by the game* $\Gamma^{du}_{P_1}$*, each agent receives the amount given by the average of Piniles' and the Dual of Piniles' rules.*

Corollary 19. *In any Nash equilibrium induced by the game* $\Gamma^{du}_{P_2}$*, each agent receives the amount given by the average of the Constrained Egalitarian and the Dual Constrained Egalitarian rules.*

Moreover, if we define $P_3 =\{$*Lorenz criterion*$\}$, then, we have a problem $(E, c, P_3) \in \mathcal{B}_P$ with *Focal* rules CEA and CEL. So,

Corollary 20. *In any Nash equilibrium induced by the game* $\Gamma^{du}_{P_3}$*, each agent receives the amount given by the average of the Constrained Equal Awards and the Constrained Equal Losses rules.*

Finally, note that the convex combination of rules preserves *Resource Monotonicity*, *Super-Modularity* and the *Midpoint* properties (Thomson and Yeh Thomson and Yeh (2008)). Hence, as theses corollaries show, our new procedure is *Admissible* for $P_t \in \{P_1, P_2, P_3\}$, while applying independently the *Diminishing Claims* and the *Unanimous Concessions* fails *Resource Monotonicity.*

6. Conclusions.

In this paper we offer the understanding of old bankruptcy rules from a new angle. Specifically, we particularize the methodology of the *Unanimous Concessions procedure* to different sets of *"Commonly Accepted Equity Principles"* by a society.

On the one hand, we have retrieved the *DPin* rule when applying the *Unanimous Concessions* procedure with the set P_1. However, this result cannot be generalized to any equity principle set P_t, as we have shown with P_2, in which we recover the DCE rule for the two-agents case, but not for the general one.

Therefore, we have shown that the allocation obtained when applying the *Unanimous Concessions* and the *Diminishing Claims* procedures, may lead "not desirable" results. Particularly, if a society agreed on choosing those rules which satisfies a determined set of equity principles, the final allocation could not satisfy the initial agreed properties.

Finally, we observe that in contexts where two *Focal* positions appear, the application of our new mechanism, which combines both procedures, retrieves the midpoint between these two focus. This fact, apart from its own logic, allows to anticipate the result. Moreover, whenever the average of these *Focal* rules fulfils the properties on which the context is based, then the *Double Concessions* procedure leads to an *Admissible* allocation.

APPENDIX 1. *General Facts*

Next we present one remark, two definitions and two facts which are used in the proofs provided in the following appendices. Moreover, from now on, $m \in \mathbb{N}$ will denote the m-*th* step of the *Unanimous Concessions* procedure (see Definition 1), and we consider, without loss of generality, $(E, c) \in \mathcal{B}_0$, where, $\mathcal{B}_0$ denotes the set of problems in which claims are increasingly ordered, that is problems with $c_1 \leq c_2 \leq ... \leq c_n$

The remark establishes, for each $P_t \in \{P_1, P_2\}$, that the order of the agents' claims is fixed along the different steps of the procedure.

Remark 1. *(Jiménez-Gómez and Marco-Gil, 2008) For each* $(E, c, P_t) \in \mathcal{B}_P$ *with* $P_t \in \{P_1, P_2\}$, *if* $c_i^m \leq c_j^m \Rightarrow c_i^{m+1} \leq c_j^{m+1}$.

The following definitions and facts provide the *P-Safety* for P_1 and P_2.

Definition 21. *(Jiménez-Gómez and Marco-Gil, 2008) Given* (E, c, P_1) *in* $\mathcal{B}_P$, *the P-Safety, s, is for each* $i \in N$,

$$s_i(E, c, P_1) = \inf\{\varphi_i^*(E, c), DPin_i(E, c)\},$$

where φ^* *denotes an Admissible rule in* P_1, *such that,* $\varphi^*(E, c) \neq DPin(E, c)$.

Definition 22. *(Jiménez-Gómez and Marco-Gil, 2008) Given* (E, c, P_2) *in* $\mathcal{B}_p$, *the P-Safety*, *s, is for each* $i \in N$,

$$s_i(E, c, P_2) = \inf\{\varphi_i^*(E, c), DCE_i(E, c)\},$$

where φ^* *denotes an Admissible rule in* P_2, *such that, , such that,* $\varphi^*(E, c) \neq DCE(E, c)$.

