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Decreases Noise.

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A Folk Theorem for Games when Frequent Monitoring Decreases Noise[*]

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Abstract

This paper studies frequent monitoring in an infinitely repeated game with imperfect public information and discounting, where players observe the state of a continuous time Brownian process at moments in time of length Δ. It shows that a limit folk theorem can be achieved with imperfect public monitoring when players monitor each other at the highest frequency, i.e., $\Delta \downarrow 0$. The approach assumes that the expected joint output depends exclusively on the action profile simultaneously and privately decided by the players at the beginning of each period of the game, but not on Δ. The strong decreasing effect on the expected immediate gains from deviation when the interval between actions shrinks, and the associated increase precision of the public signals, make the result possible in the limit.

JEL: C72/73, D82, L20.

KEYWORDS: Repeated Games, Frequent Monitoring, Public Monitoring, Brownian Motion.

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I. INTRODUCTION

In repeated games, it is common to assume that the time interval between each repetition of the stage game is of fixed length. When monitoring is perfect, letting the discount factor $\delta \uparrow 1$ either by making the players more patient (through a decrease in the discount rate r) or by shrinking the time interval between actions (through a decrease in Δ) are equivalent exercises. The former approach has been preferred to prove many folk theorems and to show the existence of efficient equilibria.[1]

When monitoring is imperfect, taking $r \downarrow 0$ or $\Delta \downarrow 0$ leads to different results. A decrease in Δ impacts on the distribution of the public signals. Abreu et al. (1991) were the first to point it out. In a setting with Poisson signals, they showed that results vary depending on whether $\delta \uparrow 1$ is due to $r \downarrow 0$ or to $\Delta \downarrow 0$. In the latter case, payoffs above the static Nash, but not fully efficient, can be sustained when the realizations of the process represent bad news which is more likely to occur when some player has deviated.[2] On the other hand, by making players increasingly patient through $r \downarrow 0$, Fudenberg et al. (1994) proved a general folk theorem, under some informational assumptions.

More recently, renewed interest in frequent monitoring has emerged, mainly due to Sannikov (2007)[3] and Faingold and Sannikov (2007). The latter work, reported a degeneracy of the set of *strongly symmetric equilibrium* (SSE) payoffs in continuous time, payoffs outside the convex hull of the static Nash equilibrium set cannot be enforced. More in the spirit of the present paper, by studying the limit of the discrete time game, Fudenberg and Levine (2007, 2009) and Sannikov and Skrzypacz (2007, 2010) reported similar degeneracy results when the noisy public signals follow a Brownian motion.

Since Brownian motion is an infinitesimal variation process, we would expect payoffs to be above the static Nash. These results received a great deal of attention and interest, but they may fail to fit with all economic situations of interest. More monitoring harms the monitor's side and produces a negative effect in terms of incentives. This goes counter to

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[1] See, e.g., Fudenberg and Maskin (1986).

[2] In an infinitesimal time interval, the absence of Poisson events is infinitely more likely. For that reason, the same result does not extend when the information arrivals represent good news. In either case, there is a loss of precision in the public signals.

[3] This paper provides a novel and elegant characterization of the set of *perfect public equilibria* payoffs using continuous time methods.

the Alchian and Demsetz (1972) theory, which defends the disciplinary effect of monitoring in terms of incentives. On the empirical side, Dickinson and Villeval (2008) showed that monitoring has a positive effect on an individual effort.[4]

Following the discussion, the present paper attempts to rationalize the intuition that frequent monitoring improves the informativeness of the public signals. As a result we develop a theory where imperfect public monitoring, in the limit, is equivalent to perfect monitoring.

We explore frequent monitoring in a simple partnership game with imperfect public monitoring and discounting.[5] We analyze the limit of a sequence of discrete time games indexed by Δ. The public signal (joint output) is the observed state of an *arithmetic Brownian motion* (ABM) process, at intervals of length Δ. Based on this information players adjust their actions for the following period.

A great deal of attention is given to SSE payoffs, not only because of their simplicity, but because with two-sided imperfect public monitoring the pairwise identifiability assumption typically fails, limiting to a great extent the provision of incentives. Destruction of value through punishments is the only way to provide incentives. Nonetheless, we show that the value of the best SSE improves monotonically with the monitoring intensity. In addition, we characterize the associated cutoff decision rule for general Δ.

Finally, we show that in the limit a folk theorem obtains, independently of how players discount the future and of the level of uncertainty.

The intuition is the following. The aggregate of players' individual decisions has associated an end of period expected joint output, which differs across effort choices but is independent on Δ. The realized joint output (the state of the ABM process) generates a noisy measure of the actual effort choices. Since, the noise component of the output process increases monotonically with the time interval between observations, the information about

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[4] They also report evidence of a Frey's (1993) crowding-out effect, which may lead to an equilibrium degeneracy in the limit. This effect is based on behavioral aspects that are not considered in the existing literature on frequent monitoring, which focuses exclusively on the informativeness of the signals with respect to the monitoring intensity.

[5] Essentially, the information structure is similar to that of Radner et al. (1986). Other, classical, situations involving imperfect public monitoring are Green and Porter (1984) and Porter (1983) where the market price is an imperfect signal of the quantities supplied by the firms. Fudenberg and Tirole (1991) and Mailath and Samuelson (2006) provided complete surveys.

the actual effort choices becomes increasingly precise when the monitoring frequency increases, i.e., Δ decreases. In addition, the expected gains associated with a deviation from the equilibrium path become less attractive. Both effects favor the provision of incentives, consequently payoffs improve monotonically when Δ decreases.

The present paper fills a gap in the existing literature by enlarging the spectrum of economic problems that can be studied using the frequent monitoring theory.[6]

Related Literature: positive or negative results in frequent monitoring are very sensitive to the modeling assumptions. Sannikov and Skrzypacz (2007),[7] and Fudenberg and Levine (2007)[8] report the impossibility of achieving payoffs higher than the static Nash. In both papers the public signals follow an ABM process and players' control the drift of the process, i.e., $\mu(a_t)\Delta$, that depends on the unknown profile of action a_t chosen by players. To understand this better, consider two distinct action profiles, a_t and a'_t, with different drifts associated, i.e., $\mu(a_t)\Delta \neq \mu(a'_t)\Delta$. In the limit, both $\mu(a_t)\Delta$ and $\mu(a'_t)\Delta$ go to zero at the same rate. Consequently, it is not possible to distinguish the drift associated with the profile a_t from the drift of the profile a'_t. Such modeling of the observed public signal becomes extremely noisy when observed at a high frequency, creating a degeneracy effect on the payoffs. These results are driven by the assumption that the process departs from the same point, independently of the players' actions. In the present paper, we make a different assumption - the ex-ante expected value varies with the action profiles.

Under the assumption that a deviation increases the volatility of the process, Fudenberg and Levine (2007) showed that fully efficient equilibria can be achieved in the limit. Inference improve with a decrease in Δ.[9] The result obtained by Fudenberg and Levine is similar in

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[6] Developments on frequent monitoring allowed the study of interesting departures from the canonical repeated game framework. Fudenberg and Olszewski (2011) studied the limit of an infinite repeated game with random asynchronous monitoring, while Osório (2008) studied infinitely repeated games, where the repetitions of the stage game are not deterministic.

[7] Sannikov and Skrzypacz (2010) bound the set of equilibrium payoffs by placing restrictions on how information from Brownian and Poisson components are used to provide incentives in the most efficient way.

[8] Fudenberg and Levine (2009) consider different ways of passing to the continuous time limit, i.e., binomial and trinomial approximations of the Brownian paths. Such construction can be applied in the context of the present paper by considering two binomial trees that start at different points (the distinct expected outputs). Then, both trees intercept in the second and following nodes.

[9] The variance parameter can be consistently estimated from the path of the process $(y_s, s \in (t, t+\varepsilon])$ for

shape to the one in the present paper. However, the assumptions that lead to efficiency differ.[10]

The rest of the paper is organized as follows. Section II integrates the present paper with the existing literature. Section III presents the repeated game model and the public information producing process. Section IV computes the bounds on the set of SSE payoffs and characterizes the optimal decision rule for varying Δ. Section V focuses on the limit case and presents the main results. Section VI discusses extensions. Section VII concludes.

II. EXAMPLES

In order to better place the present paper in the existing literature, consider the following cases. The first is in line with the present paper, the second with the existing literature.

Example 1 - Consider an infinitely repeated Cournot game, where firms' supply choices are private information and the market price is publicly observed with frequency Δ. The market price reacts not only to variations on the supply, but also to exogenous events.

