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Public Monitoring with Uncertainty in the Time Repetitions.

Antonio Osorio

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Public Monitoring with Uncertainty in the Time Repetitions*

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(Dated: December 18, 2011)

Abstract

This paper study repeated games where the time repetitions of the stage game are not known or controlled by the players. We call this feature random monitoring. Kawamori's (2004) shows that perfect random monitoring is always better than the canonical case. Surprisingly, when the monitoring is public, the result is less clear-cut and does not generalize in a straightforward way. Unless the public signals are sufficiently informative about player's actions and/or players are patient enough. In addition to a discount effect, that tends to consistently favor the provision of incentives, we found an information effect, associated with the time uncertainty on the distribution of public signals. Whether payoff improvements are or not possible, depends crucially on the direction and strength of these effects.

JEL: C73, D82, D86.

KEYWORDS: Repeated Games, Frequent Monitoring, Random Public Monitoring, Moral Hazard, Stochastic Processes.

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I. INTRODUCTION

This paper studies repeated games where the repetitions of the stage game are not known or controlled by the players. Many economic situations with repeated interaction share this feature. We call it random monitoring.

The Organization of the Petroleum Exporting Countries (OPEC), it is a well known cartel with the goal to safeguard the members' interests, i.e., influence the oil market price in their favor. OPEC's adjustments in the oil supply are typically not programmed ex-ante; the market conditions determine the timing when a strategic change is due. While in periods of stability the OPEC interventions are less frequent, in periods of instability adjustments occur more often. In theoretical terms, it is similar to a n-firms Cournot problem, where the time repetitions of the stage game (oil supply adjustments) are not predetermined.[1]

Allowing for this possibility, from an efficiency perspective, we ask; what are the effects on players' payoffs compared to the canonical setup where the repetitions are deterministic and know in advance? This paper answer these questions in the Abreu, Milgrom and Pearce (1991) public monitoring model.[2]

The perfect monitoring case with time uncertainty was studied by Kawamori's (2004), who shows that the set of strongly symmetric equilibrium payoffs is larger than in the deterministic case.[3] Even though their true discount rate remains unchanged, players' decisions are based on a "smaller discount rate", which we call the *Kawamori's effect*.

Surprisingly, when the monitoring is public, the result is less clear cut and does not generalize in a straightforward way. We identify a *discount effect* that aggregates the *Kawamori effect* and a *cross effects.* The latter captures the time correlation between discounting and the distribution of public signals. In addition to the *discount effect,* that consistently favors

[1] Explicit communication among the agents, as in the OPEC cartel example, or the observation of common events/signals, as in the present paper, facilitates the synchronization of actions.

[2] When monitoring is public, players' commonly observe noisy signals about others actions. Green and Porter (1984), Porter (1983) and Radner, Myerson and Maskin (1986) are classical examples with this information structure. See Fudenberg and Tirole (1991) and Mailath and Samuelson (2006) for complete surveys in repeated games.

[3] We call *random monitoring,* when the stage game is repeated at unknown and not equally spaced moments in time. When the stage game is repeated at known and equally spaced moments in time, we call it *deterministic monitoring.* These concepts should not be confused with perfect and public monitoring.

the provision of incentives, we found an *information effect,* associated with the time uncertainty on the distribution of public signals. Such effect, requires more demanding conditions on the informativeness of the public signals (a larger ex-ante statistical separation between cooperation and defection) in order to have a positive impact on the provision of incentives. Whether payoff improvements are or not possible, depends crucially on the direction and strength of these effects. Random public monitoring is by that fact limited and does not improve payoffs in all circumstances. These effects also alter the quality of inference extracted from the public signals.

We identify two ways in which random monitoring improves over deterministic monitoring. One is by enlarging the spectrum of frequencies of play that sustain cooperation, and the other is by improving the inference about players' actions when cooperation is enforced by both monitoring technologies. Moreover, we establish directional conditions under which these efficiency gains on the value of the best *strong symmetric equilibrium* (SSE henceforth) are possible.

Related Literature - The study of random monitoring would not be possible without the recent advances in the theory of frequent monitoring. After the seminal work of Abreu, Milgrom and Pearce (1991), renewed interest in frequent monitoring has re-emerged, in particular due to Sannikov (2007).[4]

Abreu, Milgrom and Pearce (1991) show that the value of the best strongly symmetric equilibrium degenerates at the limit when the realizations of the public process represent good news. The lack of observed signals becomes infinitely likely at the limit (when Δ takes arbitrary small values). Fudenberg and Levine (2007, 2009) and Sannikov and Skrzypacz (2007) (see also Sannikov and Skrzypacz (2009)) present similar limit results when the public signal is Brownian rather than Poisson.

Not all results obtained point to a degeneracy. When the realizations of the public process are interpreted as bad news, Abreu, Milgrom and Pearce (1991) show that equilibrium payoffs above the static Nash, but not fully efficient, can be sustained in the limit.[5]

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[4] In the same spirit, studying games in continuous time see Faingold and Sannikov (2007) and Faingold (2006).

[5] Under Brownian signals, Fudenberg and Levine (2007) and Osório (2008) show that full efficiency can emerge at the limit. The latter assumes that players control the drift of the process, and different action profiles have associated different initial conditions. The former assumes that a deviation increases the

Closer to the present paper is Fudenberg and Olszewski (2009). They study a repeated game with stochastic asynchronous monitoring. They show that at the limit, synchronous and asynchronous monitoring technologies are equivalent if the signals are exponential. However, when the signals are Brownian, in some cases, the limit value of the asynchronous games might be lower.

A common feature of these papers is the focus on the limit case. On the contrary, in the present paper, monitoring is synchronous and stochastic. In addition, we do not restrict to a particular monitoring intensity, rather the entire spectrum of frequencies of play were payoffs above the stage game Nash are possible. Concerning this aspect, this paper is the first to study the implications of time uncertainty with public monitoring and for general frequencies of play.

The rest of the paper is organized as follows. Section II describes random monitoring in repeated games. Section III characterizes the best SSE payoff. Section IV defines and verifies the existence of payoff improvements. Section V defines and decomposes the effects of random monitoring. Section VI discusses the intuition. All proofs are relegated to an appendix.

II. THE MODEL AND THE EXPECTED DISCOUNT FACTOR

We study the effects of random monitoring in the Abreu, Milgrom and Pearce's (1991) model. The infinitely repeated prisoners' dilemma payoffs are shown in Table I.[6]

We assume that $\pi' > \pi > 0$, i.e., defection is a dominant strategy for both players and that (C, C) returns the best symmetric payoff.

Let a (respectively, a') denotes the cooperative (respectively, defective) profile of actions. At moments in time $t_0, t_1, t_2, ...$, players' simultaneously take their actions. In the following period, an imperfect signal about these actions is commonly observed. Signals follow an exponential distribution with a rate parameter that depends on the action profile.

Bad (respectively, good) news model - A bad signal is the (respectively, no) occurrence of

volatility of the process. Fudenberg and Levine (2007), also show that if a deviation has the inverse effect on the noise parameter it is possible to obtain payoffs above the static Nash, but not fully efficient.

[6] We restrict our analysis to the simplest setting. We do this in order to concentrate on random monitoring effects, without extra complexities.

	C	D
C	π, π	$-(\pi' - \pi), \pi'$
D	$\pi', -(\pi' - \pi)$	$0, 0$

TABLE I: The Prisoners' Dilemma Stage Game Payoffs.

an event in a given time interval $(0, \Delta)$. If players cooperate, bad signals arrive at rate β. Otherwise, the arrival intensity increases (respectively, decreases) to $\mu > \beta > 0$ (respectively, $\mu < \beta$ with $\mu > 0$).

A. Deterministic Public Monitoring

When monitoring is deterministic, as it is usually assumed, the time interval is predetermined, i.e., $t_k - t_{k-1} = \Delta$ with $k \geq 1$.

The common discount factor is exponential,[7] i.e., $\delta^{\Delta} \equiv e^{-r\Delta}$, where $r \in (0, \infty)$ denotes the discount rate.

Punishment Probabilities - In the canonical setup of length Δ, the probability of observing a bad signal, when the profile $a = (C, C)$ is chosen, is given by

$$\overline{p}(\Delta) \equiv \int_{[0,\Delta)} f(y|a, \Delta)\, dy = \begin{cases} 1 - e^{-\beta\Delta}, \\ e^{-\beta\Delta}, \end{cases} \tag{2.1}$$

for the bad and good news models, respectively. These are mistaken punishment probabilities.