Fact 1. *Given* (E, c, P_1) *in* $\mathcal{B}_p$ *and for each* $m \in \mathbb{N}$, $s_1(E^m, c^m, P_1) = DPin_1(E^m, c^m)$ *and* $s_n(E^m, c^m, P_1) = Pin_n(E^m, c^m)$.

Fact 2. *Given* (E, c, P_2) *in* $\mathcal{B}_p$ *and for each* $m \in \mathbb{N}$, $s_1(E^m, c^m, P_2) = DCE_1(E^m, c^m)$ *and* $s_n(E^m, c^m, P_2) = CE_n(E^m, c^m)$.

APPENDIX 2. ***Proof of Proposition** 2.*

The proof of this result is based on Remark 1, Definition 21, Fact 1 and three lemmas, in which φ^* denotes an *Admissible* rule in P_1, different of the *Dual Piniles'* one, $\varphi^*(E, c) \neq DPin(E, c)$.

Lemma 23. *(Jiménez-Gómez and Marco-Gil, 2008) For each* $(E, c) \in \mathcal{B}_0$, *if there is* $m \in \mathbb{N}$ *such that* $s_i(E^m, c^m, P_1) = DPin_i(E^m, c^m)$ *then,*

$$s_i(E^{m+h}, c^{m+h}, P_1) = 0, \text{ for each } h \in \mathbb{N}.$$

Lemma 24. *(Jiménez-Gómez and Marco-Gil, 2008) For each* $(E,c) \in \mathcal{B}_0$ *and each* $i \in N$, *if* $s_i(E^m, c^m, P_1) = \varphi_i^*(E^m, c^m)$ *for each* $m \in \mathbb{N}$, *then*

$$\sum_{k=1}^{\infty} s_i(E^k, c^k, P_1) \leq DPin_i(E,c).$$

Lemma 25. *(Jiménez-Gómez and Marco-Gil, 2008) For each* $(E,c) \in \mathcal{B}_0$ *and each* $i \in N$, *if there is* $m^* \in \mathbb{N}$, $m^*{>}1$, *such that* $s_i(E^{m^*}, c^{m^*}, P_1) = DPin_i(E^{m^*}, c^{m^*})$ *and*

$$s_i(E^{m^*-1}, c^{m^*-1}, P_1) = \varphi_i^*(E^{m^*-1}, c^{m^*-1}), \textit{then}$$
$$\sum_{k=1}^{m^*} s_i(E^k, c^k, P_1) = DPin_i(E,c).$$

Proof of Proposition 2.

[**Step** 1] If all agents agree on $\varphi(E,c,P_1) = DPin(E,c)$, then $u[\varphi,(E,c,P_1)] = DPin(E,c)$. Otherwise, go to next step.

[**Step** 2] Let $s_i(E,c,P_1) = \min_{j\in N} \varphi_i^j(E,c,P_1)$, $c^2 = c - s(E,c,P_1)$, and $E^2 = E - \sum_{i\in N} s_i(E,c,P_1)$. By Lemma 23, for each agent i such that $s_i(E,c,P_1) = DPin_i(E,c)$, $s_i(E^2,c^2,P_1) = 0$. If all agents agree on $\varphi(E^2,c^2,P_1)$, by Lemma 25, $DPin(E,c) \equiv u[\varphi,(E,c,P_1)] = s(E,c,P_1)$ $+\varphi(E^2,c^2,P_1)$. Otherwise, go to next step.

[**Step** $m+1$] Let $s_i^{m+1}(E^m,c^m,P_1) = \min_{j\in N} \varphi_i^j(E^m,c^m,P_1)$, $E^{m+1} = E^m - \sum_{i\in N} s_i^m$, and $c^{m+1} = c^m - s(E^m,c^m,P_1)$. By Lemma 23, for each agent i such that $s_i(E^m,c^m,P_1) = DPin_i(E^m,c^m)$, $s_i(E^{m+1},c^{m+1},P_1) = 0$. If all agents agree on $\varphi(E^{m+1},c^{m+1},P_1)$, by Lemma 25, $DPin(E,c) \equiv u[\varphi,(E,c,P_1)] = \sum_{k=1}^{m} s(E^k,c^k,P_1) + \varphi(E^{m+1},c^{m+1},P_1)$. Otherwise, go to next step.

