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Josep E. Peris

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Departament d'Economia
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Universitat Rovira i Virgili
Facultat d'Economia i Empresa
Avgda. de la Universitat, 1
43204 Reus
Tel.: +34 977 759 811
Fax: +34 977 300 661
Email: sde@urv.cat

CREIP
www.urv.cat/creip
Universitat Rovira i Virgili
Departament d'Economia
Avgda. de la Universitat, 1
43204 Reus
Tel.: +34 977 558 936
Email: creip@urv.cat

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A Proportional Approach to Bankruptcy Problems with a guaranteed minimum.

José M. Giménez-Gómez[a], Josep E. Peris[b]

[a]*Universitat Rovira i Virgili, Dep. d'Economia, CREIP and GRODE, Av.Universitat 1, 43204 Reus, Spain.*
(e-mail: josemanuel.Gimenez@urv.cat)
[b]*Universitat d'Alacant, Dep. de Mètodes Quantitatius i Teoria Econòmica, 03080 Alacant, Spain.*
(e-mail: peris@ua.es)

Abstract

In a distribution problem, and specifically in bankruptcy issues, the *Proportional* (P) and the *Egalitarian* (EA) divisions are two of the most popular ways to resolve the conflict. The *Constrained Equal Awards* rule (CEA) is introduced in bankruptcy literature to ensure that no agent receives more than her claim, a problem that can arise when using the egalitarian division. We propose an alternative modification, by using a convex combination of P and EA. The recursive application of this new rule finishes at the CEA rule. Our solution concept ensures a minimum amount to each agent, and distributes the remaining estate in a proportional way.

Keywords: Bankruptcy problems, Proportional rule, Equal Awards, Convex combination of rules, Lorenz dominance
JEL classification: C71, D63, D71.

1. Introduction.

A bankruptcy problem is a particular case of distribution problems, in which the amount to be distributed, called the *estate*, E, is not enough to cover the agents' claims on it. This model describes the situation faced by a court that has to distribute the net worth of a bankrupt firm among its creditors, but it also corresponds with cost-sharing, taxation, or rationing problems. How should the scarce resources be allocated among its claimants? The formal analysis of situations like these, which originates in a seminal paper by O'Neill (1982), shows that a vast number of well-behaved rules have been defined for solving bankruptcy problems, being the *Proportional* and the *Equal Awards* (egalitarian) the two prominent concepts used in real world[1]. The term well-behaved reflects the idea that the considered rules might fulfill some principles of fairness, or appealing properties. Moreover, some recent works deal with (Lorenz) dominance of rules analysing those rules that favour to smaller claimants relative to larger claimants.

An illustrative example of a bankruptcy situation is the fishing quotas reduction, in which the agent's claim can be understood as the previous captures, and the estate is the new (lower) level of joint captures. A similar example is given by milk quotas among the EU members[2]. In both examples, a minimal (*survival*) amount, guaranteed to each producer, should be fixed in order to ensure the profitability of fishing (milk) industries. A similar situation can be found when a university distributes the budget to Departments. In this situation, the resources are distributed proportionally to the number of Professors, students, subjects, etc., but a minimal (fixed) amount is allocated to each regardless of size.

Although the *Proportional* division is the most used[3], whenever the smallest claim is very small compared with the largest one, a proportional division provides nearly nothing for this (these) small claimant(s). Let us consider two additional examples. A Faculty of Educational Studies at some university offers 100 places each year that are distributed among four groups: (a) graduated, (b) over 25 years, (c) from vocational studies, and (d) from baccalaureate. The number of applications received in each groups determines this group's claim. Then, academic year 2011-12, we had:

[1] The reader is referred to the survey by Thomson (2003).

[2] Quotas were introduced in 1984. Each member state was given a reference quantity which was then allocated to individual producers. The initial quotas were not sufficiently restrictive as to remedy the surplus situation and so the quotas were cut in the late 1980s and early 1990s. Quotas will end on April 1, 2015.

[3] "In western society, for example, the customary solution would be to split the asset in proportion to the claims", see Young (1994), pg 123,

group	*applications*	*proposed admissions*	*proportional rule*
graduates	5	2	0
over 25	9	2	0
vocational studies	486	25	25
baccalaureate	1500	71	75

However, minimum amount should always be granted for each group, and final admissions could differ from the proportional division. In this context, an egalitarian solution (*Constrained Equal Awards rule*) proposes the distribution $(5, 9, 43, 43)$ that would not be considered fair by baccalaureate students.

An alternative example of using the proportional approach is the way in which seats in the Spanish Parliament are allocated to each electoral district (province). This is made proportionally to the population in each province, but a minimal number of seats (2) is guaranteed to each. We shall return to this example later.