In case of a deviation, the detection probabilities in the bad and good news models, are respectively,

$$\overline{q}(\Delta) \equiv \int_{[\Delta,\infty)} f(y|a', \Delta)\, dy = \begin{cases} 1 - e^{-\mu\Delta}, \\ e^{-\mu\Delta}, \end{cases} \tag{2.2}$$

where $f(y|a, x) = \beta e^{-\beta x}$ and $f(y|a', x) = \mu e^{-\mu x}$ are the conditional exponential densities.

[7] We can consider other discounting functions. The qualitative features of the model remain, providing that discounting is convex in time.

B. Random Public Monitoring

When monitoring is random, $t_k - t_{k-1} = x$ is an *i.i.d.* continuous random variable with *c.d.f.* $G_X(x)$. To simplify, the time random variable is also assumed exponentially distributed, i.e. $X \sim Exp(1/\Delta)$.[8] Since time is non-negative $x \in [0, \infty)$. The *i.i.d.* assumption implies that the length of each time interval is independent of the length of the previous and subsequent intervals.

Definition 1 *A repeated game is of random monitoring if the time repetitions $t_{k \geq 1}$ of the stage game are stochastic.*

Random as opposed to deterministic public monitoring, requires uncertainty in the repetitions of the stage game. Perfect or imperfect monitoring refers to the signals informativeness.

Meaningful comparisons require that the expected time interval length associated with random monitoring, matches the deterministic monitoring frequency Δ, i.e. $E_X(x) = \Delta < \infty$.

Consequently, the discount factor is a random function of time. We talk about an expected discount factor, i.e.

$$E_X(\delta^x) = \int_{[0,\infty)} e^{-rx} dG_X(x) = 1/(1 + r\Delta).^{9} \tag{2.3}$$

Punishment Probabilities - Payoff are discounted from the random time x at which the value of the process is observed. Consequently, we cannot separate discounting from the distribution of signals. This is the main difference w.r.t. the deterministic setup. The punishment "probabilities" have to be adapted. In this case we speak about "discounted punishment probabilities", i.e.,

$$\widetilde{p}(\Delta) \equiv \int_{[0,\infty)} \int_{[0,x)} e^{-rx} g(x) f(y|a,x) \, dy dx = \begin{cases} \beta\Delta/(1 + r\Delta)(1 + r\Delta + \beta\Delta), \\ 1/(1 + r\Delta + \beta\Delta), \end{cases} \tag{2.4}$$

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[8] The exponential distribution is interesting, not only because of its tractability, but it also maximizes the entropy of random monitoring for distributions with support $x \in [0, \infty)$.

[9] For $t_k - t_{k-1}, ..., t_1 - t_0$, we have a sequence of k *i.i.d.* time intervals, and we can write

$$E\left(\delta^{t_k}\right) = E_X\left(\delta^{(t_k - t_{k-1}) + ... + (t_1 - 0)}\right) = \prod_{j=1}^{k} E_X(\delta^x) = E_X(\delta^x)^k.$$

Consequently, each payoff can be discounted and treated independently of the previous period payoff.

and

$$\widetilde{q}(\Delta) \equiv \int_{[0,\infty)} \int_{[x,\infty)} e^{-rx} g(x) f(y|a', x)\, dy dx = \begin{cases} \mu\Delta / (1 + r\Delta)(1 + r\Delta + \mu\Delta), \\ 1/(1 + r\Delta + \mu\Delta), \end{cases} \tag{2.5}$$

for the event of mistaken and correct punishment, respectively. The bad and good news models are simultaneously presented. Note that now, the density $f(y|., x)$ is also conditional on the realized random time. $g(x) = e^{-x/\Delta}/\Delta$ is the density of the time random variable.

We look at strategies profiles that form a *perfect public equilibrium.*[10,11]

III. THE BEST STRONGLY SYMMETRIC EQUILIBRIUM

The value of the best SSE of the prisoners' dilemma of Table I is well known for the canonical public monitoring case. To accommodate random monitoring, we need to take into account that discounting cannot be separated from the signals distribution.

Players' employ $\alpha - grim$ strategies.[12] If a bad signal is observed, players coordinate the punishment decisions on a public random device, which effectively punish with probability $\alpha \in (0, 1]$ and forgives otherwise. When monitoring is random (respectively, deterministic), we denote this probability as $\widetilde{\alpha}$ (respectively, $\overline{\alpha}$)

The continuation value is a convex combination between the expected normalized payoff associated with the observation of a good signal $\widetilde{v}$, and the expected normalized payoff associated with the observation of a bad signal $\widetilde{\underline{v}}$. The players' expected payoff is

$$\widetilde{v} = [1 - E_X(\delta^x)]\pi + E_X(\delta^x)\left[(1 - \widetilde{\alpha}\widetilde{p}(\Delta)/E_X(\delta^x))\widetilde{v} + \widetilde{\alpha}\widetilde{p}(\Delta)\widetilde{\underline{v}}/E_X(\delta^x)\right]. \tag{3.1}$$

Even though that discounting and signals are convolved and the repetitions of the stage game are random, as in (2.4) and (2.5), we apply the recursive dynamic programming methods of Abreu, Pearce and Staccetti (1986, 1990).

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[10] A strategy is public if it depends only on the public histories and not on the private history of player i. Given a public history, a profile of public strategies that induces a Nash equilibrium, on the continuation game from that time on, is called a PPE.

[11] The publicly observed history is $h^{t_k} \equiv \{y^{t_0}, y^{t_1}, ..., y^{t_{k-1}}\}$ with $h^{t_0} \equiv \varnothing$. Player i has also a private history $h_i^{t_k} \equiv \left\{y^{t_0}, a_i^{t_0}, ..., y^{t_{k-1}}, a_i^{t_{k-1}}\right\}$.

[12] $\alpha - grim$ strategies are required when public signals are exponential in order to make the enforceability constraint to bind. Optimal behavior with Brownian signals endogenously sets $\alpha = 1$.

Lemma 2 *Under random monitoring, the value of the best SSE is given by*

$$\widetilde{v}(\Delta) = \pi - (\pi' - \pi)\,\widetilde{p}(\Delta) / (\widetilde{q}(\Delta) - \widetilde{p}(\Delta)), \tag{3.2}$$

which is enforceable while

$$(0,1] \ni \widetilde{\alpha}(\Delta) = (1 - E_X(\delta^x))(\pi' - \pi) / (\widetilde{q}(\Delta)\,\pi - \widetilde{p}(\Delta)\,\pi'). \tag{3.3}$$

The payoff depends directly on the discount rate r through $\widetilde{p}(\Delta)$ and $\widetilde{q}(\Delta)$. It is a consequence that discounting and signals are not independent. When the enforceability condition (3.3) fails, i.e. $\widetilde{\alpha}(\Delta) \notin (0,1]$, perpetual defection is the equilibrium of the game.

When monitoring is deterministic but public, we can replace $\widetilde{q}(\Delta)$ for $\delta^{\Delta}\overline{q}(\Delta)$, $\widetilde{p}(\Delta)$ for $\delta^{\Delta}\overline{p}(\Delta)$, and $E_X(\delta^x)$ for δ^{Δ}, in (3.2) and (3.3), to obtain the expressions that characterize the value of the best SSE, i.e.,

$$\overline{v}(\Delta) = \pi - (\pi' - \pi)\,\overline{p}(\Delta) / (\overline{q}(\Delta) - \overline{p}(\Delta)), \tag{3.4}$$

and

$$(0,1] \ni \overline{\alpha}(\Delta) = \left(\left(1 - \delta^{\Delta}\right)/\delta^{\Delta}\right)(\pi' - \pi) / (\overline{q}(\Delta)\,\pi - \overline{p}(\Delta)\,\pi'), \tag{3.5}$$

respectively. Note that $\overline{v}(\Delta)$ is independent of the discount rate.

IV. PAYOFF GAINS WITH RANDOM MONITORING

With perfect informative signals, random monitoring is able to enforce the same payoff as in the deterministic setup with a higher discount rates. Players' decisions are based on a larger discount factor (the expected discount factor). The effect is similar as if players had become "more patient". We call it the *Kawamori's effect*.