[**Limit case**] Compute $\sum_{k=1}^{\infty} s_i(E^k,c^k,P_1)$. Let us note that, by Lemmas 23 and 25 and the definition of the *DPin* rule, for each agent $i \in N : c_i \neq c_n$, $\sum_{k=1}^{\infty} s_i(E^k,c^k,P_1) = DPin_i(E,c)$. Moreover, for the rest of agents, l, by Lemma 24, $\sum_{k=1}^{m} s_l(E^k,c^k,P_1) \leq DPin_l(E,c)$. Furthermore, by Fact 1 and the definition of the *DPin* rule,

$$s_l(E,c,P_1) \geq \ E/n \geq \frac{DPin_l(E,c)}{n};$$

$$s_l(E^2, c^2, P_1) \geq DPin_l(E,c) - \frac{s_l(E,c,P_1)}{n};$$

thus, $\sum\limits_{k=0}^{m} s_l\left(E^k, c^k, P_1\right) \geq \frac{DPin_l(E,c)}{n}\left(\frac{n-1}{n}\right)^m$, i.e.,

$$\sum_{k=1}^{\infty} s_l\left(E^k, c^k, P_1\right) \geq DPin_l(E,c).$$

Therefore, $\sum\limits_{k=1}^{\infty} s_l\left(E^k, c^k, P_1\right) = DPin_l(E,c)$. **q.e.d.**

APPENDIX 3. *Proof of Proposition 3.*

By Remark 1, for each $m \in \mathbb{N}$, $c_1^m \leq c_2^m \leq ... \leq c_n^{m+1}$.

Moreover, let us note that for each (E, c, P_1) in $\mathcal{B}_P$ and each $\varphi \in \Phi(P_1)$,

$$DPin_n(E,c) \geq \varphi_n(E,c).$$

Finally, by Lemmas 23, 24 and 25, $\sum\limits_{k=1}^{\infty} s_n(E^k, c^k, P_1) \leq DPin_n(E,c)$.

Therefore, $DPin_n(E, c, P_1) \geq u_n[\varphi, (E, c, P_1)]$, i.e., the *DPin* rule is a weakly dominant strategy for the agent with the highest claim. **q.e.d.**

APPENDIX 4. *Proof of Theorem 4.*

Let us consider $(E, c, P_1) \in \mathcal{B}_P$. Then, each agent's outcome in any Nash equilibrium of $\Gamma_{P_1}^u$ satisfies $DPin_i(E,c) \leq u_i[\varphi, (E, c, P_1)]$, for each $i \in N$. Otherwise if for some $i \in N$, $DPin_i(E,c) > u_i[\varphi, (E, c, P_1)]$ then, by Proposition 2, agent i could deviate to choose *DPin*, which gives her more awards, contradicting the Nash equilibrium. Finally, if for each $i \in N$, $DPin_i(E,c) \leq u_i[\varphi, (E, c, P_1)]$, then, $u[\varphi, (E, c, P_1)] = DPin(E,c)$, since $\sum_{i \in N} u_i[\varphi, (E, c, P_1)] \leq E$.

APPENDIX 5. *Proof of Proposition 5.*

The proof of this result is based on Remark 1, Definition 22, Fact 2 and the following three lemmas, in which φ^* denotes an *Admissible* rule in P_2, different of the *Dual Constrained Egalitarian* one $\varphi^*(E,c) \neq DCE(E,c)$.

Lemma 26. *(Jiménez-Gómez and Marco-Gil, 2008) For each* $(E, c, P_2) \in \mathcal{B}_P$*, and each* $i \in \{1, 2\}$*, if* $s_i(E, c, P_2) = DCE_i(E, c)$*, then for* $m \geq 2$*,* $s_i(E^m, c^m, P_2) = DCE_i(E^m, c^m) = 0$.