If one of the firms deviates from some collusive arrangement, by increasing its own production, then the noisy market price should adjust instantaneously to the new aggregate joint output. This is true if information is perfect, but also when it is imperfect (with the addition of a noisy term). Since noisy infinitesimal variations in prices occur continuously, their aggregated sum is more likely to be relevant for larger values of Δ. To enforce collusion, mistaken punishment becomes more likely. On the other hand, when the end-of-period market price is observed at a high frequency (small Δ), the aggregate sum of exogenous noisy perturbations is less likely to hide a deviation. In such a case, a decrease in market price is almost surely a signal of that some firm is supplying larger quantities than it should.

The approaches of Fudenberg and Levine (2007, 2009) and Sannikov and Skrzypacz (2007, 2010) do not fit in this framework, because in the limit the market price does not adjust

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a small but measurable ε: the same does not happen with the drift of the process. In the latter case, a relatively large Δ is needed in order for the players' actions to be statistically distinguishable. See Prakasa Rao (1999) for a formal treatment of the statistical methods for diffusion processes.

[10] The result does not generalize when a deviation decreases the uncertainty parameter and extreme realizations represent "good news". To preserve the incentives players incur too often in mistaken punishments

instantaneously to changes in the supply. The following example illustrates a situation where these models are more adequate.

Example 2 - Suppose that a worker is expected to have produced the amount of output $\mu(E)\Delta$ at the end of a period of length Δ. The worker effort is costly and the observed output is a noisy measure of it. When the worker shirks, the end-of-period expected output is $0 = \mu(S)\Delta$, while if he provides effort $\mu(E)\Delta > \mu(S)\Delta$. The difference from the previous example, is that output does not react instantaneously to a no effort decision taken at the beginning of the period, because $\mu(E)\Delta \downarrow 0$ when $\Delta \downarrow 0$, which is the same as $\mu(S)\Delta = 0$.

If the monitoring events are very frequent, i.e., for small Δ, there is nothing to monitor. The supervisor must wait, i.e., larger Δ, in order to obtain reliable information about the worker's effort. The informativeness of the public signals increases if the monitoring is less frequent.

III. THE REPEATED GAME MODEL

We explore frequent monitoring in a simple partnership game with two long-run players $i \in \{1, 2\}$. The history of the game is the following. At moments in time $t = 0, \Delta, 2\Delta, ...,$ players can choose from two different effort levels $a_{it} = 1$ or $a_{it} = 0$. In the former, player i provides effort E to the partnership; in the latter case, she is shirking S. Let $a_t = (a_{1t}, a_{2t})$ denotes a profile of actions.

Independently, of their private effort decisions, at moments in time $t = \Delta, 2\Delta, ...,$ players observe and split the realized joint output $y_{t+\Delta}$, generated during the time interval of length Δ. The observed joint output (the public signal) at $t + \Delta$,[11] is driven by the following ABM process,

$$y_{t+\Delta} = y_t(a_t) + \sigma \int_t^{t+\Delta} dZ_s, \text{ with } Z_t = 0 \text{ and } t = 0, \Delta, 2\Delta, ..., \tag{3.1}$$

where $y_t(a_t) = 2\pi'(a_{1t} + a_{2t})$ is the initial condition of the process at time t, a function of the unknown profile of actions. The parameter σ measures the noise of the process. Uncertainty is generated by the standard Brownian motion $\{Z_s; s \geq 0\}$. The joint output

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[11] We could have considered the possibility that at the end of each period of length Δ, players observe the full path of the process $\{y_s, s \in (t, t + \Delta]\}$ realized from t to $t + \Delta$. This case provides more information to the monitor. Consequently, it has associated larger payoffs. In the limit both cases are equivalent.

	$E \Leftrightarrow 1$	$S \Leftrightarrow 0$
$E \Leftrightarrow 1$	π, π	$-\left(\pi'-\pi\right), \pi'$
$S \Leftrightarrow 0$	$\pi', -\left(\pi'-\pi\right)$	$0, 0$

TABLE I: "Prisoner's dilemma type" stage game payoffs.

evolves continuously. In every infinitesimal instant of time a noise realization is added. The constant π' is a productivity measure.

All the relevant information about players' actions is contained in $y_t\left(a_t\right)$, which is also the end-of-period expected joint output.[12] Since players cannot revise their actions during the time interval of length Δ, we removed the drift from the process (3.1).[13]

Player $i's \in \{1,2\}$ realized payoff (ex-post) from the partnership is,

$$r_i\left(a_{it}, y_{t+\Delta}\right) \equiv y_{t+\Delta}/2 - \left(2\pi' - \pi\right) a_{it},$$

where $\pi' > \pi > 0$.[14] The second term on the *right-hand side* (RHS) measures the cost of providing effort for player i. The constant π is conveniently placed to obtain the prisoner's-dilemma-like payoffs of table I. Player $i's$ ex-ante expected payoff from the partnership is then,

$$\pi_i\left(a_t\right) \equiv E\left(r_i\left(a_{it}, y_{t+\Delta}\right)|y_t\left(a_t\right)\right) = \pi'\left(a_{1t} + a_{2t}\right) - \left(2\pi' - \pi\right) a_{it}.$$

After considering all the possible effort profiles, we obtain the stage game payoff matrix of Table I.

Shirking is a dominate strategy for both players. The minimax value of the game coincides with the stage game Nash's payoff and equals 0 for both players. For convenience, we assume $2\pi > \pi'$.

A great deal of attention is given to SSE payoffs. In a strongly symmetric public strategy, after every public history the same action is chosen by both players.[15] More generally, a

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[12] The public process is a martingale with respect to some filtration, i.e., $E\left(y_{t+\Delta}|y_t\left(a_t\right)\right) = y_t\left(a_t\right)$. Examples of other processes with the same property are $\mu\left(y_t, t\right) = 0$ for the geometric Brownian motion and $\mu\left(y_t, t\right) = \rho\left(y_0 - y_t\right)$ for the Ornstein-Uhlenbeck process.

[13] With different initial conditions, a process with drift leads to the same results.

[14] The public signal represents the evolution of the aggregate output of the partnership. Other types of a public signals can be considered, provided that they depend on both players' actions. It is also important that $r_i\left(.\right)$ does not depend on a_{-i} explicitly.

[15] A public history, a time t, is a sequence of realizations of the observed state of the process, denoted by

strategy is public if depends only on the public histories and not on player $i's$ private history. Given a public history, a strategies that induces a Nash equilibrium on the continuation game, from time t on, is called a *perfect public equilibrium* (PPE).

Without loss of generality, we assume a common exponential discount factor $\delta \equiv e^{-r\Delta}$, where r is the discount rate.

A.   The Expected Joint Output and its Distribution

In this section, we examine in more detail the monitoring technology employed in the present paper. From now on, to keep the notation simple, we drop the t index. The subscript Δ refers to an end-of-period object.

The signal space generated by (3.1) can take any value in $\mathbb{R}$. Players use a threshold strategy to distinguish realizations suggesting equilibrium play, i.e., $\{y_\Delta > b\}$ which we call "good signals", from realizations suggesting defection, i.e., $\{y_\Delta \leq b\}$ which we call "bad signals".[16]

In general, for a given expected joint output $y = 2\pi' (a_1 + a_2)$, the probability that the state of the public process (3.1) appears below b in the end of a period of length Δ is

$$\Pr (y_\Delta \leq b) = \Phi \left((b - 2\pi' (a_1 + a_2)) / \sigma \sqrt{\Delta} \right),$$

where $\Phi (.)$ is the standard zero mean and unit variance Gaussian distribution.

There are four possible action profiles: the strongly symmetric profile $a = (E, E)$, the asymmetric profiles $a' = (E, S) = (S, E)$ and the Nash profile $a^N = (S, S)$, which is trivially self-enforceable.[17] At the end of a period of length Δ, the expected joint output associated with a particular effort profile appears disturbed by some noise. Decreasing Δ, we are more likely to observe the process around the expected joint output. This way, players' inference about the others private effort choices becomes more precise.

$h^t = (y_t, y_{t-\Delta}, ..., y_\Delta) \in Y^t$, with $h^0 = Y^0 \equiv \varnothing$. The sequence of player $i's$ private effort choices is player $i's$ private history.

[16] Sannikov and Skrzypacz (2007), showed that a threshold is the best rule to detect unilateral deviations. For now, we contend with an arbitrary threshold, and further on we focus on the optimal threshold value.

[17] Since the setting is symmetric, there is no loss in generality in placing no distinction between the profiles (E, S) and (S, E). Also note that to keep the notation standard, until now a denoted a general action profile. With a slight abuse of notation, a now denotes the strongly symmetric effort profile.

Case (i) - For the SSE profile $a = (E, E)$, the ex-ante expected joint output is $y = 4\pi'$. The punishment probability is

$$F^{EE} = \Phi\left((b - 4\pi') / \sigma\sqrt{\Delta}\right).$$

This is a type I error probability; even though no player has deviated, in each period, punishment is initiated with positive probability. We want to protect this profile against a the deviation $a' = (S, E) = (E, S)$. The profile a' has associated an expected joint output of $y = 2\pi'$. The probability of punishment is

$$F^{ES} = \Phi\left((b - 2\pi') / \sigma\sqrt{\Delta}\right).$$

Analogously a type II error is given by $1 - F^{ES}$.[18] Clearly, we must have $F^{ES} > F^{EE}$.