Surprisingly, when monitoring is imperfect the result is less clear cut. In particular, because uncertainty on the time repetitions of the stage game may adversely affect the informational content of the public signals. Before any other considerations, we define how random monitoring can improve payoffs w.r.t. the canonical case.

Definition 3 *The best SSE under random monitoring* $\widetilde{v}(\Delta)$, *is larger than the best SSE under deterministic monitoring* $\overline{v}(\Delta)$ *when:*

(i) $\widetilde{\alpha}(\Delta) \in (0,1]$, $\overline{\alpha}(\Delta) \in (0,1]$ *and*

$$\widetilde{q}(\Delta)/\widetilde{p}(\Delta) > \overline{q}(\Delta)/\overline{p}(\Delta), \tag{4.1}$$

holds.

(ii) $\widetilde{\alpha}(\Delta) \in (0,1]$ *and* $\overline{\alpha}(\Delta) \notin (0,1]$.

In part (i), both monitoring technologies enforce cooperation, i.e., conditions (3.3) and (3.5) are simultaneously satisfied. In this case, both $\widetilde{v}(\Delta)$ and $\overline{v}(\Delta)$ are at least weakly above zero. Consequently, random monitoring returns higher payoffs if $\widetilde{v}(\Delta) \geq \overline{v}(\Delta)$, or equivalently if the inequality (4.1) is satisfied. In statistical terms, if uncertainty in the time domain leads to an increase in the ratio $\widetilde{q}(\Delta)/\widetilde{p}(\Delta)$ w.r.t. $\overline{q}(\Delta)/\overline{p}(\Delta)$, it becomes easier to separate defection from cooperative.

Part (ii) states that cooperation is enforced exclusively with random monitoring. Consequently, we have a gain equal to $\widetilde{v}(\Delta) \geq 0$ because $\overline{v}(\Delta) = 0$. In this case, random monitoring expands the spectrum of frequencies of play that sustain cooperation.

Outside Definition 3, either deterministic monitoring leads to higher payoffs or no monitoring technology can improve over the static Nash.

Definition 3 is general and ambiguous w.r.t. a particular model. For that reason we can think about multiple improvement structures that satisfy either statement.

Let Δ_0 be the Δ that solves (4.1) with equality. In addition let $\widetilde{\Delta}$ and $\overline{\Delta}$ be the Δ that solve $\widetilde{\alpha}(\Delta) = 1$ and $\overline{\alpha}(\Delta) = 1$, respectively.

A. The Bad News Model

A minimal requirement to perform any analysis is that both $\widetilde{v}(\Delta)$ and $\overline{v}(\Delta)$ are larger than zero and enforceable at least in the limit $\Delta \downarrow 0$. Consequently we have the upper bound on the discount rate $r < (\mu\pi - \beta\pi')/(\pi' - \pi)$.

Random monitoring and inference - We want to know for which values of Δ the inequality (4.1) is satisfied. Since Δ_0 cannot be expressed explicitly and approximations to the exponential are not useful, we cannot clearly state whether $\widetilde{\alpha}(\Delta_0) \in (0,1]$ and $\overline{\alpha}(\Delta_0) \in (0,1]$ are satisfied. Nonetheless, we can guarantee the existence of Δ_0, a necessary condition for the statement (i) of Definition 3 to hold.

Proposition 4 *There exist a* $\Delta_0 \in \Re_{++}$ *such that inequality (4.1) holds for* $\Delta \in (\Delta_0, \infty)$.

Since $\Delta_0 > 0$, the immediate conclusion is that random public monitoring cannot improve for all frequencies of play in the bad news model. The result contrasts with the deterministic perfect monitoring scenario, where improvements are possible for all enforceable frequencies of play, as shown by Kawamori (2004).

Random Monitoring and the Provision of Incentives - While the solution $\widetilde{\Delta}$ for $\widetilde{\alpha}(\Delta) = 1$ can be found in close form, the solution $\overline{\Delta}$ to $\overline{\alpha}(\Delta) = 1$ cannot. Nonetheless, the following result sheds light on the effects of random monitoring in terms of incentives. In particular, whether the statement (ii) of Definition 3 applies.

Proposition 5 *If* r *is sufficiently small w.r.t. the ratio* $\mu\pi/\beta\pi'$, *random monitoring has a positive impact on the provision of incentives.*

Improvements are possible if the arrival rates of bad news in case of cooperation and defection are sufficiently distinct and/or players' are enough patience. This observation is persistent for all result. For that reason and to not repeat constantly the same arguments, Section VI elaborates on the intuition.

The last two results allow us to narrow the forms of payoff improvements that might be observed. The following result for the bad news model, replace Definition 3.

Corollary 6 *In the bad news model, if payoff improvements of type* (i) *and* (ii) *of Definition 3 are possible, they are in one of the following forms, respectively:*

(i) $\Delta \in \left(\Delta_0, \overline{\Delta}\right)$ *with* $\Delta_0 < \overline{\Delta} \leq \widetilde{\Delta}$, *or* $\Delta \in \left(\Delta_0, \widetilde{\Delta}\right)$ *with* $\Delta_0 < \widetilde{\Delta} \leq \overline{\Delta}$.

(ii) $\Delta \in \left(\overline{\Delta}, \widetilde{\Delta}\right)$ *with* $\overline{\Delta} < \widetilde{\Delta}$.

The first statement is based on the observation that (4.1) holds in the direction $\Delta > \Delta_0 > 0$. The second statement relies on the fact that both $\overline{\alpha}(\Delta)$ and $\widetilde{\alpha}(\Delta)$ are positive, strictly convex and increasing in $\Delta > 0$ (until an asymptote is reached). In addition, for small Δ, both $\overline{\alpha}(\Delta)$ and $\widetilde{\alpha}(\Delta)$ are in $(0, 1)$.[13] Consequently, there is a single positive real root, and improvement structures of the type $\Delta \in \left(\widetilde{\Delta}, \overline{\Delta}\right)$ with $\widetilde{\Delta} \leq \overline{\Delta}$ are impossible.

[13] We have $\widetilde{\alpha}(\Delta) > \alpha(\Delta)$ for small Δ. Consequently, $\alpha(\Delta)$ and $\widetilde{\alpha}(\Delta)$ must cross once in order for an improvement to be possible.

B. The Good News Model

This model highlights further the limitations of random monitoring in improving payoffs.

Random monitoring and inference -The following result states that when the occurrence of an event is interpreted as good news, improvements in the sense of part (i) of Definition 3 are impossible.

Proposition 7 *There is no $\Delta_0 \in \Re_{++}$ that satisfies (4.1) with equality.*

An additional implication of Proposition 7, is that $\overline{v}(\Delta) \geq \widetilde{v}(\Delta)$ for all $\Delta > 0$.[14]

Random monitoring and the provision of incentives - In face of the previous result, we turn our attention to the existence of improvements of the kind stated in part (ii) of Definition 3.

Proposition 8 *For small Δ (i.e., the smaller positive real root of $\widetilde{\alpha}(\Delta) = 1$) random monitoring has a negative impact on the provision of incentives. Otherwise, if r is sufficiently small w.r.t. the ratio $\beta\pi/\mu\pi'$, random monitoring has a positive impact on the provision of incentives.*

In the good news model, $\alpha(\Delta) = 1$ delivers either two (the interesting case) or zero roots positive real roots. In the interesting scenario there is an interval in $\Re_{++}$ that enforces cooperation.

The following result aggregates the implications of our findings in the good news model and replaces Definition 3.

Corollary 9 *In the good news model, if payoff improvements of type (ii) of Definition 3 are possible, they are of the form $\Delta \in \left(\overline{\Delta}, \widetilde{\Delta}\right)$ with $\overline{\Delta} < \widetilde{\Delta}$.*

The potential improvement structures are narrowed to a single case. Proposition 7 exclude any improvements of the type (i) in Definition 3. On the other hand, improvements with the structure $\widetilde{\Delta} < \overline{\Delta}$, are excluded by the first part of Proposition 8.

The reader is referred to the numerical Examples 14 and 17 below.

[14] As shown by Abreu, Milgrom and Pearce (1991), the good news model degenerates in the limit $\Delta \downarrow 0$.

V. DECOMPOSING THE RANDOM MONITORING EFFECTS

We have two effects associated with random public monitoring. An *information effect* caused by the introduction of time uncertainty on the signals observation, and a *discount effect* cause by the addition of discount on the time uncertain structure of the model. The latter includes on it the *Kawamori effect*.