The previous comments and examples show that real world, when applying proportional distributions, try to ensure an egalitarian amount to each agent, to avoid that larger claims left without anything small claimants. In this paper we will define a new solution concept that captures this behaviour. This solution can be understood as a compromise between the proportional and the egalitarian division. Particularly, our rule:

- modifies the *Proportional* rule and considers a *minimal* amount that each agent should receive[4];
- modifies the *Equal Awards* division, so that the proposal satisfies the claim-boundedness condition and it is a bankruptcy rule.

The paper is organized as follows: Section 2 contains the preliminaries. Section 3 presents our solution concept. Sections 4 and 5 contain the axiomatic analysis and main results. Finally, Section 6 contains some comments and an example of application of our solution. The Appendix gathers the proofs.

[4]Our proposal satisfies a lower bound on awards property; see Section 4.

2. Preliminaries. Bankruptcy problems.

Throughout the paper we will consider a set of agents $N = \{1, 2, ..., n\}$. Each agent is identified by her *claim*, c_i, $i \in N$, on the *estate* E. A *bankruptcy problem* appears whenever the estate is not enough to satisfy all the claims; that is, $\sum_{i=1}^{n} c_i > E$. Without loss of generality, we will order the agents according to their claims: $c_1 \leq c_2 \leq \cdots \leq c_n$. The pair (E, c) represents the bankruptcy problem, and we will denote by $\mathcal{B}$ the set of all bankruptcy problems. A *bankruptcy rule* (*solution*) is a single valued function $\varphi : \mathcal{B} \to \mathbb{R}^n_+$ such that,

$$0 \leq \varphi_i(E, c) \leq c_i \qquad \forall i \in N$$

(non-negativity and claim-boundedness), and

$$\sum_{i=1}^{n} \varphi_i(E, c) = E$$

(efficiency).

Many solution concepts have been defined in the literature about bankruptcy problems (see for instance Thomson (2003), and Bosmans and Lauwers (2011)). Two of the most important solution concepts are the *Proportional* and the *Egalitarian* ones.

Definition 1. *The* ***Proportional*** *rule, P. For each $(E, c) \in \mathcal{B}$ and each $i \in N$, $P_i(E, c) = \lambda c_i$, where λ is chosen so that $\sum_{i \in N} \lambda c_i = E$.*

Definition 2. *The* ***Equal Awards*** *division, EA. For each $(E, c) \in \mathcal{B}$ and each $i \in N$, $EA_i(E, c) = \frac{E}{n}$.*

It is easy to find examples in which the equal distribution of the *estate* exceeds the claim of some agent. So that, the EA division is not a bankruptcy rule, in the sense we have defined it (EA may not satisfy the second part of the first condition of a solution: claim-boundedness). In order to solve this situation the following modification of the EA division has been introduced.

Definition 3. *The **Constrained Equal Awards** rule, CEA. For each $(E,c) \in \mathcal{B}$ and each $i \in N$, $CEA_i(E,c) \equiv \min\{c_i, \mu\}$, where μ is chosen so that $\sum_{i \in N} \min\{c_i, \mu\} = E$.*

An interesting tool to compare the behaviour of solution concepts is that of Lorenz dominance. Let $\mathbb{R}^n_+$ be the set of positive n-dimensional vectors $x = (x_1, x_2, \ldots, x_n)$ ordered from small to large, i.e., $0 < x_1 \leq x_2 \leq \ldots \leq x_n$. Let x and y be in $\mathbb{R}^n_+$. We say that x Lorenz dominates y, $x \succ_L y$, if for each $k = 1, 2, \ldots, n-1$,:

$$x_1 + x_2 + \cdots + x_k \geq y_1 + y_2 + \ldots + y_k$$

and $x_1 + x_2 + \ldots + x_n = y_1 + y_2 + \ldots + y_n$. If x Lorenz dominates y and $x \neq y$, then at least one of these $n-1$ inequalities is a strict inequality. The following definition extends the notion of Lorenz dominance to bankruptcy rules.

Definition 4. *Given two bankruptcy rules φ and ψ it is said that φ Lorenz dominates ψ, $\varphi \succ_L \psi$, if for any bankruptcy problem (E,c) the vector $\varphi(E,c)$ Lorenz dominates $\psi(E,c)$.*

Lorenz domination is a used criterion to check whether a rule is more favourable to smaller claimants relative to larger claimants. So, in some sense, a Lorenz dominant rule can be understood as more equitable. In a recent paper, Bosmans and Lauwers (2011) obtain a Lorenz dominance comparison among several rules and they obtain that CEA is the more equitable rule, in the sense that it Lorenz dominates any other bankruptcy rule. More precisely, the dominance relation they obtain is as follows[5]:

$$CEA \succ_L CE \succ_L Pin \succ_L P \succ_L CEL$$

Then, the *Proportional* rule only dominates[6] to CEL, which is the most favourable rule for larger claimants relative to smaller ones (so, the less equitable one).