Lemma 27. *(Jiménez-Gómez and Marco-Gil, 2008) For each* $(E, c) \in \mathcal{B}_0$ *and each* $i \in N$*, if* $s_i(E^m, c^m, P_2) = \varphi_i^*(E^m, c^m)$ *for each* $m \in \mathbb{N}$*, then*

$$\sum_{k=1}^{\infty} s_i(E^k, c^k, P_2) \leq DCE_i(E, c).$$

Lemma 28. *(Jiménez-Gómez and Marco-Gil, 2008) For each* $(E, c) \in \mathcal{B}_0$ *and each* $i \in N$*, if there is* $m^* \in \mathbb{N}$*,* $m^*>1$*, such that* $s_i(E^{m^*}, c^{m^*}, P_2) = DCE_i(E^{m^*}, c^{m^*})$ *and*

$$s_i(E^{m^*-1}, c^{m^*-1}, P_2) = \varphi_i^*(E^{m^*-1}, c^{m^*-1}), then$$

$$\sum_{k=1}^{m^*} s_i(E^k, c^k, P_2) = DCE_i(E, c).$$

Proof of Proposition 5.

[**Step** 1] If the two agents agree on $\varphi(E, c, P_2) = DCE(E, c)$, then $u[\varphi, (E, c, P_2)] = DCE(E, c)$. Otherwise, go to next step.

[**Step** 2] Let $s_i(E, c, P_2) = \min_{j \in N} \varphi_i^j(E, c, P_2)$, $c^2 = c - s(E, c, P_2)$ and $E^2 = E - \sum_{i \in N} s_i(E, c, P_2)$. In this case, by Definition 22, $s_1(E, c, P_2) = DCE_1(E, c)$, and by Lemma 26, $s_1(E^2, c^2, P_2) = 0$. So that, if all agents agree on $\varphi(E^2, c^2, P_2)$, then, by Lemma 28, $u[\varphi, (E, c, P_2)] = s(E, c, P_2) + \varphi(E^2, c^2, P_2) = DCE(E, c)$. Otherwise, go to next step.

[**Step** $m+1$] Let $s_i^{m+1}(E^m, c^m, P_2) = \min_{j \in N} \varphi_i^j(E^m, c^m, P_2)$, $E^{m+1} = E^m - \sum_{i \in N} s_i^m$, and $c^{m+1} = c^m - s(E^m, c^m, P_2)$. By Lemma 26, $s_1(E^m, c^m, P_2) = 0$. So that, if all agents agree on $\varphi(E^m, c^m, P_2)$, then, by Lemma 28, $u[\varphi, (E, c, P_2)] = \sum_{k=1}^{m} s\left(E^k, c^k, P_2\right) + \varphi(E^{m+1}, c^{m+1}, P_2) = DCE(E, c)$. Otherwise, go to next step.

[**Limit case**] Compute $\sum_{k=1}^{\infty} s_i\left(E^k, c^k, P_2\right)$. Let us note that, by Lemmas 26 and 28 and the definition of the DCE rule, for agent 1, $\sum_{k=1}^{\infty} s_1\left(E^k, c^k, P_2\right) = DPin_1(E, c)$.

Moreover, by Lemma 27, $\sum_{k=1}^{m} s_2(E^k, c^k, P_2) \leq DCE_2(E, c)$. Furthermore, by Fact 2 and the definition of the DCE rule,

$$s_2(E, c, P_2) \geq \; E/2 \geq DCE_2(E, c)/2,$$
$$s_2(E^2, c^2, P_2) \geq \frac{DCE_2(E, c) - s_2(E, c, P_2)}{2};$$

thus,

$$\sum_{k=0}^{m} s_2\left(E^k, c^k, P_2\right) \geq \; \frac{DCE_2(E, c)}{2}\left(\frac{1}{2}\right)^m,$$

i.e.,

$$\sum_{k=1}^{\infty} s_2\left(E^k, c^k, P_2\right) \geq DCE_2(E, c).$$

Therefore, $\sum_{k=1}^{\infty} s_2\left(E^k, c^k, P_2\right) = DCE_2(E, c)$. **q.e.d.**

APPENDIX 6. ***Proofs of Propositions 6 and 7.***

These proofs are based on Remark 1, and Lemmas 26, 27 and 28.