These effects alter the distribution of the public signals and the provision of incentives. Whether random monitoring generates payoff improvements, depends on the direction and magnitude of these effects. The goal of the present Section is to define and disaggregate these effects.

To separate the *information* from the *discount effect* we need to define the "undiscounted mistaken punishment probability"

$$\widehat{p}(\Delta) \equiv \int_{[0,\infty)} \int_{\widetilde{Y}^-} g(x) f(y|a,x)\, dydx = \begin{cases} \beta\Delta / (1+\beta\Delta), \\ 1/(1+\beta\Delta), \end{cases} \tag{5.1}$$

and the "undiscounted correct punishment probability"

$$\widehat{q}(\Delta) \equiv \int_{[0,\infty)} \int_{\widetilde{Y}^-} g(x) f(y|a',x)\, dydx = \begin{cases} \mu\Delta / (1+\mu\Delta), \\ 1/(1+\mu\Delta), \end{cases} \tag{5.2}$$

for the bad and good news models, respectively. These probabilities take into account the time uncertainty on the distribution of the public signals but with discount removed.

To decompose the effects of random monitoring on the provision of incentives, we define $\widehat{\alpha}(\Delta)$. Replace in (3.3), $\widetilde{p}(\Delta)$ by $\delta^{\Delta}\widehat{p}(\Delta)$ and $\widetilde{q}(\Delta)$ by $\delta^{\Delta}\widehat{q}(\Delta)$, to obtain

$$(0,1] \ni \widehat{\alpha}(\Delta) = \frac{1-\delta^{\Delta}}{\delta^{\Delta}} \frac{\pi' - \pi}{\widehat{q}(\Delta)\pi - \widehat{p}(\Delta)\pi'}. \tag{5.3}$$

Denote $\widehat{\Delta}$, as the Δ value that solves $\widehat{\alpha}(\Delta) = 1$. In addition, let Δ_I and Δ_{II}, be Δ values in $\Re_+$ that solve

$$\widehat{q}(\Delta)/\widehat{p}(\Delta) > \overline{q}(\Delta)/\overline{p}(\Delta), \tag{5.4}$$

and

$$\widetilde{q}(\Delta)/\widetilde{p}(\Delta) > \widehat{q}(\Delta)/\widehat{p}(\Delta), \tag{5.5}$$

with equality, respectively.

Random monitoring effects and the distribution of public signals - We start from the canonical public monitoring likelihood ratio $\overline{q}(\Delta)/\overline{p}(\Delta)$. The next step is to add uncertainty in the time domain to obtain $\widehat{q}(\Delta)/\widehat{p}(\Delta)$. If this ratio is larger (respectively, smaller) than $\overline{q}(\Delta)/\overline{p}(\Delta)$, we say that random monitoring produces a positive (respectively, negative) *information effect*. At this point, if we add discounting to $\widehat{q}(\Delta)/\widehat{p}(\Delta)$, we obtain $\widetilde{q}(\Delta)/\widetilde{p}(\Delta)$. In this case, there is a positive (respectively, negative) *discount effect* if the latter ratio results larger (respectively, smaller) than the former. Since, we are dealing with ratios the *Kawamori's effect* is null. Nonetheless, discounting plays an indirect role, through the time correlation with the public signals.

Given Corollaries 6 and 9, we define the random monitoring effects on the distribution of the public signals.

Definition 10 *Random monitoring has a positive:*

(*i*) *information effect on the distribution of the public signals if* $\Delta_I < \Delta_0$.

(*ii*) *discount effect on the distribution of the public signals if* $\Delta_{II} < \Delta_0$.

Otherwise, these effects are negative or adverse.

Random monitoring effects and the provision of incentives - Variations in the likelihood ratio is just half of the story. Random monitoring also impacts on the provision of incentives.

Start with the deterministic monitoring $\overline{\alpha}(\Delta)$ punishment probability and associated cutoff $\overline{\Delta}$. The addition of time uncertainty in the distribution of public signals results in the enforceability condition $\widehat{\alpha}(\Delta)$ defined in (5.3) and the associated cutoff $\widehat{\Delta}$. If $\widehat{\Delta} > \overline{\Delta}$, then the spectrum of monitoring frequencies that enforce cooperation has increased due to positive *information effects* on the provision of incentives. Discounting is present, through the discount factor, but it does not play a role.

Finally, we extend the time uncertainty to the discount factor and replace the punishment probabilities by the discounted analogue, defined in (2.4) and (2.5), to obtain $\widetilde{\alpha}(\Delta)$. The difference between $\widetilde{\Delta}$ and $\widehat{\Delta}$ establishes the *discount effect*. If $\widetilde{\Delta} > \widehat{\Delta}$, then discounting expands the enforceable spectrum.

Definition 11 *Random monitoring has a positive:*

(*i*) *information effect on the provision of incentives if* $\overline{\Delta} < \widehat{\Delta}$.

(*ii*) *discount effect on the provision of incentives if* $\widehat{\Delta} < \widetilde{\Delta}$.

Otherwise, these effects are negative or adverse.

We can have both effects in the same or in opposed directions. In the latter case, its the magnitude of each effect that determines the impact of random monitoring on the provision of incentives.

A. The Bad News Model

Proposition 12 *The information (respectively, discount) effect on the distribution of the public signals is negative (respectively, positive) and bounded.*

The positive *discount effect* parallels with the perfect random monitoring case. Now, in addition, we have an *information effect* that weakens the inference about players' actions.

Nonetheless, the *information effect* is not necessarily negative when we consider the provision of incentives. The reason is that these effects are measured at different points. Note that, for $\Delta \in (0, \Delta_I)$ inequality (5.4) fails while for $\Delta \in (\Delta_I, \infty)$ it holds.

The following result provides crucial knowledge on how random monitoring impacts on the provision of incentives through the *discount* and the *information effect.*

Proposition 13 (i) *If the ratio $\mu\pi/\beta\pi'$ is sufficiently large, the information effect has a positive and bounded impact on the provision of incentives.*

(ii) *If r is sufficiently small w.r.t. the ratio $\mu\pi/\beta\pi'$, the discount effect has a positive impact on the provision of incentives.*

The *information effect* depends crucially on the relation $\overline{q}(\Delta)\pi - \overline{p}(\Delta)\pi'$ w.r.t. $\widehat{q}(\Delta)\pi - \widehat{p}(\Delta)\pi'$. If the former difference is lower than the latter, we have the guarantee of a positive effect. However, when $\mu\pi/\beta\pi'$ decreases the relation tends to reverse.

The expressions $\widehat{\alpha}(\Delta)$ and $\widetilde{\alpha}(\Delta)$, and consequently the *discount effect*, can be divided in two components. The first part $\left(1 - e^{-r\Delta}\right)/e^{-r\Delta} > r\Delta$, reflects the *Kawamori effect.* It incorporates the immediate impact of discounting on the provision of incentives, which is positive and increasing.

Proposition 12 states that $\widetilde{q}(\Delta)/\widetilde{p}(\Delta) > \widehat{q}(\Delta)/\widehat{p}(\Delta)$ for all r,[15] still the *discount effect* is not positive for all r. In this case, the second part of $\widehat{\alpha}(\Delta)$ and $\widetilde{\alpha}(\Delta)$ must satisfy

$$\frac{\pi' - \pi}{\widehat{q}(\Delta)\pi - \widehat{p}(\Delta)\pi'} < \frac{\pi' - \pi}{\widetilde{q}(\Delta)\pi/E_X(\delta^x) - \widetilde{p}(\Delta)\pi'/E_X(\delta^x)},$$

at least for sufficiently large r. Such favours an increase in $\widehat{\Delta}$ w.r.t. $\widetilde{\Delta}$ and its against the existence of a positive *discount effect*. To the latter, we call it *cross effect.* Consequently, we are lead to the conclusion that better inference does not necessarily result in stronger incentive.

However, our sense is that the *discount effect* is always positive when improvements of the type (ii) of Corollary 6 are possible.

The following numerical example attempts to precise some of the statements made before.