Case (ii) - Asymmetric equilibrium payoffs require the profile $a' = (E, S) = (S, E)$. In this case, the probability of mistaken punishment is

$$G^{ES} = \Phi\left((\underline{b} - 2\pi') / \sigma\sqrt{\Delta}\right).$$

A deviation by player i must occur when the worst profile is due. A deviation by the effort providing player leads to the profile $a^N = (S, S)$. The expected joint output is null, i.e., $y = 0$. The probability of punishment is

$$G^{SS} = \Phi\left((\underline{b} - 0) / \sigma\sqrt{\Delta}\right).$$

Again, we must have $G^{SS} > G^{ES}$.

Notice that, depending on the profile we want to enforce, different incentives are required. For that reason, we need to distinguish between b and $\underline{b}$. Then, F^{ES} and G^{ES} do not have the same value, since they have associated different decision rules.

IV. THE BEST STRONGLY SYMMETRIC EQUILIBRIA

This section presents a set of general results that are independent of the monitoring intensity. They are useful for the following section when we focus on the limit case.[19] It also

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[18] A type II error is the event of not punishing a deviator.

[19] "on the limit" means $\Delta \downarrow 0$, sometimes also referred to as the "highest monitoring intensity" or "continuous monitoring". Throughout the paper, we frequently mention "an increase in the monitoring intensity" or "an increase in the monitoring frequency"; they refer to a decrease in Δ.

presents a characterization of the optimal decision rule associated with the value of the best SSE of the infinitely repeated partnership game.

The equilibrium profile we want to sustain is $a = (E, E)$. A profile where a single player deviates is denoted as a', and the Nash profile is denoted as a^N. These profiles have associated the stage game payoffs, π, π' and 0, respectively. See the payoffs matrix of Table I.

We apply the Abreu et al. (1986, 1990) bang-bang result to compute the best SSE payoff. Since the distribution of public signals is not convex, optimality requires an infinite punishment length, see Mirrlees (1974) and Porter (1983).

To find the expression that characterizes the best SSE payoff, we need to solve the following dynamic programing problem, which by symmetry is the same for both players:

$$\overline{v} = (1 - \delta) \pi + \delta \left[\left(1 - F^{EE} \right) \overline{v} + F^{EE} \underline{v} \right], \tag{4.1}$$

$$\overline{v} \geq (1 - \delta) \pi' + \delta \left[\left(1 - F^{ES} \right) \overline{v} + F^{ES} \underline{v} \right], \tag{4.2}$$

$$\underline{v} = (1 - \delta) 0 + \delta \left[p\underline{v} + (1 - p) \overline{v} \right], \tag{4.3}$$

$$p \in [0, 1]. \tag{4.4}$$

Expression (4.1) is the normalized discounted value of the relation when both players provide effort. Players receive the expected payoff π associated with the mutual effort, as well as a discounted expectation over the expected values $\overline{v}$ and $\underline{v}$, associated with the two types of signals that might be observed. Constraint (4.2) is an enforceability condition. The expected value of the game associated with the mutual effort has to be at least as good as the expected value of the game associated with a potential unilateral deviation. To minimize the probability of mistaken punishment, in equilibrium (4.2) must bind.

Expression (4.3) is the normalized discounted value of the punishment phase, where p is the probability with which the relation remains in this state. A value $p = 1$ means perpetual punishment, and $p = 0$ requires a single punishment period. Since a^N is a Nash equilibrium, punishment is trivially enforced.

Expression (4.3) can be solved for $\underline{v}$ to obtain

$$\underline{v} = (\delta (1 - p) \overline{v}) (/1 - \delta p). \tag{4.5}$$

Plugging $\underline{v}$ into (4.1) and (4.2) and making the latter hold with equality, we obtain the

enforceability condition

$$1/\delta - \left(F^{ES}\pi - F^{EE}\pi' \right) / \left(\pi' - \pi \right) = p \in [0, 1], \tag{4.6}$$

which we require in order to satisfy (4.4).

If there is no way to satisfy $p \le 1$, no equilibrium other than the infinite repetition of the static Nash can be sustained, i.e., $\overline{v} = 0$. In this case, we say that the set of SSE payoffs degenerates. On the other hand, $p < 0$ does not pose a problem, we can adjust the optimal cutoff rule to keep $p \in [0, 1]$.

After replacing (4.5) and the enforceability condition (4.6) into (4.1), we can solve for $\overline{v}$ to obtain the expression for the best SSE,

$$\overline{v} = \pi - F^{EE} \left(\pi' - \pi \right) / \left(F^{ES} - F^{EE} \right). \tag{4.7}$$

The following result characterizes the optimal cutoff $b^* (\Delta)$ that maximizes (4.7).

Lemma 1 *Under (3.1), the strategy that achieves the best SSE payoff $\overline{v}^*$, requires perpetual punishment $\underline{v} = 0$ the first time the process is observed below $b^* (\Delta)$. Where $b^* (\Delta) \le b^p (\Delta)$ is called the optimal threshold and solves*

$$F^{ES}\pi - F^{EE}\pi' = (1 - \delta) \left(\pi' - \pi \right) / \delta, \tag{4.8}$$

and

$$b^p (\Delta) = 3\pi' + \sigma^2 \Delta \ln \left(\pi/\pi' \right) / 2\pi', \tag{4.9}$$

is an upper bound on the optimal decision rule.

The bang-bang solution in this case is trivial, it maximizes $\overline{v}$ while pushing $\underline{v}$ to its lowest feasible value. This is because the ABM is a Gaussian process and the distribution of the public signals is not convex.

However, not explicitly mentioned in order to keep the notation simple, the solution $b^* (\Delta)$ depends on all the parameters of the model, i.e., π', π, r, Δ and σ. Consequently, since both F^{ES} and F^{EE} depend on $b^* (\Delta)$, they also depend on these parameters. The equality (4.8) gives an implicit function to compute $b^* (\Delta)$. It is simply the enforceability

condition (4.6) evaluated at $p = 1$ (or equivalently $\underline{v} = 0$, see (A1) in the Appendix). In addition, if an optimal decision rule $b^*(\Delta)$ exist it must take a value below $b^p(\Delta)$.[20]

Existence and Uniqueness of the Optimal Decision Rule - The function p in (4.6) is strictly convex in b with a unique minimum value at $b^p(\Delta)$. When $p(b^p(\Delta)) < 1$, we have two threshold values, say b_1 and b_2, that satisfy $p(b_1) = p(b_2) = 1$. In this case the cutoff value is not unique. Suppose $b_1 \leq b_2$, since $\partial \overline{v}/\partial b < 0$, the associated SSE payoffs are respectively $\overline{v}_1 \geq \overline{v}_2$. Then, it is not admissible to choose a threshold other than the one associated with the larger SSE payoff, i.e., b_1.

Definition 2 *We say that a threshold value b is admissible when it has associated the largest payoff $\overline{v}$. If in addition $p(b) \in [0, 1]$ we say that such threshold is also feasible.*

Expression (4.9) establishes an upper bound and the existence of a $b^*(\Delta)$ is in the interval $(-\infty, b^p(\Delta))$. Admissibility is then a proxy to uniqueness. However, not all $b < b^p(\Delta)$ are feasible. In particular, if the minimum value $p(b^p(\Delta))$ is larger than one, we cannot enforce the profile (E, E). The set of feasible and admissible thresholds is then $[b^*(\Delta), b^p(\Delta))$.[21]

Even though there may be a continuum of feasible thresholds, since $\partial \overline{v}/\partial b < 0$ the optimal choice is $b^*(\Delta)$. The problem is that for large Δ, the set of feasible threshold values vanishes. Denote this monitoring intensity by $\overline{\Delta}$. It corresponds to the cutoff monitoring frequency below (above) which we can (cannot) enforce the profile (E, E).

Lemma 3 *Let σ and r be constant. There exist a nonempty interval $(0, \overline{\Delta})$ where cooperation can be enforced. The value $\overline{\Delta}$ is the Δ solution of (4.8) evaluated at (4.9).*

Then, exist always some interval where cooperation can be enforced, even if that requires monitoring at very high frequencies. Depending on the model's parameters, $\overline{\Delta}$ might take a larger or a smaller value.

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[20] The two point set $\{0, \overline{v}^*\}$ associated with (4.8) in Lemma 1 is self-generating, since the continuation values $\underline{v} = \pi^N = 0$ and $\overline{v}^*$ are elements of the set. Public correlation allows us to convexify the set into an interval $[0, \overline{v}^*]$.