By Remark 1, for each $m \in \mathbb{N}$, $c_1^m \leq c_2^m \leq ... \leq c_n^m$.

Proof of Proposition 6.

Let us note that for each (E, c, P_2) in $\mathcal{B}_P$ and each $\varphi \in \Phi(P_2)$, $DCE_n(E, c) \geq \varphi_n(E, c)$.

By Lemmas 26, 27 and 28, $\sum_{k=1}^{\infty} s_n(E^k, c^k, P_2) \leq DCE_n(E, c)$.

Therefore, $DCE_n(E, c, P_2) \geq u_n[\varphi, (E, c, P_2)]$, i.e., the DCE rule is a weakly dominant strategy for the agent with the highest claim.

Proof of Proposition 7.

Let us note that for each (E, c, P_2) in $\mathcal{B}_P$ and each $\varphi \in \Phi(P_2)$, $CE_1(E, c) \geq \varphi_1(E, c)$.

By Lemmas 26, 27 and 28, $\sum_{k=1}^{\infty} s_1(E^k, c^k, P_2) \leq DCE_1(E, c) \leq CE_1(E, c)$.

Therefore, $CE_1(E, c, P_2) \geq u_1[\varphi, (E, c, P_2)]$, i.e., the CE rule is a weakly dominant strategy for the agent with the smallest claim. **q.e.d.**

APPENDIX 7. ***Proof of Theorem 8.***

Let us consider $(E, c, P_2) \in \mathcal{B}_P$. Then, each agent's outcome, in any Nash equilibrium of $\Gamma^u_{P_2}$ satisfies $DCE_i(E, c) \leq u_i[\varphi, (E, c, P_2)]$ for each $i \in N$, with $|N| = 2$. Otherwise if for some $i \in \{1, 2\}$, $DCE_i(E, c) > u_i[\varphi, (E, c, P_2)]$ then, by Proposition 5, agent i could deviate to choose DCE, which gives her more awards, contradicting the Nash equilibrium. Finally, if for each $i \in \{1, 2\}$, $DCE_i(E, c) \leq u_i[\varphi, (E, c, P_2)]$, then, $u[\varphi, (E, c, P_2)] = DCE(E, c)$, since $\sum_{i \in N} u_i[\varphi, (E, c, P_2)] \leq E$.

APPENDIX 8. ***Proof of Theorems 9 and 10.***

First of all, let us note the following fact.

Fact 3. *(Jiménez-Gómez and Marco-Gil, 2008) By the definition for the DCE rule, we know that it can be written as follows,*

given $(E, c) \in \mathcal{B}_0$, $i \in N$,

$$DCE_i(E, c) \equiv \begin{cases} c_i - \gamma_i & \text{if } E \leq C/2 \\ c_i - \gamma_i & \text{if } E \geq C/2 \end{cases},$$

where γ_i *is chosen such that* $\sum\limits_{i \in N} DCE_i(E, c) = E$.

Therefore,

Case a: $E \leq C/2$. *We can compute* γ_i *as:*

$$\gamma_i = \begin{cases} c_i & \forall\ i < l \\ \max\{c_i/2, \alpha_i\} & \forall\ i \geq l \end{cases},$$

where agent l *is that one such that* $\sum\limits_{j>l} \min\{c_j - c_l; c_j/2\} < E$, *and either* $\sum\limits_{j>l-1} \min\{c_j - c_{l-1}; c_j/2\} \geq E$, *either* $l = 1$. *Otherwise,* $l = n$.

Then, $\forall i \geq l$,

$$\alpha_i = \frac{L - \sum\limits_{j=1}^{k-1} c_j - \sum\limits_{j>i} \gamma_j}{i - l + 1}.$$

Note also that we should compute α *from the highest claimant to the smallest one.*

Case b: $E \geq C/2$. *Then,* γ_i *will denote the losses incurred by agent i when the losses from the claim vector are equal to all agents subject to no-one obtaining less than her half-claim.*

Proof of Theorem 9.