Example 14 *Suppose that $\pi' = 3$, $\pi = 2$ and $r = 0.1$.*

When $\beta = 1$ and $\mu = 3$, we have $\Delta_0 = 0.890$ (with $\Delta_I = 1.012$ as stated by Proposition 12).

$$\overline{\Delta} = 0.891 \quad \underbrace{(-)}_{\text{information effect}} \quad \widehat{\Delta} = 0.791 \quad \underbrace{(+)}_{\text{discount effect}} \quad \widetilde{\Delta} = 0.850$$

Following Corollary 6, since $\widetilde{\Delta} < \Delta_0$ and $\widetilde{\Delta} < \overline{\Delta}$, there is no feasible payoff improvements. The strong negative information effect on the provision of incentives is not compensated by the discount effect, see Proposition 13.

The situation is different when $\beta = 1$ and $\mu = 10$. In this case, we have $\Delta_0 = 0.472$ (and $\Delta_I = 0.521$).

$$\overline{\Delta} = 0.999 \quad \underbrace{(+)}_{\text{information effect}} \quad \widehat{\Delta} = 1.340 \quad \underbrace{(+)}_{\text{discount effect}} \quad \widetilde{\Delta} = 1.521$$

For $\Delta \in (0.472, 0.999)$ and $\Delta \in (0.999, 1.521)$ we have improvements in the sense of part (i) and (ii) of Corollary 6, respectively. Both effects favour the provision of incentives. In relation to the previous parametrization, there is an increase in the ratio $\mu\pi/\beta\pi'$ (Proposition 13).

[15] We have the relations $\overline{q}(\Delta) > \widehat{q}(\Delta) > \widetilde{q}(\Delta)$ and $\overline{p}(\Delta) > \widehat{p}(\Delta) > \widetilde{p}(\Delta)$. When μ/β increases these inequalities tend to become more tight and more concave.

B. The Good News Model

Proposition 7 and Corollary 9 state that in the good news model random monitoring cannot improve in the sense of part (i) of Definition 3. The following result provides the explanation.

Proposition 15 *The discount and the information effects on the distribution of the public signals are always negative.*

Following Proposition 8, random monitoring improvements as stated in Corollary 9, might be possible for sufficiently large values of Δ (the large positive real root of $\widetilde{\alpha}(\Delta) = 1$). We now try to separate the forces underlying this result.[16]

Proposition 16 (i) *If the ratio $\beta\pi/\mu\pi'$ is sufficiently large the information effect has a positive impact on the provision of incentives.*

(ii) *If r is sufficiently small w.r.t. the ratio $\beta\pi/\mu\pi'$, the discount effect has a positive impact on the provision of incentives.*

It is important to note that the cutoff associated with each statements are not the same (this is true for all presented results). It just states that such cutoffs must increase (respectively, decrease) with β and π (respectively, μ and π').

The following numerical example attempts to clear some remaining ambiguity associated with the inexistence of explicit expressions.

Example 17 *Suppose that $\pi' = 3$, $\pi = 2$, $\beta = 3$ and $\mu = 1$. When $r = 0.1$, the two positive real roots associated with each threshold are $\overline{\Delta} = \{0.209, 2.116\}$, $\widehat{\Delta} = \{0.370, 2.103\}$ and $\widetilde{\Delta} = \{0.385, 2.405\}$. For example, the latter pair of roots means that random monitoring enforces cooperation for $\Delta \in (0.385, 2.405)$. The interpretation is similar for the former pair of roots, in which case deterministic monitoring enforces cooperation.*

$$\overline{\Delta} = 2.116 \quad \underbrace{(-)}_{\textit{information effect}} \quad \widehat{\Delta} = 2.103 \quad \underbrace{(+)}_{\textit{discount effect}} \quad \widetilde{\Delta} = 2.405$$

[16] For the smaller root, the information and the discount effects are always negative. Explaining the inexistence of payoff improvements on the provision of incentives, as stated in Proposition 8.

Following Proposition 8, improvements due to random monitoring that satisfy Corollary 9 are possible for the large root only, i.e., when $\Delta \in (2.116, 2.405)$. *Proposition 7 excludes other improvement. The negative information effect is cancelled by a strong discount effect, see Proposition 16.*

In the case where $r = 0.2$, *payoff improvements are no possible. The negative information effect is too strong to be compensated by the discount effect.*

$$\overline{\Delta} = 1.603 \quad \underbrace{(-)}_{information\ effect} \quad \widehat{\Delta} = 1.139 \quad \underbrace{(+)}_{discount\ effect} \quad \widetilde{\Delta} = 1.25$$

Note that, for the lower roots the interpretation of a positive and a negative effect is reversed. A decrease in a cutoff has a positive effect since it expands the enforcement interval to the left.

VI. COMMENTS ON THE INTUITION

Random monitoring in the bad and good news models present very tractable structures of interest in applied work. However, payoff comparisons with the deterministic setting are tricky. In particular, because we cannot establish most relations in close form. The goal of this Section is to establish some intuition on a complex network of effects.

Both, the bad and good news models present similar relations w.r.t. the differences $\overline{q}(\Delta) - \overline{p}(\Delta)$ and $\widetilde{q}(\Delta) - \widetilde{p}(\Delta)$. For small Δ, the former is large than the latter, while for larger Δ, the relation is reversed. In the bad news model, a increase in the ratio μ/β favours the provision of incentives (Proposition 13), because it lowers the inequality shifting point, increasing $\widetilde{\Delta}$ w.r.t. $\overline{\Delta}$.

A variation in β/μ of the same magnitude in the good news model delivers exactly the same result (Proposition 8). That is the reason why payoff improvements of the type (*ii*) of Definition 3 have the same structure (compare Corollaries 6 and 9).

When we decompose the effect of random monitoring the equivalence between both models remain. In both cases we have $\widehat{q}(\Delta) - \widehat{p}(\Delta) > \widetilde{q}(\Delta) - \widetilde{p}(\Delta)$. Consequently, the crossing point with $\overline{q}(\Delta) - \overline{p}(\Delta)$ is lower, and the *information effect* is positive if $\widetilde{\Delta}$ is sufficiently large (or equivalently, for large μ/β and β/μ, respectively).

The introduction of discounting increases the crossing point, the more the value of r (which is negative in terms of incentives provision). On same time, the direct effect on the discount tends to increase the value of $\widetilde{\Delta}$ w.r.t. $\widehat{\Delta}$. Consequently, the direction of the *discount effect* depends crucially on the strength of these opposed forces (Propositions 13 and 16).

The equivalence between both models is broken when we look for improvements of the type (i) of Definition 3. In the good news model, for any parametrization, $\overline{q}(\Delta)/\overline{p}(\Delta)$ is larger than $\widetilde{q}(\Delta)/\widetilde{p}(\Delta)$ (Proposition 7). However, in the bad news model, if μ/β is sufficiently large the relation can be reversed (Proposition 4) and improvements are possible (Corollary 6(i)).[17] Definitive conclusions have to verify for the existence of incentives.

We close by stressing that in both models more informative signals (larger values of μ/β and β/μ, respectively) favor the existence of positive *discount* and *information effects*, and consequently payoff improvements.

APPENDIX: PROOF OF LEMMAS AND PROPOSITIONS

Proof of Lemma 2. Following Abreu, Pearce and Stacchetti (1986, 1990), to find the best SSE payoff, we need to solve the dynamic programing problem: composed expression (3.1),

$$\widetilde{v} \geq [1 - E_X(\delta^x)]\,\pi' + E_X(\delta^x)\left[(1 - \widetilde{\alpha}\widetilde{q}(\Delta)/E_X(\delta^x))\,\widetilde{v} + \widetilde{\alpha}\widetilde{q}(\Delta)\,\underline{\widetilde{v}}/E_X(\delta^x)\right], \tag{6.1}$$

and $\widetilde{\alpha} \in [0,1]$. With $\widetilde{p}(\Delta)$ and $\widetilde{q}(\Delta)$ are defined, respectively in (2.4) and (2.5).

Expression (3.1) is the cooperation value. Players receive the stage game payoff associated with cooperation, plus a discounted expectation over the expected payoffs $\widetilde{v}$ and $\underline{\widetilde{v}}$, suggesting cooperation and defection respectively. Constraint (6.1) imposes that the cooperation payoff is at least as good as the defection payoff. If $\widetilde{\alpha} \notin (0,1]$, we cannot enforce the profile (C,C). $\widetilde{\alpha} \in (0,1]$ is the punishment probability after the observation of a defective signal. With $\alpha - grim$ strategies we can set $\underline{\widetilde{v}} = 0$, the static Nash payoff which is trivially enforced. Punishment is then an absorbing state.