[21] We could have defined admissibility in a different but equivalent way. Notice that the function $\underline{v}$ is strictly concave in b, taking a unique unrestricted maximum value for some $b^v \in (-\infty, b^p]$. When at b^v we have $\underline{v}(b^v) > 0$, there are two thresholds that satisfy the optimality condition $\underline{v}(b_1) = \underline{v}(b_2) = 0$. The lower of the two is the admissible one. The set of feasible and admissible threshold would be $\{b \leq b^v : \underline{v}(b) \geq 0\} \subseteq \{b \leq b^p : p(b) \in [0, 1]\}$.
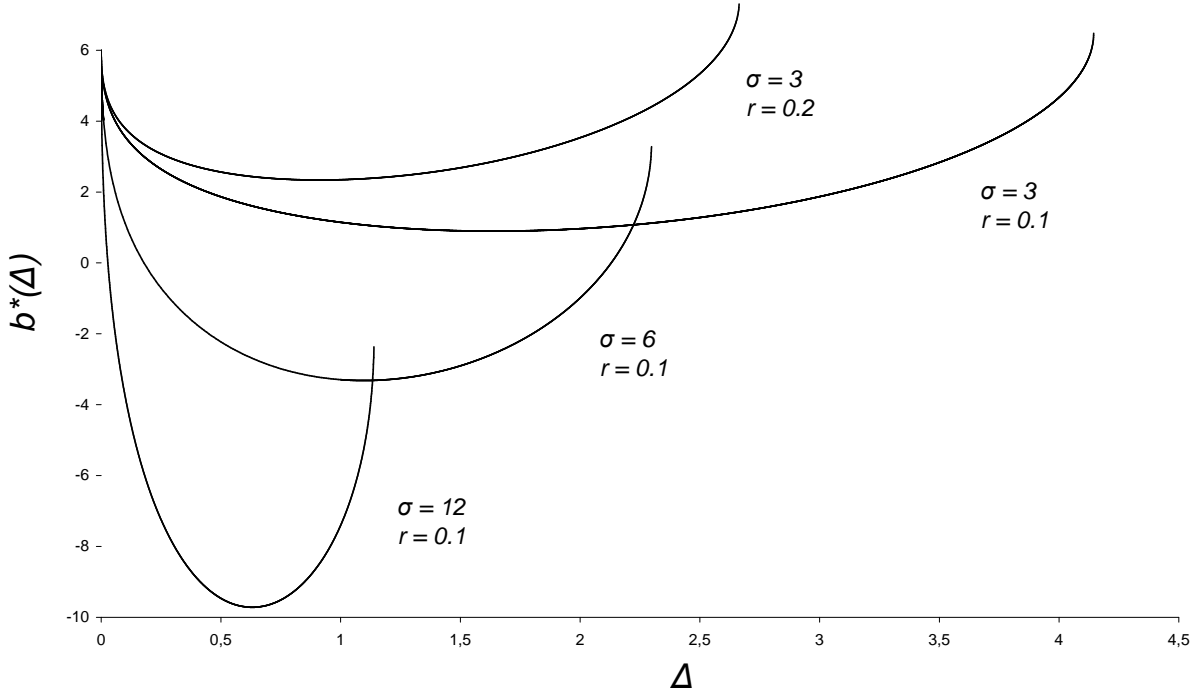
FIG. 1: The optimal threshold as a function of Δ $(\pi' = 3$ and $\pi = 2)$.

Note that $b^*(\Delta) = b^p(\Delta)$ at $\Delta = \overline{\Delta}$, while if $\Delta < \overline{\Delta}$ we can enforce (E, E) with $b^p(\Delta)$ but we can do better, if we choose $b^*(\Delta) \leq b^p(\Delta)$. On the other hand, for values of $\Delta > \overline{\Delta}$ we cannot enforce (E, E), because neither $b^p(\Delta)$ nor $b^*(\Delta)$ exist.

By Lemma 1, if a solution $b \leq b^p(\Delta)$ to (4.8) exists, it is an optimal threshold. The following result establishes some additional properties.

Lemma 4 *For $\Delta \in \left(0, \overline{\Delta}\right)$, a solution $b^*(\Delta) \in (-\infty, b^p(\Delta))$ to (4.8) exists and is unique and differentiable.*

The threshold $b^*(\Delta)$ adjusts smoothly to changes in Δ. When $\Delta > \overline{\Delta}$ we cannot guarantee the existence of the function $b^*(\Delta)$.

Numerical Illustration of the Optimal Threshold - We finish the Section with an illustration, Figure 1, of most results presented.

The value $\overline{\Delta}$ is the point (to the right) where each curve ends. It increases, either because players become more patient or because the public signals are less noisy.

In addition, the more impatient (or lower the noise) the players are, the tighter the monitoring has to be in order to create incentives.

The strict convex shape of the threshold function for $\Delta \in \left(0, \overline{\Delta}\right)$ is caused by two effects that operate in the same direction. As the monitoring intensity increases, i.e., Δ becomes small, the public signals become more informative and the expected immediate gains from deviation become less important. For that reason, the optimal threshold approaches $2\pi'$, the value to which the output would fall in the perfect monitoring case when some player deviates. Such a result is formally shown in Lemma 5 of Section V.[22]

As the monitoring intensity decreases, the sum of infinitesimal variations of the process becomes more likely to generate "bad signals". Wrong punishments in the equilibrium path become more likely, because the public signal loses precision. The threshold relaxes, in order to minimize such a possibility. At the same time, the expected immediate gains from a deviation become more attractive. At a certain point, after reaching its minimum value, the optimal threshold starts increasing at an increasing rate, creating the U-shape.

V. MONITORING FREQUENCY AND LIMIT EFFICIENCY

In this section, we focus on the limit case. We show that under Brownian uncertainty monitoring intensity has a positive effect on the payoffs. Finally, we present the main result of this paper, a Δ-limit folk theorem.

A. The Limit Value of the Optimal Threshold

We start by studying the limit value of the optimal decision rule $b^* \left(\Delta\right)$. Figure 1 above provides an illustration, and the following result formalizes it.

Lemma 5 *When $\Delta \downarrow 0$ the optimal threshold $b^* \left(\Delta\right)$ converges to $2\pi'$, i.e., the expected signal associated with the deviation with less impact on the distribution of the public signals.*

Because different action profiles have associated differing expected output levels, in the limit, signals become perfectly informative about players' actions. We have asymptotic

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[22] It is also true that $\partial b^*/\partial \Delta \downarrow -\infty$ when $\Delta \downarrow 0$ and $\partial b^*/\partial \Delta \uparrow \infty$ when $\Delta \uparrow \overline{\Delta}$.

perfect monitoring. Under perfect monitoring, after a deviation, the associated deterministic output would be observed at the level $2\pi'$. That is exactly the point to where it converges $b^*(\Delta)$. The result is more general; $b^*(\Delta)$ must converge to the expected signal associated with the deviation with less impact on the distribution of the public signals.

As mentioned, $b^*(\Delta)$ cannot be obtained in a close form. For that reason, it is nontrivial to verify the rate at which $b^*(\Delta)$ converges to $2\pi'$. Suppose that the optimal threshold function has the following structure: $b^*(\Delta) = 2\pi' - \Delta^\alpha k(.)$ with $\alpha > 0$, where $k(.)$ is some function of the parameters of the model. If $\alpha \geq 1/2$, efficient results obtain in the limit, but enforceability holds with slack. On the other hand, if $\alpha < 1/2$, the enforceability condition (4.8) fails. These experiments suggest that $b^*(\Delta) \uparrow 2\pi'$ at the rate $\Delta^{0.49(9)}$. We do not develop this idea further, because of its complexity and low relevance for the present paper. Nonetheless, we point this out.

B. Monotonicity of the Best SSE Payoff

A relevant question, is how does the value of $\overline{v}^*$ change with Δ? The following result establishes a monotonic relation between the best SSE payoff and monitoring intensity.

Proposition 6 *In the interval $\left(0, \overline{\Delta}\right)$, the best SSE payoff $\overline{v}^*$ increases monotonically with the monitoring intensity, i.e., with a decrease in Δ.*

More monitoring improves the payoffs and has a positive effect on the incentives. Figure 2 illustrates the strict monotonic improvement in the best SSE payoffs towards efficiency.

In another context, Kandori (1992) showed a similar result, where an exogenous improvement in the precision of the public signals expanded the set of PPE payoffs. A monotonic relation between monitoring intensity and payoffs can also be found in the Abreu et al. (1991) "bad news" model and in the Fudenberg and Levine (2007) volatility sensitive model. As in the present paper, in these models the information quality improves with a decrease in Δ.