Let us consider the following problem $(E,c)\in\mathcal{B}=(21,(5;19.5;20))$, by Propositions 6 and 7, for each step $m\in\mathbb{N}$,

$$\psi\left(E^m,c^m,P_2\right)=\left(CE\left(E^m,c^m\right),\varphi_2^m\left(E^m,c^m\right),DCE\left(E^m,c^m\right)\right).$$

Thus, given the definitions of the CE rule and its dual and Fact 3, we get at step $m=1$, $(E^1,c^1)=(21,(5;19.5;20))$, $CE(E^1,c^1)=(2.5;9.25;9.25)$, and $DCE(E^1,c^1)=(1.25;9.75;10)$.

[**Step** 1] Since there is no agreement, go to next step.

[**Step** 2] $s(E,c,P_2)=(1.25;9.25;9.25)$, and $E^2=1.25$. So, $(E^2,c^2)=(1.25,(3.75;10.25;10.75))$, $CE(E^2,c^2)=(0.416;0.416;0.416)$, and $DCE(E^2,c^2)=(0;0.375;0.875)$, and since there is no agreement, go to next step.

[**Step** 3] $s(E^2,c^2,P_2)=(0;0.375;0.416)$, and $E^3=0.459$. So, $(E^3,c^3)=(0.459,(3.75;9.875;10.334))$, $CE(E^3,c^3)=(0.153;0.153;0.153)$, and $DCE(E^3,c^3)=(0;0;0.459)$, and since there is no agreement, go to next step.

[**Step** 4] $s(E^3,c^3,P_2)=(0;0;0.153)$, and $E^4=0.306$. So, $(E^4,c^4)=(0.306,(3.75;9.875;10.181))$, $CE(E^3,c^3)=(0.102;0.102;0.102)$, and $DCE(E^3,c^3)=(0;0;0.306)$, and since there is no agreement, go to next step.

[**Limit case**] Let us note that, since $E^m\le c_3^m-c_2^m$, and $c_i^m\ge E^m/3$, for each step $m\ge 3$, $DCE(E^m,c^m)=(0;0;E^m)$ and $CE(E^m,c^m)=(E^m/3;E^m/3;E^m/3)$. Thus, $s(E^m,c^m,P_2)=(0;0;E^m/3)$. Therefore, $u\left[\varphi,(E,c,P_2)\right]=\sum_{k=1}^{\infty}s\left(E^k,c^k,P_2\right)=(1.25;9.625;10.125)\neq DCE(E,c)=(1.25;9.75;10)$. **q.e.d.**

Proof of Theorem 10.

Let us consider the two following *Problems with Legitimate Principles:*

$$(E,c,P_2)=(21,(5;19.5;20),P_2),$$
$$(E',c,P_2)=(22.25;(5;19.5;20),P_2).$$

In this case,

$$u\left[\varphi,(E,c,P_2)\right]=(1.25;9.625;10.125),$$

and

$$u\left[\varphi,\left(E',c,P_2\right)\right]=\left(2.5;9.75;10\right).$$

Obviously, these two distributions contradict *Resource Monotonicity* since the highest claimant receives less when the endowment increases. **q.e.d.**

APPENDIX 9. *Proof of Theorem 17.*

The proof of this result is based on a fact, two lemmas and a remark.

Fact 4. *By the definition of the Double Concessions procedure, it can be easily see that, if any Admissible rule, φ, fulfils that:* $\min\{f_i(E,c,P_t),g_i(E,c,P_t)\}$ $\leq\ \varphi_i\left(E,c,P_t\right)\ \leq\ \max\{f_i(E,c,P_t),g_i(E,c,P_t)\}$*, and* $f(E,c,P_t)=c-g(L,c,P_t)$*, then, the weakly dominant strategies for the smallest and the highest claimant are dual. In other words, if the weakly dominant strategy for the smallest claimant is f, then, the weakly dominant strategy for the highest agents will be g, and vice verse.*

The first lemma shows that, whenever there are two *Focal* rules, which are dual to each other, in any step $m\in\mathbb{N}$, $m>1$, the sum of minimum and the maximum amounts recommended by these two focus coincides with the sum of the claims.