We solve (3.1) w.r.t. $\widetilde{v}$, to obtain

$$\widetilde{v} = (1 - E_X(\delta^x))\,\pi/\left[1 - E_X(\delta^x) + \widetilde{\alpha}\widetilde{p}(\Delta)\right]. \tag{6.2}$$

Similarly, we solve the inequality (6.1) w.r.t. $\widetilde{v}$, to obtain

$$\widetilde{v} \geq (1 - E_X(\delta^x))\,\pi'/\left[1 - E_X(\delta^x) + \widetilde{\alpha}\widetilde{q}(\Delta)\right]. \tag{6.3}$$

[17] The *discount* and *information effects* decomposition is trivial, Propositions 12 and 15.

Plug (6.2) into (6.3), with the latter holding with equality, and solve for $\widetilde{\alpha}$ to obtain (3.3), which must satisfy $\widetilde{\alpha}(\Delta) \in (0,1]$. Finally, replace (3.3) into (6.2), to obtain (3.2).

The deterministic monitoring case is essentially the same with $E_X(\delta^x)$, $\widetilde{p}(\Delta)$ and $\widetilde{q}(\Delta)$ replaced by δ^Δ, $\delta^\Delta \overline{p}(\Delta)$ and $\delta^\Delta \overline{q}(\Delta)$, respectively. ■

Proof of Proposition 4. In the bad news model of frequency Δ, inequality (4.1) is written as

$$\mu(1 + r\Delta + \beta\Delta) / \beta(1 + r\Delta + \mu\Delta) > \left(1 - e^{-\mu\Delta}\right) / \left(1 - e^{-\beta\Delta}\right). \tag{6.4}$$

When $\Delta \downarrow 0$, the limits of $\widetilde{q}(\Delta)/\widetilde{p}(\Delta)$ (LHS of the inequality) and $\overline{q}(\Delta)/\overline{p}(\Delta)$ (RHS of inequality) equal to μ/β. However, they converge at different rates. The limit of $\partial(\widetilde{q}(\Delta)/\widetilde{p}(\Delta))/\partial\Delta \to -(\mu - \beta)\mu/\beta$, and the limit of $\partial(\overline{q}(\Delta)/\overline{p}(\Delta))/\partial\Delta \to -(\mu - \beta)\mu/2\beta$. Consequently, for small Δ the latter is larger, and inequality (6.4) fails.

On the other hand, the limits of $\widetilde{q}(\Delta)/\widetilde{p}(\Delta)$ and $\overline{q}(\Delta)/\overline{p}(\Delta)$ for $\Delta \uparrow \infty$, return respectively $\mu(\beta + r)/\beta(\mu + r)$ and 1. Since $\mu > \beta$ and $r > 0$, for large Δ the inequality (6.4) is satisfied.

Since both sides of (6.4) are convex and monotonically decreasing in Δ, there must exist a crossing point $\Delta = \Delta_0$ that solves (6.4) with equality. ■

Proof of Proposition 5. In the bad news model, the probability $\widetilde{\alpha}(\Delta)$ in (3.3) is given by

$$\widetilde{\alpha}(\Delta) = \frac{r(\pi' - \pi)(1 + r\Delta + \beta\Delta)(1 + r\Delta + \mu\Delta)}{\mu(1 + r\Delta + \beta\Delta)\pi - \beta(1 + r\Delta + \mu\Delta)\pi'}. \tag{6.5}$$

The value of Δ that satisfy $\widetilde{\alpha}(\Delta) = 1$ has the close form

$$\widetilde{\Delta} = (\pi(r + \mu) - \pi'(r + \beta)) / (\pi' - \pi)(r + \beta)(r + \mu). \tag{6.6}$$

The sign depends on the numerator, which is strictly positive because $r < (\pi\mu - \pi'\beta)/(\pi' - \pi)$. It also implies that $\widetilde{\alpha}(\Delta) > 0$ for $r > 0$. In the bad news model, the probability $\overline{\alpha}(\Delta)$ in (3.5) is given by

$$\overline{\alpha}(\Delta) = \left(e^{r\Delta} - 1\right) \frac{\pi' - \pi}{(1 - e^{-\mu\Delta})\pi - (1 - e^{-\beta\Delta})\pi'}. \tag{6.7}$$

In order for $\widetilde{\Delta} \geq \overline{\Delta}$ we must have $\overline{\alpha}\left(\widetilde{\Delta}\right) \notin (0,1]$, we want to show it for low r. Plug $\widetilde{\Delta}$ into $\overline{\alpha}(\Delta)$, and take the limit $r \downarrow 0$ of $\overline{\alpha}\left(\widetilde{\Delta}\right) \notin (0,1]$ to obtain 0, an indetermination.

Consequently, we need to look at the limit of $\partial\overline{\alpha}\left(\widetilde{\Delta}\right)/\partial r$, which equals to

$$\frac{\mu\pi-\beta\pi'}{\pi\left(1-e^{-\frac{\pi\mu-\pi'\beta}{\beta(\pi'-\pi)}}\right)-\pi'\left(1-e^{-\frac{\pi\mu-\pi'\beta}{\mu(\pi'-\pi)}}\right)}.$$

A negative derivative implies that $\overline{\alpha}\left(\widetilde{\Delta}\right)<0$ for small r, i.e., $\overline{\alpha}\left(\widetilde{\Delta}\right)$ is decreasing in this region. Consequently $\widetilde{\Delta}\geq\overline{\Delta}$, i.e., improvement in incentives are possible for small r. If the term in the LHS of the denominator is large than the term in RHS, cæteris páribus, then the sign is positive (respectively, negative), i.e., when μ/β is sufficiently small (respectively, large). Similarly, since the effects of π and π' on the exponential part (denominator) is stronger, if π/π' is sufficiently small (respectively, large) the sign is positive (respectively, negative). We can aggregate this information into $\mu\pi/\beta\pi'$ to prove the result.

The limit of $\overline{\alpha}\left(\widetilde{\Delta}\right)$ when $r\uparrow(\pi\mu-\pi'\beta)/(\pi'-\pi)$ (the upper bound on r) converge to $1\in(0,1]$ from below. Then on the lower neighborhood of $(\pi\mu-\pi'\beta)/(\pi'-\pi)$ we have $\overline{\alpha}\left(\widetilde{\Delta}\right)\in(0,1]\Longrightarrow\widetilde{\Delta}\leq\overline{\Delta}$ and increasing. Consequently, there must exist a fixed point $r^*\in(0,(\pi\mu-\pi'\beta)/(\pi'-\pi))$ s.t. $\overline{\alpha}\left(\widetilde{\Delta}\right)=1$. ■

Proof of Proposition 7. Inequality (4.1) for the good news model is given by

$$(1+r\Delta+\beta\Delta)/(1+r\Delta+\mu\Delta)>e^{-\mu\Delta}/e^{-\beta\Delta}. \tag{6.8}$$

The LHS (respectively, RHS) ratio is strictly concave (respectively, convex) and monotonically increasing in $\Delta>0$. Since $e^{(r+\beta)\Delta}\geq1+r\Delta+\beta\Delta+O(\Delta)^2$ for $\Delta\geq0$, we have that $e^{(r+\beta)\Delta}/e^{(r+\mu)\Delta}$ is an upper bound on the LHS when $\beta>\mu$. After we cancel the discount rate, it equals to the RHS. Consequently, inequality (6.8) must fail for all $\Delta>0$. ■

Proof of Proposition 8 . Start by notice that $\overline{v}(\Delta)\geq0$ and $\widetilde{v}(\Delta)\geq0$ require, respectively

$$\Delta\geq\ln(\pi'/\pi)/(\beta-\mu)\equiv\Delta_\dagger,$$

and

$$\Delta\geq(\pi'-\pi)/(\beta\pi-\mu\pi'-r(\pi'-\pi))\equiv\Delta_\ddagger.$$

The RHS of the former inequality is strictly positive. While the RHS of the latter, is larger than zero when $r<(\beta\pi-\mu\pi')/(\pi'-\pi)$. Moreover, by Proposition 7, $\overline{v}(\Delta)\geq\widetilde{v}(\Delta)$ for all $\Delta>0$. In addition, both expressions are monotonically increasing in Δ. Then, we must have $\overline{v}(\Delta_\ddagger)\geq\widetilde{v}(\Delta_\ddagger)=0$, which implies that $\Delta_\ddagger\geq\Delta_\dagger$.