When $r \downarrow 0$, i.e., the players become arbitrarily patient, the value $\overline{v}^* \uparrow \pi$. The best SSE payoff is fully efficient. In addition, we have $b^*(\Delta) \uparrow \infty$. Such a result is well known and
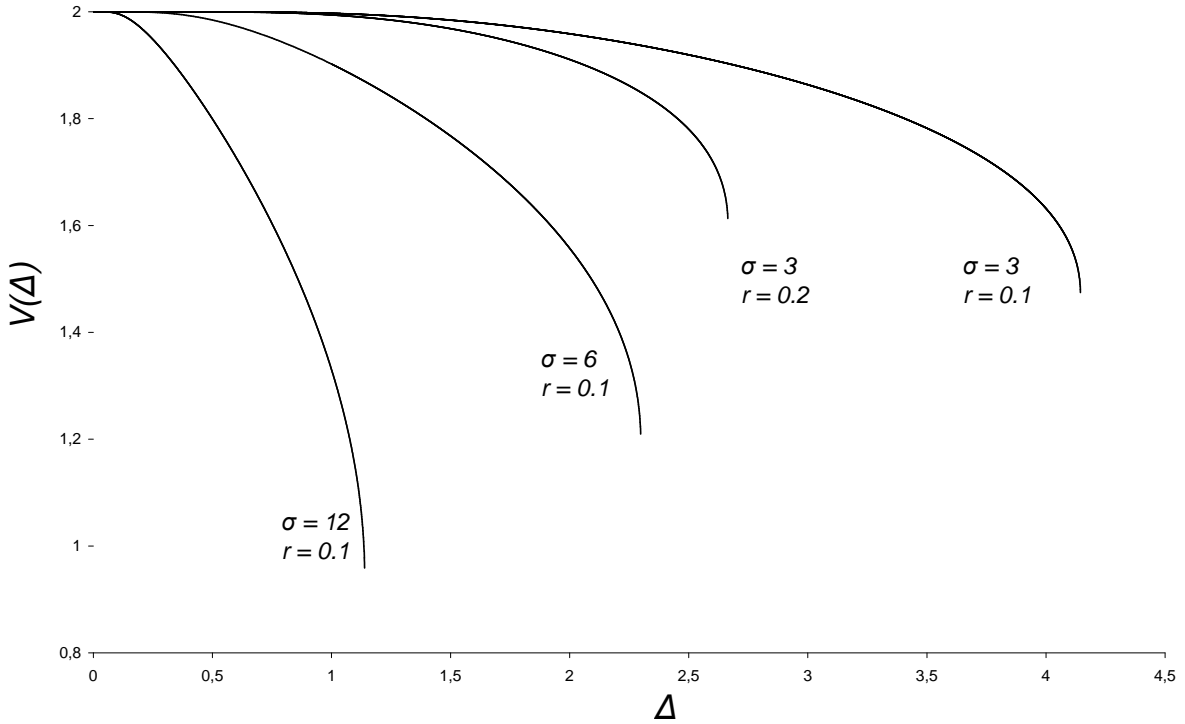
FIG. 2: The best SSE payoff as a function of Δ.

leads to a folk theorem in the limit as $r \downarrow 0.$[23] In fact, whatever efficiency is achieved by letting $r \downarrow 0$ can be achieved all the more by letting $\Delta \downarrow 0$. The reason is that a decrease in Δ has associated an informational gain that is not present when r decreases.[24] Consequently, we have the following result.

Proposition 7 *The best SSE payoff $\overline{v}^*$ converges to π faster with Δ than with r.*

C. The Δ-Limit Folk Theorem

In Lemmas 1, 3, 4 and 5, we developed a great knowledge about the threshold $b^*(\Delta)$. A threshold larger than $b^*(\Delta)$, but below $b^p(\Delta)$, still satisfies the enforceability conditions (4.8) but with slack. Limit efficiency does not require an optimal value for b, as in the limit,

[23] It happens because the distribution of the public signals is Gaussian and has unbounded support. See Mirrlees (1974).

[24] I am thankful to an anonymous referee who has pointed out to me this reasoning.

the same result holds with a threshold that is feasible and admissible according to Definition 2. Consequently, we have $F^{EE} \downarrow 0$ and $F^{ES} \uparrow 1$.[25]

For the optimal threshold $\underline{b}^*(\Delta)$ associated with an asymmetric path that starts with the profile (E, S), similar reasoning applies.[26] Notice that F^{ES} and G^{ES} do not take the same value, because they have associated different decision rules, $b^*(\Delta)$ and $\underline{b}^*(\Delta)$, respectively. Following Lemma 5, the threshold $\underline{b}^*(\Delta)$ must converge to 0, the joint output associated with a deviation from the effort providing player. Consequently, according to Definition 2, any choice $\underline{b} \in [\underline{b}^*(\Delta), \underline{b}^p(\Delta))$ is feasible and admissible. In this case, we have $G^{ES} \downarrow 0$ and $G^{SS} \uparrow 1$.[27]

Then a Δ-limit folk theorem must holds for the partnership game.

Since the claim does not rely on any feature of the prisoners' dilemma, but on the informativeness of the public signals, a Δ-limit folk theorem must generalize to games with more players and richer action spaces. Let $V(\Delta, r)$, denote the set of PPE payoffs for a given discount rate r and a monitoring intensity Δ. Let V^+ be the set that contains every feasible and individual rational payoff, with nonempty interior. Assume that every distinct action profile is pairwise identifiability, i.e., has associated a different probability distribution.

Since signals are unbounded, we have $\lim_{r \downarrow 0} V(\Delta, r) \uparrow V^+$, see Mirrlees (1974). Consequently, and because a decrease in Δ has the same effect has a decrease in r plus a informational gain, see Proposition 7, we must have $\lim_{\Delta \downarrow 0} V(\Delta, r) \uparrow V^+$.

Proposition 8 (Δ-limit folk theorem) *Providing that the r and σ are bounded, a folk theorem obtains when $\Delta \downarrow 0$.*

In the limit, the public signals become perfectly informative about players' actions.[28] In addition, the infinitesimal expected gains from a deviation become irrelevant with respect to the potential punishments.

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[25] Enforceability holds with slack, but we are still able to obtain perfectly informative signals when $\Delta \downarrow 0$.

[26] Notice that, in general, an asymmetric path does not need to start with the profile (E, S). However, the optimal moment for a deviation is when the profile (E, S) is due.

[27] In order for an efficient result to obtain, the probabilities F^{ES} and G^{SS} need not converge to one. A limit efficient result is possible with F^{EE} and G^{ES} converging to zero, wile F^{ES} and G^{SS} converging to some value larger than zero, or even to zero but at a lower rate. Fudenberg and Levine (2007) discussed the necessity of similar conditions for the existence of an efficient limit equilibrium.

[28] The result holds true for other well-known Gaussian processes, as e.g., the geometric Brownian motion or the Ornstein-Uhlenbeck process.

The increase informativeness of the Brownian signals for high monitoring intensities is the key aspect. It is due to the measurable distance between the expected joint output values associated with each effort profile. Such a distance in a process of infinitesimal variation is critical. Relevant uncertainty arises only if players cannot observe the public process during some measurable time interval. Then the accumulated sum of infinitesimal normal events may be misleading, which is more likely, the larger the time interval is during which the process was left unattended.

In our setting, more monitoring cannot harm the monitor's side. As a result we developed a theory where imperfect public monitoring, in the limit, is equivalent to perfect monitoring.[29]

VI. POSSIBLE EXTENSIONS: SOME COMMENTS

In this section, we briefly discuss the case where players have a continuum of available actions.[30]

A. A Game with a Continuous Action Space

When the action space is discrete, in the limit, deviations from the equilibrium path are similar to jumps in the process. Since Brownian paths are continuous but not smooth, such defective behavior is almost surely detected. In this section, we briefly discuss the repeated Cournot game. This game is of interest since it has a continuum of actions, and deviations can be of infinitesimal magnitude.

In brief, the stage game expected payoffs (ex-ante) are given by $\pi_i(q_1, q_2) = q_i P(Q)$. Let $P(Q) = 1 - Q$ be the inverse demand function and $Q = q_1 + q_2$ the aggregate supply. Without loss of generality, we assume that production costs are zero and no capacity constraints, i.e.,

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[29] As discussed in the Introduction, these results are in line with the Alchian and Demsetz (1972) theory that defended the disciplinary effect of monitoring and the Dickinson and Villeval (2008) empirical findings that monitoring has a positive effect on the individuals' effort. See also the references therein.