Lemma 29. *For each $(E,c,P_t)\in\mathcal{B}_P$, such that for each $i\in N$, if any Admissible rule, φ, fulfils that:* $\min\{f_i(E,c,P_t),g_i(E,c,P_t)\}\leq\varphi_i\left(E,c,P_t\right)\leq$ $\max\{f_i(E,c,P_t),g_i(E,c,P_t)\}$*, and* $f(E,c,P_t)=c-g(L,c,P_t)$*, and* $m\in\mathbb{N},m>1$,

$$\sum_{i\in N}\left[ce_i(E^m,c^m,P_t)+s_i(E^m,c^m,P_t)\right]=C^m.$$

Proof. Let each $(E,c,P_t)\in\mathcal{B}_P$, such that for each $i\in N$, any *Admissible* rule, φ, fulfils that: $\min\{f_i(E,c,P_t),g_i(E,c,P_t)\}\leq\varphi_i\left(E,c,P_t\right)\leq$ $\max\{f_i(E,c,P_t),g_i(E,c,P_t)\}$, $f(E,c,P_t)=c-g(L,c,P_t)$, and $m\in\mathbb{N},m>1$. Then,

$s_i(E,c,P_t)=\min\left\{f_i\left(E,c,P_t\right),g_i\left(E,c,P_t\right)\right\}$, and
$ce_i(E,c,P_t)=\max\left\{f_i\left(E,c,P_t\right),g_i\left(E,c,P_t\right)\right\}$.

By duality, for each agent we are adding the two *Focal* rules. So next expression comes straightforwardly.

$$\sum_{i \in N} \left[\frac{ce_i(E^m, c^m, P_t) + s_i(E^m, c^m, P_t)}{2} \right] = E^m.$$

Finally, we know that

$$\begin{aligned} E^m &= E^{m-1} - \sum_{i \in N} s_i(E^{m-1}, c^{m-1}, P_t) = \\ &= \sum_{i \in N} \left[\frac{ce_i(E^{m-1}, c^{m-1}, P_t) + s_i(E^{m-1}, c^{m-1}, P_t)}{2} \right] - \\ &- \sum_{i \in N} s_i(E^{m-1}, c^{m-1}, P_t) = \\ &= \sum_{i \in N} \left[\frac{ce_i(E^{m-1}, c^{m-1}, P_t) - s_i(E^{m-1}, c^{m-1}, P_t)}{2} \right] = C^m/2, \end{aligned}$$

by the definition of the *Double Concessions* procedure. **q.e.d.**

The following remark is a direct consequence of Lemma 29 and it says that for each *Bankruptcy Problem with Legitimate Principles*, whenever there are two *Focal* rules, which are dual to each other, and at any step $m \in \mathbb{N}, m > 1$, the half of the claims sum at every step of the *Double Concesions* procedure coincides with both the endowment and the total loss at every step of the process.

Remark 2. *For each* $(E, c, P_t) \in \mathcal{B}_P$*, such that for each* $i \in N$*, if any Admissible rule,* φ*, fulfils that:* $\min\{f_i(E, c, P_t), g_i(E, c, P_t)\} \leq \varphi_i(E, c, P_t) \leq \max\{f_i(E, c, P_t), g_i(E, c, P_t)\}$*, and* $f(E, c, P_t) = c - g(L, c, P_t)$*, and* $m \in \mathbb{N}, m > 1$*,* $E^m = L^m = C^m/2$*.*

Proof. Let each $(E, c, P_t) \in \mathcal{B}_P$, such that for each $i \in N$, any *Admissible* rule, φ, fulfils that: $\min\{f_i(E, c, P_t), g_i(E, c, P_t)\} \leq \varphi_i(E, c, P_t) \leq \max\{f_i(E, c, P_t), g_i(E, c, P_t)\}$, $f(E, c, P_t) = c - g(L, c, P_t)$, and $m > 1 \in \mathbb{N}$. We know that, $L^m = C^m - E^m$. By Lemma 29, $E^m = C^m/2$. Therefore, $L^m = C^m - C^m/2 = C^m/2$. **q.e.d.**

Finally, next lemma says that, whenever there are two *Focal* rules, which are dual to each other, each agent's claim at each step different of the initial one coincides with sum of both the minimum and the maximum amounts recommended by these two focus.