In the good news model, the expression for $\overline{\alpha}(\Delta)$ and $\widetilde{\alpha}(\Delta)$ are, respectively

$$\overline{\alpha}(\Delta) = \left(e^{r\Delta} - 1\right)(\pi' - \pi) / \left(e^{-\mu\Delta}\pi - e^{-\beta\Delta}\pi'\right), \tag{6.9}$$

and

$$\widetilde{\alpha}(\Delta) = \frac{r\Delta}{1 + r\Delta} \frac{(\pi' - \pi)(1 + r\Delta + \mu\Delta)(1 + r\Delta + \beta\Delta)}{(1 + r\Delta + \beta\Delta)\pi - (1 + r\Delta + \mu\Delta)\pi'}. \tag{6.10}$$

The asymptote of $\overline{\alpha}(\Delta)$ occurs at $\Delta = \Delta_{\dagger}$, while the asymptote of $\widetilde{\alpha}(\Delta)$ occurs at $\Delta = \Delta_{\ddagger}$. On the right neighborhood of their asymptotes, both expression are monotonically decreasing in Δ. Then, for $\Delta \downarrow \Delta_{\ddagger}$ we must have $\widetilde{\alpha}(\Delta) \uparrow \infty$, while $\overline{\alpha}(\Delta_{\ddagger})$ takes some bounded value. We have to consider two cases. *i)* $\overline{\alpha}(\Delta_{\ddagger}) \leq 1$: implying that $\overline{\Delta} \leq \widetilde{\Delta}$, and there is no possible improvement (for the small root, the interpretation is reversed). *ii)* $\overline{\alpha}(\Delta_{\ddagger}) \geq 1$: implying that $\widetilde{\Delta} \leq \overline{\Delta}$, and improvements might be possible. But Proposition 7 states that $\overline{v}(\Delta_{\ddagger}) \geq \widetilde{v}(\Delta_{\ddagger}) = 0$. Consequently, improvements through incentives are not possible as well.

For the large positive real root $\widetilde{\Delta}$ that solves $\widetilde{\alpha}(\Delta) = 1$, the argument is similar to the one employed in the proof of Proposition 5. The difference is that now, $\widetilde{\Delta}$ has a more complex expression. The following is a resume of the findings. Because of the difference in the denominator of (6.10), for large Δ, an increase in r causes a larger increase in $\widetilde{\alpha}(\Delta)$ w.r.t. $\overline{\alpha}(\Delta)$, favouring a higher $\overline{\Delta}$ w.r.t. $\widetilde{\Delta}$. An increase in β (respectively, μ) causes a larger decrease (respectively, increase) in $\widetilde{\alpha}(\Delta)$ w.r.t. $\overline{\alpha}(\Delta)$, favouring a lower (respectively, higher) $\overline{\Delta}$ w.r.t. $\widetilde{\Delta}$. An increase in π (respectively, π') causes a larger decrease (respectively, increase) in $\widetilde{\alpha}(\Delta)$ w.r.t. $\overline{\alpha}(\Delta)$, favouring a lower (respectively, higher) $\overline{\Delta}$ w.r.t. $\widetilde{\Delta}$. We aggregate this information into the ratio $\beta\pi/\mu\pi'$. Larger ratios favour the provision of incentives, i.e., $\overline{\Delta} < \widetilde{\Delta}$, and the other way around. ■

Proof of Proposition 12. We have seen that for $\Delta > \Delta_0 > 0$, inequality (6.4) is satisfied. Similarly, for $\Delta > \Delta_I$ inequality (5.4), that is written as

$$\mu(1 + \beta\Delta)/\beta(1 + \mu\Delta) > \left(1 - e^{-\mu\Delta}\right) / \left(1 - e^{-\beta\Delta}\right),$$

is also satisfied. Then, since $r > 0$, we have for all $\Delta \neq 0$ that

$$\mu(1 + r\Delta + \beta\Delta)/\beta(1 + r\Delta + \mu\Delta) > \mu(1 + \beta\Delta)/\beta(1 + \mu\Delta),$$

where the latter inequality correspond to (5.5). Then, the *discount effect* is positive, because $\Delta_{II} = 0$. Consequently, we must have $\Delta_0 < \Delta_I$, i.e., the *information effect* must be negative.

Otherwise, a positive *discount effect* would be impossible. Inequality (5.4) fails for $\Delta \in (0, \Delta_I)$.

The derivative of the difference of the ratios in the latter inequality is strictly increasing in Δ. Consequently, the maximal difference is obtained for $\Delta \uparrow \infty$, and is bounded, because the LHS converges to $(r+\beta)/(r+\mu)$ while the RHS goes to β/μ. Similarly, note that $\widehat{q}(\Delta)/\widehat{p}(\Delta)$ and $\overline{q}(\Delta)/\overline{p}(\Delta)$ are monotonically decreasing, and have supremum and infimum values equal to μ/β and 1, respectively. Consequently, both effects are bounded. ■

Proof of Proposition 13. *(ii)* For the bad news model, the expression $\widetilde{\alpha}(\Delta)$ is given by (6.5) and $\widehat{\alpha}(\Delta)$ by

$$\widehat{\alpha}(\Delta) = \left(e^{r\Delta} - 1\right) \frac{(\pi' - \pi)(1+\beta\Delta)(1+\mu\Delta)}{\mu\Delta(1+\beta\Delta)\pi - \beta\Delta(1+\mu\Delta)\pi'}. \tag{6.11}$$

From zero to the respective asymptote, these expressions are convex and monotonically increasing in Δ, where both take the value ∞. The asymptote of $\widetilde{\alpha}(\Delta)$ occurs at

$$\widetilde{\Delta}_a = (\mu\pi - \beta\pi') / (\beta(r+\mu)\pi' - \mu(r+\beta)\pi),$$

while the asymptote of $\widehat{\alpha}(\Delta)$ does not depend on r and it is always positive, i.e.,

$$\widehat{\Delta}_a = (\mu\pi - \beta\pi') / \beta\mu(\pi' - \pi).$$

Since both $\widehat{\Delta}$ and $\widetilde{\Delta}$ occur before their asymptote value, it rules out the possibility of an unbounded *discount effect*. In addition, since $r > 0$ we have $\widetilde{\Delta} < \widehat{\Delta}_a < \widetilde{\Delta}_a$.

The value r that makes $\widehat{\alpha}(\Delta) \geq 1$ is given by

$$r > \ln\left(\frac{(1+\mu\Delta)\pi' - (1+\beta\Delta)\pi}{(\pi'-\pi)(1+\beta\Delta)(1+\mu\Delta)}\right)/\Delta.$$

Plug the solution $\widetilde{\Delta}$, of $\widetilde{\alpha}(\Delta) = 1$, given in (6.6) to get

$$r > R(r) \equiv \frac{\ln\left(\frac{\left(1+\mu\frac{\pi(r+\mu)-\pi'(r+\beta)}{(\pi'-\pi)(r+\beta)(r+\mu)}\right)\pi' - \left(1+\beta\frac{\pi(r+\mu)-\pi'(r+\beta)}{(\pi'-\pi)(r+\beta)(r+\mu)}\right)\pi}{(\pi'-\pi)\left(1+\beta\frac{\pi(r+\mu)-\pi'(r+\beta)}{(\pi'-\pi)(r+\beta)(r+\mu)}\right)\left(1+\mu\frac{\pi(r+\mu)-\pi'(r+\beta)}{(\pi'-\pi)(r+\beta)(r+\mu)}\right)}\right)}{(\pi(r+\mu) - \pi'(r+\beta)) / (\pi'-\pi)(r+\beta)(r+\mu)}.$$

The limit of $R(r)$ when $r \downarrow 0$ equals to zero. To deal with the indetermination, differentiate the both sides of the inequality and take the limit to obtain

$$1 > (\pi' - \pi)\left(\pi\mu^2 - \pi'\beta^2\right) / \left(\pi\pi'(\mu-\beta)^2\right),$$

which is always satisfied, since $0 > -(\pi\mu - \pi'\beta)^2$. Then, for small r we must have a positive *discount effect*.