[30] It's also interesting to think about the relation between the limit of a sequence of discrete time games and the continuous time version. A number of technical issues arise when trying to define a continuous time version of the model presented in this paper. See Simon and Stinchcombe (1989), Bergin and MacLeod (1993), and Fudenberg and Levine (1986).
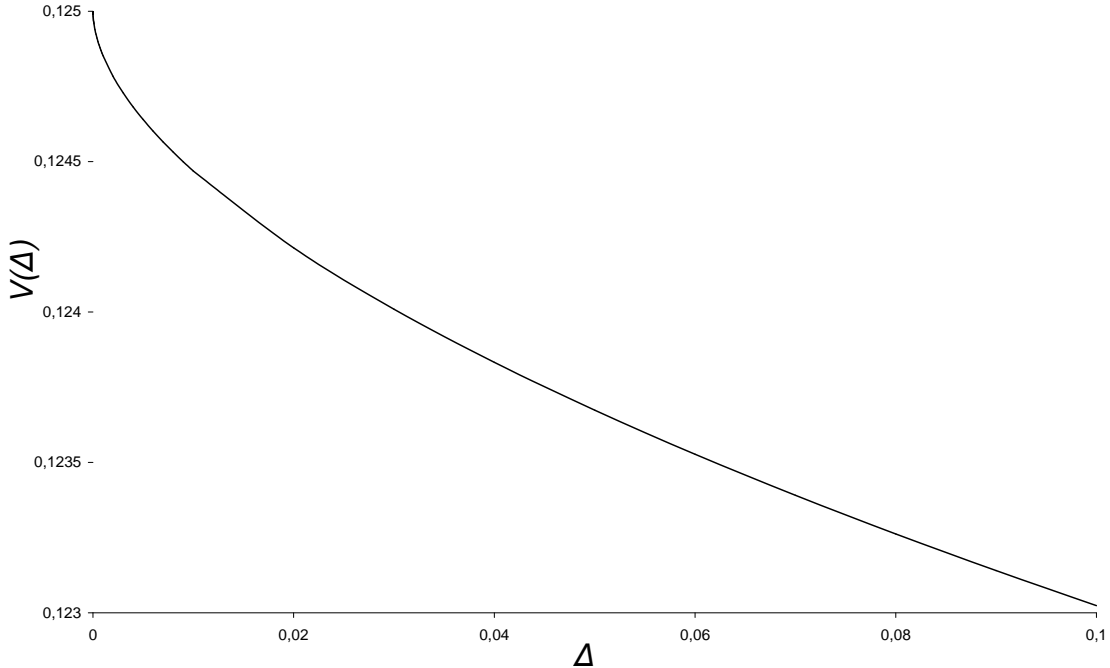
FIG. 3: The best SSE payoff in the Cournot game ($\sigma = r = 0.1$).

$q_i \in [0, \infty)$. The two firms decide their supply quantities simultaneously and independently at moments in time $t = 0, \Delta, 2\Delta...$, and observe the market price $y_{t+\Delta}$ (the state of the public process (3.1)) at times $t = \Delta, 2\Delta....$[31] Note that $y_t = P(Q_t)$ is the expected end-of-period market price associated with the individual private supply decisions chosen by each firm in the beginning of the period t.

Our goal is to enforce the SSE payoff. In a Cournot duopoly with imperfect public information, it is never optimal to produce exactly the monopoly quantities $q_i^M = 1/4$, but rather, an amount slightly larger (except in the limit). Denote this quantity as q_i^*. To find it, we apply the Abreu et al. (1986, 1990) bang-bang result.[32]

Figure 3 shows the value of the best SSE payoff for varying Δ. In particular, when Δ becomes small, the best SSE payoff converges to the full efficient value $1/8$, i.e., the perfect monitoring payoff. The numerical approximation suggest that the efficient limit result of

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[31] The observed state of an ABM price process may take negative values. There is no loss of generality when allowing for such possibility. A geometric Brownian motion process solves the problem.

[32] I thank Andrzej Skrzypacz for providing me with the material needed to compute and understand the mechanics of the best SSE in the Cournot game.

Propositions 8 holds with a continuum of actions. A monotonic improvement in the payoffs is clear when Δ becomes small, in line with Proposition 6.

It is interesting to contrast the evolution of the optimal threshold $b^*(\Delta)$ with the expected signal of the process $P(Q^*)$. We found that as Δ gets small, the threshold value becomes tighter. Converging in the limit, to the expected signal associated with the most collusive equilibrium $P(Q^M) = 1/2$. The observation is an extension of Lemma 5 for games with a continuous action space, where the deviation with less impact on the distribution of the public signals is infinitesimal, for that reason $b^*(\Delta) \to P(Q^M)$.

VII.   FINAL COMMENTS

In economics it is hard to think of situations where information is continuously available. The price of very liquid stocks or certain commodities are available at high frequencies, but not continuously. In spite of that fact, if such a possibility were available, this paper shows that the most efficient outcomes might be achieved by continuously monitoring the state of the process.

In practice, we observe agents monitoring at discrete moments in time either because it unfeasible or it does not compensate the potential benefits. Quoting Alchian and Demsetz (1972, p. 780), *"If detecting such behavior were costless, neither party would have an incentive to shirk, because neither could impose the cost of his shirking on the other."*.

On many occasions, monitoring events might be random, i.e., an agent does not control the timing at which the supervisor accedes to the available information. Osório (2008) and Fudenberg and Olszewski (2011) studied problems of this kind. In addition, when a partner continuously monitors the other, she cannot devote her time to other activities. Such requires specialization of the monitoring activities.

We stress that while players' impatience is typically exogenous, monitoring frequency has an enormous appeal to be endogenously determined, opening new research avenues and bringing new tools to better understand the complexity of real economic problems.

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APPENDIX A: PROOFS OF THE LEMMAS AND PROPOSITIONS.

Proof of Lemma 1. After solving the system composed by (4.1), (4.2), (4.3) and (4.4), we obtain p, $\overline{v}$, and $\underline{v}$, given respectively by (4.6), (4.7), and

$$\underline{v} = \overline{v} - (1 - \delta)(\pi' - \pi) / \left(\delta \left(F^{ES} - F^{EE} \right) \right),\tag{A1}$$

which is obtained after replacing (4.6) and (4.7) into (4.5). Expression (4.7) and (A1) are the upper and lower bounds on the set of SSE payoffs, respectively. Our goal is to find the optimal cutoff b, which maximizes $\overline{v}$ subject of $p \in [0, 1]$. We start by showing that $\overline{v}$ increases monotonically with a decrease in b. Differentiate $\overline{v}$ with respect to b, we obtain

$$\partial \overline{v} / \partial b = -(\pi' - \pi) \left(F^{ES} F_b^{EE} - F^{EE} F_b^{ES} \right) / \left(F^{ES} - F^{EE} \right)^2,$$

which is always negative, since for the Gaussian distribution, $F^{ES} F_b^{EE} - F^{EE} F_b^{ES} > 0$ for all b, where, we denote the partial derivatives of F^{EE} and F^{ES} with respect to b as $F_b^{EE} \equiv \partial F^{EE} / \partial b$ and $F_b^{ES} \equiv \partial F^{ES} / \partial b$, respectively. Consequently, b must be as small as possible, but constrained to satisfy (4.6). For that we need to know how b changes with p; in particular, we are interested in finding out under which conditions $\partial b / \partial p \leq 0$. Implicitly differentiating (4.6), we obtain

$$\partial b / \partial p = -(\pi' - \pi) / \left(F_b^{ES} \pi - F_b^{EE} \pi' \right).$$

This derivative is negative if

$$F_b^{EE} / F_b^{ES} = \exp \left(\left(4b\pi' - 12\pi'^2 \right) / 2\Delta\sigma^2 \right) < \pi / \pi'.\tag{A2}$$

The RHS is a number in the interval $(1/2, 1)$. Recall that we assume $2\pi > \pi'$. The LHS is continuous and monotonically increasing in b. When $b = 3\pi'$, it takes the value 1, which does not satisfy the inequality. When $b \to -\infty$, the LHS goes to 0, which satisfies the inequality. Then, by continuity and monotonicity of the LHS in b, there must be a value b below $3\pi'$, at which the inequality (A2) holds. This value is given by

$$b^p (\Delta) = 3\pi' + \sigma^2 \Delta \ln (\pi/\pi') / 2\pi',$$

and it is an asymptote of $\partial b / \partial p$. Note that $b^p (\Delta)$ is also the unique value that minimizes the LHS of (4.6) with respect to b. To see it, when (A2) holds with equality, we have $\partial p / \partial b = 0$. The second order condition is

$$\partial^2 p / \partial b^2 = - \left(F_{bb}^{ES} \pi - F_{bb}^{EE} \pi' \right) / (\pi' - \pi),\tag{A3}$$

23

where, the second derivatives of F^{EE} and F^{ES} with respect to b are denoted respectively as $F_{bb}^{EE} \equiv \partial^2 F^{EE} / \partial b^2$ and $F_{bb}^{ES} \equiv \partial^2 F^{ES} / \partial b^2$. The derivative in (A3) is strictly positive. To see it, notice that $F_{bb}^{EE} (b, \Delta) = - (b - 4\pi') F_b^{EE} / \Delta \sigma^2$ and $F_{bb}^{ES} (b, \Delta) = - (b - 2\pi') F_b^{ES} / \Delta \sigma^2$. Replace these expressions in (A3) and use the first order condition (A2) holding with equality, to obtain

$$\partial^2 p / \partial b^2 = 2\pi'^2 F_b^{EE} / \left(\Delta \sigma^2 (\pi' - \pi) \right) > 0.$$