Lemma 30. *For each* $(E, c, P_t) \in \mathcal{B}_P$*, such that for each* $i \in N$*, if any Admissible rule,* φ*, fulfils that:* $\min\{f_i(E, c, P_t), g_i(E, c, P_t)\} \leq \varphi_i(E, c, P_t) \leq \max\{f_i(E, c, P_t), g_i(E, c, P_t)\}$*, and* $f(E, c, P_t) = c - g(L, c, P_t)$*, and* $m > 1 \in \mathbb{N}$*,*

$$c_i^m = ce_i(E^m, c^m, P_t) + s_i(E^m, c^m, P_t).$$

Proof. Let each $(E, c, P_t) \in \mathcal{B}_P$, such that for each $i \in N$, any *Admissible* rule, φ, fulfils that: $\min\{f_i(E, c, P_t), g_i(E, c, P_t)\} \leq \varphi_i(E, c, P_t) \leq \max\{f_i(E, c, P_t), g_i(E, c, P_t)\}$, $f(E, c, P_t) = c - g(L, c, P_t)$, and $m > 1 \in \mathbb{N}$, by Remark 2 we know that for $m > 1 \in \mathbb{N}$, $L^m = E^m$, so, $s_i(E^m, c^m, P_t) = s_i((L^m, c^m, P_t)^d)$. By duality $ce_i(E^m, c^m, P_t) = c_i^m - s_i((L^m, c^m, P_t)) = c_i^m - s_i(E^m, c^m, P_t)$, then, $c_i^m = ce_i(E^m, c^m, P_t) + s_i(E^m, c^m, P_t)$. **q.e.d.**

Proof of Theorem 17.

Let $(E, c, P_t) \in \mathcal{B}_P$, such that for each $i \in N$, $\min\{f_i(E, c, P_t), g_i(E, c, P_t)\} \leq \varphi_i(E, c, P_t) \leq \max\{f_i(E, c, P_t), g_i(E, c, P_t)\}$, $f(E, c, P_t) = c - g(L, c, P_t)$, for each $i \in N$, and each $m \in \mathbb{N}$,

$$du_i\left[\varphi, (E, c, P_t)\right] = \lim_{m \to \infty} \sum_{k=1}^{m} s(E^k, c^k, P_t)) = s_i(E, c, P_t) + \sum_{m=2}^{\infty} s_i(E^m, c^m, P_t).$$

By the definition of the *Double Concessions* procedure,

$$\begin{aligned}
\sum_{m=2}^{\infty} c_i^m &= \sum_{m=2}^{\infty} \left[ce_i(E^{m-1}, c^{m-1}, P_t) - s_i(E^{m-1}, c^{m-1}, P_t)\right] = \\
&= ce_i(E^m, c^m, P_t) + \sum_{m=2}^{\infty} ce_i(E^m, c^m, P_t) - s_i(E^m, c^m, P_t) - \\
&- \sum_{m=2}^{\infty} s_i(E^m, c^m, P_t).
\end{aligned}$$

By Lemma 30,

$$\sum_{m=2}^{\infty} c_i^m = \sum_{m=2}^{\infty} \left[ce_i(E^m, c^m, P_t) + s_i(E^m, c^m, P_t) \right].$$

So,

$$ce_i(E, c, P_t) + \sum_{m=2}^{\infty} ce_i(E^m, c^m, P_t) - s_i(E, c, P_t) - \sum_{m=2}^{\infty} s_i(E^m, c^m, P_t)$$

$$= \sum_{m=2}^{\infty} \left[ce_i(E^m, c^m, P_t) + s_i(E^m, c^m, P_t) \right].$$

Thus,

$$\sum_{m=2}^{\infty} s_i(E, c, P_t) = \left(ce_i(E^m, c^m, P_t) - s_i(E, c, P_t) \right)/2, \text{and}$$

$$s_i(E, c, P_t) + \frac{ce_i(E, c, P_t) - s_i(E, c, P_t)}{2} = \frac{s_i(E, c, P_t) + ce_i(E, c, P_t)}{2}.$$

Therefore, by Fact 4

$$du\left[\varphi, (E, c, P_t)\right] = \frac{s(E, c, P_t) + ce(E, c, P_t)}{2}$$

$$= \frac{f(E, cP_t) + g(E, c, P_t)}{2}.$$ **q.e.d.**

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