Observe that when $r \uparrow (\mu\pi - \beta\pi') / (\pi' - \pi)$ we again obtain an indetermination. To deal with this more complex indetermination, note that for small $\Delta \downarrow 0$, both $\widetilde{\alpha}(\Delta)$ and $\widehat{\alpha}(\Delta)$ converge to $r(\pi' - \pi) / (\mu\pi - \beta\pi')$ which is below the unit if $r < (\mu\pi - \beta\pi') / (\pi' - \pi)$. The same condition also guarantee that $\widetilde{\Delta} > 0$. The limit $\Delta \downarrow 0$ of the derivatives when $r = (\mu\pi - \beta\pi') / (\pi' - \pi)$ are

$$\partial\widetilde{\alpha}(\Delta) / \partial\Delta \to \pi\pi' (\mu - \beta)^2 / (\pi' - \pi)(\mu\pi - \beta\pi'),$$

and

$$\partial\widehat{\alpha}(\Delta) / \partial\Delta \to \left(2\pi\pi' \left(\mu^2 + \beta^2\right) - (\mu\pi + \beta\pi')^2\right) / 2(\pi' - \pi)(\mu\pi - \beta\pi'),$$

respectively. Consequently, we have $\partial\widetilde{\alpha}(\Delta) / \partial\Delta > \partial\widehat{\alpha}(\Delta) / \partial\Delta$ for $\Delta \downarrow 0$ and $r = (\mu\pi - \beta\pi') / (\pi' - \pi)$. Then, for larger r we must have $\widetilde{\Delta} < \widehat{\Delta}$, i.e., a negative *discount effect*. In addition, a fixed point $r^* = R(r^*)$ is guaranteed to exist in the interval $(0, (\mu\pi - \beta\pi') / (\pi' - \pi))$, below which a positive *discount effect* is guaranteed.

(i) The expression $\widehat{\alpha}(\Delta)$ is given above in (6.11), while $\overline{\alpha}(\Delta)$ is given by (6.7). Both $\widehat{\alpha}(\Delta)$ and $\overline{\alpha}(\Delta)$ are strictly convex and monotonically increasing in Δ, taking the value one before their asymptotes. Notice that both expressions differ only in the denominator, consequently r plays no role.

Notice that $\overline{p}(\Delta) > \widehat{p}(\Delta)$ and $\overline{q}(\Delta) > \widehat{q}(\Delta)$, i.e., respectively, $1 - e^{-\beta\Delta} > \beta\Delta / (1 + \beta\Delta)$ and $1 - e^{-\mu\Delta} > \mu\Delta / (1 + \mu\Delta)$. The same increase in π (respectively, π'), decreases (respectively, increases) $\overline{\alpha}(\Delta)$ more than $\widehat{\alpha}(\Delta)$, because $\overline{q}(\Delta) > \widehat{q}(\Delta)$ (respectively, $\overline{p}(\Delta) > \widehat{p}(\Delta)$). So larger π (respectively, π'), increase (respectively, decrease) $\overline{\Delta}$ w.r.t. $\widehat{\Delta}$.Also, the same increase in μ (respectively, β), decreases (respectively, increases) $\overline{\alpha}(\Delta)$ more than $\widehat{\alpha}(\Delta)$, because $\overline{q}(\Delta)$ (respectively, $\overline{p}(\Delta)$) is exponential. So larger μ (respectively, β), tends to increase (respectively, decrease) $\overline{\Delta}$ w.r.t. $\widehat{\Delta}$. We can aggregate this information into the ratio $\mu\pi/\beta\pi'$. Larger ratios favour the existence of a positive *information effect*, i.e., $\overline{\Delta} < \widehat{\Delta}$, and the other way around. ■

Proof of Proposition 15. In the good news model, the *information effect* improves inference if inequality (5.4) is satisfied, i.e.,

$$(1 + \beta\Delta) / (1 + \mu\Delta) > e^{-\mu\Delta} / e^{-\beta\Delta}.$$

Similarly, the *discount effect* improves inference if inequality (5.5) is satisfied, i.e.,

$$(1 + r\Delta + \beta\Delta) / (1 + r\Delta + \mu\Delta) > (1 + \beta\Delta) / (1 + \mu\Delta).$$

Both sides of each inequality are monotonically increasing in Δ. The first inequality is never satisfied. Note that the LHS is concave while the RHS in convex. The RHS can be approximated by the lower bound $1 + \beta\Delta - \mu\Delta + O(\Delta^2) \lesssim e^{(\beta-\mu)\Delta}$. After some algebraic manipulations we obtain $0 \geq \mu\Delta^2(\beta - \mu) + O(\Delta^2)$, an impossibility, because $\beta > \mu$ and $O(\Delta^2)$ are strictly positive.

In the latter inequality, the LHS differs from the RHS when $r > 0$. Differentiate the LHS w.r.t. r to obtain $-(\beta - \mu)\Delta^2/(1 + r\Delta + \mu\Delta)^2$. Consequently, we have another impossibility, the LHS is always lower than the RHS. ■

Proof of Proposition 16. In the good news model

$$\widehat{\alpha}(\Delta) = \left(e^{r\Delta} - 1\right) \frac{(\pi' - \pi)}{\frac{1}{(1+\mu\Delta)}\pi - \frac{1}{(1+\beta\Delta)}\pi'}.$$

While $\overline{\alpha}(\Delta)$ and $\widetilde{\alpha}(\Delta)$ are respectively, given by (6.9) and (6.10). We are interested on the larger real roots $\overline{\Delta}$, $\widehat{\Delta}$ and $\widetilde{\Delta}$. Around these values, $\overline{\alpha}(\Delta)$, $\widetilde{\alpha}(\Delta)$ and $\widehat{\alpha}(\Delta)$, are monotonically increasing and convex in Δ. It easy to show that, for the large real roots we have we have $\widehat{q}(\Delta) > \widetilde{q}(\Delta) > \overline{q}(\Delta)$.

(i) When looking at *information effects*, the difference between $\overline{\alpha}(\Delta)$ and $\widehat{\alpha}(\Delta)$ is on the denominator and r plays no role. Suppose that μ increases (respectively, β), since $\widehat{q}(\Delta) > \overline{q}(\Delta)$ (respectively, $\widehat{p}(\Delta) > \overline{p}(\Delta)$) it tends to increase (respectively, decrease) $\widehat{\alpha}(\Delta)$ more than $\overline{\alpha}(\Delta)$, i.e., favours a decrease (respectively, increase) of $\widehat{\Delta}$ w.r.t. $\overline{\Delta}$. For the same reason, an increase in π (respectively, π'), tends to decrease (respectively, increase) $\overline{\alpha}(\Delta)$ less than $\widehat{\alpha}(\Delta)$, i.e., favours a decrease (respectively, increase) of $\overline{\Delta}$ w.r.t. $\widehat{\Delta}$. Putting together this information, if the ratio $\beta\pi/\mu\pi'$ is sufficiently large we must have a positive *information effect*, i.e. $\overline{\Delta} < \widehat{\Delta}$.

(ii) Now, we look at the *discount effect*. In comparisons between $\widehat{\alpha}(\Delta)$ and $\widetilde{\alpha}(\Delta)$, the value of r plays a role. Briefly, an increase in r causes a larger increase in $\widehat{\alpha}(\Delta)$ w.r.t. $\widetilde{\alpha}(\Delta)$, i.e., favours a increase of $\widehat{\Delta}$ w.r.t. $\widetilde{\Delta}$. Consequently, for any parametrization, lower r favours the existence of a positive *discount effect*. Since $\widehat{q}(\Delta) > \widetilde{q}(\Delta)$, an increase in μ tends to increase $\widetilde{\alpha}(\Delta)$ more than $\widehat{\alpha}(\Delta)$, i.e., favors a decrease of $\widetilde{\Delta}$ w.r.t. $\widehat{\Delta}$. An increase in β has the opposite effect. Similarly, an increase in π, tends to decrease $\widetilde{\alpha}(\Delta)$ more than $\widehat{\alpha}(\Delta)$, i.e., favours an increase of $\widetilde{\Delta}$ w.r.t. $\widehat{\Delta}$. An increase in π' has the opposed effect. Putting together this information, when the ratio $\beta\pi/\mu\pi'$ is sufficiently large w.r.t. r we must have a positive *discount effect*, i.e. $\widehat{\Delta} < \widetilde{\Delta}$. ■

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