So, $b^p (\Delta)$ is the value that minimizes (4.6), but note that this is an unconstrained minimum. For that reason, p might take a negative value at $b^p (\Delta)$, or even be larger than 1. In either case, the condition (4.6) is not satisfied since $p \in [0, 1]$. While, in the latter case we cannot enforce the profile (E, E), in the former case we can decrease b below $b^p (\Delta)$, until p be at least equal to 0, i.e., feasible, because $\partial b / \partial p < 0$ for $b < b^p (\Delta)$. But we can do even better in this direction, because the value of b that maximizes $\bar{v}$ and satisfies (4.6) can be pushed even lower, until $p = 1$. Such a value of $b < b^p (\Delta)$ is the optimal threshold $b^* (\Delta)$ and by (4.5) has associated the optimal punishment $\underline{v} = 0$, i.e., perpetual punishment. ∎

Proof of the Lemma 3. From Lemma 1 we know that $b^* (\Delta) = \{b : \underline{v} = 0\} = \{b : p = 1\}$ and that $b^* (\Delta) \leq b^p (\Delta) = \arg \min_b p$. When $\min_b p \leq 1$, we can enforce (E, E) with $b^p (\Delta)$, but we can do better in payoff terms if we choose $b^* (\Delta) \leq b^p (\Delta)$. However, there is a values of Δ for which $\min_b p > 1$. Consequently, we cannot enforce (E, E) - neither with $b^p (\Delta)$ nor with $b^* (\Delta)$ - because it does not exist. Similarly, we can define a new unconstrained threshold $b^v (\Delta) = \arg \max_b \underline{v}$. In this case, when $\max_b \underline{v} \geq 0$, we can enforce (E, E) with $b^v (\Delta)$, but we can do better in payoff terms if we choose $b^* (\Delta) \leq b^v (\Delta)$. Again there must be a monitoring frequency for which $\max_b \underline{v} < 0$. Since the equilibrium condition is $\underline{v} = 0$, which is equivalent to $p = 1$, there must be a common monitoring frequency $\Delta = \overline{\Delta}$ at which $\min_b p = 1$ and $\max_b \underline{v} = 0$. Consequently, for $\Delta < \overline{\Delta}$, we must have both $\min_b p < 1$ and $\max_b \underline{v} > 0$, while for $\Delta > \overline{\Delta}$, we have both $\min_b p > 1$ and $\max_b \underline{v} < 0$, and $b^* (\Delta)$ does not exist.

We now search for the value $\Delta = \overline{\Delta}$. In the proof of Lemma 1 we found that p is strictly convex in b. In addition $\min_b p$ gives the first order condition,

$$F_b^{EE} / F_b^{ES} = \pi / \pi',$$

where $b^p (\Delta)$ solve the equality. It can be shown that $\underline{v}$ is strictly concave with a unique maximum - to simplify let's assume that it is. After some arrangements, the first order

condition of $\max_b \underline{v}$ can be written as

$$\delta \left(F^{ES} F_b^{EE} - F^{EE} F_b^{ES} \right) = (1 - \delta) \left(F_b^{ES} - F_b^{EE} \right),$$

where $b^v (\Delta)$ solves the equality. Because at $\overline{\Delta}$ we have $b^p (\Delta) = b^v (\Delta)$, the two first order conditions must be satisfied at the same value. Manipulating both equalities we obtain

$$F^{ES} (b^p (\Delta)) \pi - F^{EE} (b^p (\Delta)) \pi' = (1 - \delta) (\pi' - \pi) / \delta,$$

which is (4.8) evaluated at $b = b^p (\Delta)$, where $b^p (\Delta)$ is given by (4.9). The value of Δ that solves this equality is $\overline{\Delta}$.

The value $\overline{\Delta}$ has the following asymptotic properties when σ or r goes to ∞ we have $\overline{\Delta} \to 0$, while if $\sigma \to 0$ we obtain $\overline{\Delta} \to \ln (\pi'/ (\pi' - \pi)) /r$, and if $r \to 0$ we get $\overline{\Delta} \to \infty$. Consequently, providing that σ and r are bounded, the interval $\left(0, \overline{\Delta} \right)$ is always guaranteed to be nonempty, i.e., $\overline{\Delta} > 0$. ∎

Proof of Lemma 4. We extend the local implicit function theorem to hold in the convex interval $\left(0, \overline{\Delta} \right)$. Lemma 3 tells us how to find the value $\overline{\Delta}$. Moreover, for bounded σ and r, a value of $\overline{\Delta} > 0$ always exists. Rewrite the equality (4.8) in the following way and denote it as $I (b, \Delta)$, i.e.,

$$I (b, \Delta) \equiv F^{ES} \pi - F^{EE} \pi' - (1 - \delta) (\pi' - \pi) / \delta. \tag{A4}$$

Since F^{EE}, F^{ES} and δ are continuous and differentiable with respect to $\Delta \in \left(0, \overline{\Delta} \right)$ and $b \in (-\infty, b^p (\Delta)]$, so thus the mapping $I (b, \Delta)$ is continuous and differentiable.

From, Lemma 1 and Lemma 3, for any $\Delta_0 \in \left(0, \overline{\Delta} \right)$ there is exactly one $b_0 \in (-\infty, b^p (\Delta)]$ such that $I (b_0, \Delta_0) = 0$. If $\Delta_0 > \overline{\Delta}$ there is no such value of b_0, and if $b \in \mathbb{R}$, b_0 is not unique, because p given in (4.6) is strictly convex with a minimum at $b^p (\Delta)$. Consequently, point (i) of Sandberg's (1981, p. 146) global implicit function theorem is satisfied.

Additionally, we need to verify that $I (b, \Delta)$ is locally solvable in the neighborhood of the point (b_0, Δ_0) - in which case, by continuity of $b (\Delta)$, Sandberg's theorem holds for all $\Delta \in \left(0, \overline{\Delta} \right)$. The condition for local solvability is $\partial I (b_0, \Delta_0) / \partial b \neq 0$, implying that $I (b_0, \Delta_0) = 0$. The differentiability condition is written as

$$F_b^{ES} \pi - F_b^{EE} \pi' \neq 0,$$

where, the partial derivatives of F^{EE} and F^{ES} with respect to b are denoted respectively as $F_b^{EE} \equiv \partial F^{EE} / \partial b$ and $F_b^{ES} \equiv \partial F^{ES} / \partial b$. In the proof of Lemma 1 we have seen that this

equation is satisfied for all $b_0 \neq b^p(\Delta)$, with $b^p(\Delta)$ given by (4.9). However, to improve on the value of the best SSE payoff we are interested in $b_0 \leq b^p(\Delta)$, implying that $\partial I(b_0, \Delta_0)/\partial b \geq 0$. Note that $b_0 = b^p(\Delta)$ (and consequently, $\partial I(b_0, \Delta_0)/\partial b = 0$) only at $\Delta = \overline{\Delta}$. But for all $\Delta_0 \in (0, \overline{\Delta})$, we always have $b < b^p(\Delta)$ (and consequently, $\partial I(b_0, \Delta_0)/\partial b > 0$). Then for each point inside the interval $(0, \overline{\Delta})$ there is a unique, continuous and differentiable function $b(\Delta)$ on an open ball around (b_0, Δ_0) that locally satisfies $I(b, \Delta) = 0$.

Now, apply Sandberg's implicit function theorem: by continuity of $b(\Delta)$, for each $S \in A$ there is a $T \in B$, where A and B are the families of compact subsets of $(0, \overline{\Delta})$ and $(-\infty, b^p(\Delta))$, respectively. Then $b(S)$ is also compact and belongs to T. Implying that there is a unique, continuous and differentiable function $b^*(\Delta)$, such that $I(b^*(\Delta), \Delta) = 0$ for all $\Delta \in (0, \overline{\Delta})$. ∎

Proof of Lemma 5. In the proof of Lemma 1, we found that $\partial\overline{v}/\partial b < 0$, i.e., the lower the value b is, the larger is the payoff $\overline{v}$. Moreover, Proposition 6 states that $\overline{v}^*$ improves monotonically as Δ gets small. Consequently, the largest value $\overline{v}^*$ is reached in the limit $\Delta \downarrow 0$. Now, we want to find the lowest feasible limit value of b that maximizes $\overline{v}$ and satisfies (4.8). We rewrite (4.8) here, i.e.,

$$F^{ES}\pi - F^{EE}\pi' = (1 - \delta)(\pi' - \pi)/\delta.$$

The RHS goes to 0 with Δ, so in the limit the LHS must also go to 0 as well. Since, the upper bound $b^p(\Delta) \to 3\pi'$ when $\Delta \to 0$, we don't need to consider values of $b > 3\pi'$. Let's start by considering the case where $b \to x \in (2\pi', 3\pi']$. Then, we have $F^{ES} \to 1$ and $F^{EE} \to 0$, implying that the LHS goes to $(\pi - \pi^N) > 0$. The enforceability condition (4.8) is satisfied but with slack. So the limit value of $b^*(\Delta)$ must be lower.

Now, suppose that $b \to x \in (-\infty, 2\pi')$, in this case we have $F^{ES} \to 0$ and $F^{EE} \to 0$, both the LHS and the RHS goes to 0. We to check whether the LHS and RHS go to 0 at the same rate. When we differentiate both sides with respect to Δ we obtain

$$-\frac{b - 2\pi'}{2\Delta^{3/2}\sigma\sqrt{2\pi}}e^{-\frac{(b-2\pi')^2}{2\Delta\sigma^2}}\pi + \frac{b - 4\pi'}{2\Delta^{3/2}\sigma\sqrt{2\pi}}e^{-\frac{(b-4\pi')^2}{2\Delta\sigma^2}}\pi' = r(\pi' - \pi)/\delta. \quad (A5)$$

The limit on LHS is smaller than the limit on the RHS, i.e., $0 < r(\pi' - \pi)$. Meaning that the LHS of (4.8) is smaller than the RHS, i.e., we lose enforceability. So, when $\Delta \to 0$, we cannot have $b \to x \in (-\infty, 2\pi')$. Consequently, we must have $\lim_{\Delta \to 0} b^*(\Delta) \to 2\pi'$. The order of convergence must be such that (4.8) and (A5) hold with equality. ∎

Proof of Proposition 6. Start by defining $\overline{v}^* (\Delta) = \left\{ \max_{b \leq b^p (\Delta)} \overline{v} (b, \Delta) : I (b, \Delta) = 0 \right\}$ and write the Lagrangian $\mathcal{L} (b, \Delta) = \overline{v} (b, \Delta) - \lambda I (b, \Delta)$. By Lemma (4) the solution $b^* (\Delta)$ is a continuous and differentiable function of Δ; assume that the same holds for the Lagrangian multiplier λ. Then by the envelope theorem for constrained maximization, we can write $\partial \overline{v}^* (\Delta) / \partial \Delta = \partial \mathcal{L} (b^* (\Delta), \Delta) / \partial \Delta$. Our goal is to show that

$$\partial \overline{v}^* (\Delta) / \partial \Delta = \partial \overline{v} (b^* (\Delta), \Delta) / \partial \Delta - \lambda \partial I (b^* (\Delta), \Delta) / \partial \Delta < 0, \tag{A6}$$

where λ is obtained from solving $\partial \mathcal{L} (b, \Delta) / \partial b = 0$. Expression (A6) has the following three components:

$$\partial \overline{v} (b^* (\Delta), \Delta) / \partial \Delta = - \left(F^{ES} F_\Delta^{EE} - F^{EE} F_\Delta^{ES} \right) (\pi' - \pi) / \left(F^{ES} - F^{EE} \right)^2,$$
$$\partial I (b^* (\Delta), \Delta) / \partial \Delta = F_\Delta^{ES} \pi - F_\Delta^{EE} \pi' - r (\pi' - \pi) / \delta,$$

and

$$\lambda = - \left(F^{ES} F_b^{EE} - F^{EE} F_b^{ES} \right) (\pi' - \pi) / \left(\left(F^{ES} - F^{EE} \right)^2 \left(F_b^{ES} \pi - F_b^{EE} \pi' \right) \right), \tag{A7}$$

where $F_\Delta^{EE} = \partial F^{EE} / \partial \Delta$, $F_\Delta^{ES} = \partial F^{ES} / \partial \Delta$, $F_b = \partial F^{EE} / \partial b$ and $F_b^{ES} = \partial F^{ES} / \partial b$, are evaluated at $b = b^* (\Delta)$. Replace these expressions into (A6), after some algebra we get

$$-r (\pi' - \pi) \left(F^{ES} F_b^{EE} - F^{EE} F_b^{ES} \right) / \delta < \left(F^{ES} \pi - F^{EE} \pi' \right) \left(F_b^{ES} F_\Delta^{EE} - F_b^{EE} F_\Delta^{ES} \right).$$

Apply (4.8), so that we can simplify further, to obtain

$$-r \left(F^{ES} F_b^{EE} - F^{EE} F_b^{ES} \right) < (1 - \delta) \left(F_b^{ES} F_\Delta^{EE} - F_b^{EE} F_\Delta^{ES} \right).$$

Note that $F_\Delta^{EE} = - (b^* (\Delta) - 4\pi') F_b^{EE} / 2\Delta$ and $F_\Delta^{ES} = - (b^* (\Delta) - 2\pi') F_b^{ES} / 2\Delta$; then after replacing, we obtain

$$-r \left(F^{ES} F_b^{EE} - F^{EE} F_b^{ES} \right) < (1 - \delta) F_b^{EE} F_b^{ES} \pi' / \Delta.$$

The LHS is always negative, while the RHS is always positive since for the Gaussian distribution $F^{ES} F_b^{EE} - F^{EE} F_b^{ES} > 0$ for any $\Delta > 0$. ∎

Proof of Proposition 7. Following the same arguments as in the Proof of Proposition 6. We can define $\overline{v}^* (r) = \left\{ \max_{b \leq b^p (r)} \overline{v} (b, r) : I (b, r) = 0 \right\}$, and write the Lagrangian $\mathcal{L} (b, r) = \overline{v} (b, r) - \lambda I (b, r)$. Similarly to $b^* (\Delta)$, it can be shown that a solution $b^* (r)$ exists, and it is

27

a continuous and differentiable function of r. Then by the envelope theorem for constrained maximization, and since $\lambda < 0$, we can show that

$$\partial \overline{v}^* (r) / \partial r = \lambda \Delta (\pi' - \pi) / \delta < 0,$$

where $\partial \overline{v} (b^* (r), r) / \partial r = 0$, $\partial I (b^* (r), r) / \partial r = -\Delta (\pi' - \pi) / \delta$ and λ is given by (A7). In order to infer the existence of an informational effect, we set $r = \Delta \in (0, \overline{\Delta})$. Consequently, if

$$\partial \overline{v}^* (r) / \partial r_{|r=\Delta} > \partial \overline{v}^* (\Delta) / \partial \Delta_{|r=\Delta},$$

is satisfied, a decrease on Δ has associated an extra gain associated with better inference. The previous inequality is then written as

$$\lambda \frac{\Delta}{\delta} (\pi' - \pi) > -\frac{F^{ES} F_\Delta^{EE} - F^{EE} F_\Delta^{ES}}{(F^{ES} - F^{EE})^2} (\pi' - \pi) - \lambda \left(F_\Delta^{ES} \pi - F_\Delta^{EE} \pi' - \frac{\Delta}{\delta} (\pi' - \pi) \right).$$

After some algebraic manipulations we obtain

$$F_b^{ES} \pi - F_b^{EE} \pi' > F_\Delta^{ES} \pi - F_\Delta^{EE} \pi'.$$

Since $F_\Delta^{EE} = - (b^* (\Delta) - 4\pi') F_b^{EE} / 2\Delta$ and $F_\Delta^{ES} = - (b^* (\Delta) - 2\pi') F_b^{ES} / 2\Delta$, and after making some rearrangements we obtain

$$F_b^{ES} \pi (2\Delta + b^* (\Delta) - 2\pi') / 2\Delta > F_b^{EE} \pi' (2\Delta + b^* (\Delta) - 4\pi') / 2\Delta.$$

The term inside brackets in the LHS is clearly larger than the term inside brackets in the RHS. By (A2), the remaining terms reinforce the inequality. ∎

Proof of Proposition 8. Let A_i be the action set of player i and $y : A_i \times A_{-i} \longrightarrow \mathbb{R}^d$ an arbitrary function satisfying a suitable pairwise identifiability condition. Suppose the public signal is distributed according to a d-dimensional multivariate Gaussian distribution with mean $y (a_i, a_{-i})$ and variance-covariance matrix given by $I_{d \times d} \sigma^2 \Delta$, where $I_{d \times d}$ is the d-dimensional identity matrix. Let the stage game payoffs satisfy the full dimensionality condition, i.e., V^+ has nonempty interior.

Following Mirrlees (1974), since the distribution of the public signals is Gaussian, it has unbounded support. Then, we must have $\lim_{r \downarrow 0} V (\Delta, r) \uparrow V^+$. Now, we fix r and σ. Since by Proposition 7, information improves with a decrease in Δ, we must have

$$\lim_{r \downarrow 0} V (\Delta, r) \subseteq \lim_{\Delta \downarrow 0} V (\Delta, r).$$

Whatever efficiency is achieved by letting $r \downarrow 0$ can be achieved all the more by letting $\Delta \downarrow 0$. Consequently, we have $\lim_{\Delta \downarrow 0} V (\Delta, r) \uparrow V^+$. ∎