[5]Hereinafter, Pin, T, CE, A, RA, MO, and CEL will denote the *Piniles', Talmud, Constrained Egalitarian, Adjusted Proportional, Random Arrival, Minimal Overlap and Constrained Equal Losses* rules, respectively. See Thomson (2003) for their formal definitions.

[6]See Bosmans and Lauwers (2011) for additional relationships.

3. A proposal of solution: $\alpha_{\min}$-Egalitarian

Given the *Proportional* and the *Egalitarian* divisions, we consider now the family of convex combinations:

$$\varphi_\alpha = \alpha P + (1-\alpha)\, EA \qquad \alpha \in [0,1]$$

Example 1. *Consider* $(E, c) = (100, (40, 50, 70))$.

Claims	$\alpha = 0$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1$
40	100/3	31.25	29.17	27.08	25
50	100/3	32.81	32.29	31.77	31.25
70	100/3	35.94	38.54	41.15	43.75

As we have already mentioned, when $\alpha = 0$ the division may not satisfy the conditions of a solution (claim boundedness fails)[7]. In order to avoid this problem, we can obtain for every problem (E, c) the minimum value of $\alpha \in [0,1]$ such that φ_α is a bankruptcy rule:

$$\alpha^*(E,c) = \min\left\{\alpha \in [0,1] \text{ such that } \left(\varphi_\alpha(E,c)\right)_1 \leq c_1\right\}$$

Remark 1. *It must be noticed that if the claim boundedness is fulfilled by the agent with lowest claim, it is fulfilled by any agent (see the proof in the Appendix).*

Definition 5. *The $\alpha_{\min}$-Egalitarian rule is defined for every bankruptcy problem (E, c), with $c_i > 0 \quad \forall i \in N$, as:*

$$\varphi_{\min}(E,c) = \varphi_{\alpha^*}(E,c)$$

where $\alpha^* = \alpha^*(E,c)$

Note that α^* varies from a bankruptcy problem to another. However, by the way it is defined, the $\alpha_{\min}$-*Egalitarian* rule is continuous. In some sense, this rule is defined as the *smallest convex combination for the P division with respect to the EA one, that makes it a rule.* Next, we consider a *consistent* extension of our rule in the presence of null claims, and we propose an easy way of obtaining the α^*.

[7] For instance, consider the claims vector $c = (20, 50, 60)$ and the estate $E = 100$.

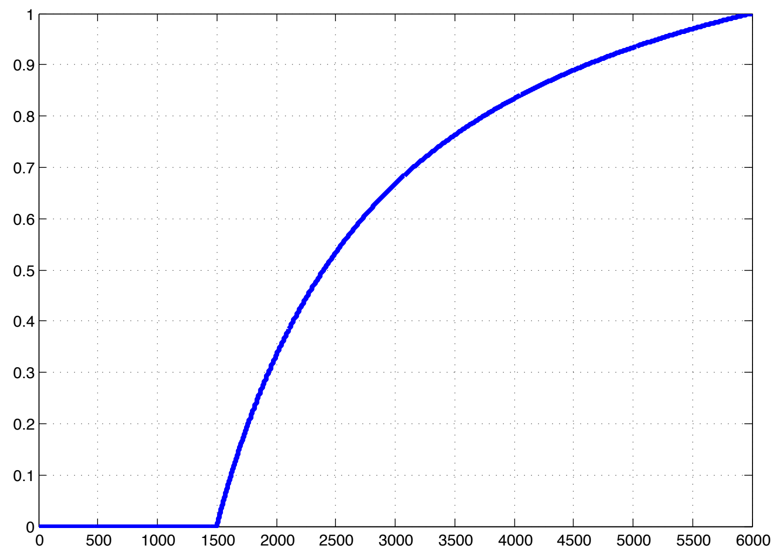

Figure 1: $\alpha^*(E, c)$ as a function of E for fixed claims ($c = (500, 2000, 3500)$).

Remark 2. *If there are some zero claims, $c_1 = c_2 = \ldots = c_k = 0, c_{k+1} > 0$, we extend our solution in a consistent way:*

$$\varphi_{\min}(E, c) = (\boldsymbol{0}, \varphi_{\min}(E, \bar{c})) \qquad \boldsymbol{0} = (0, \ldots, 0)_{1\times k} \quad \bar{c} = (c_{k+1}, \ldots, c_n)$$

Remark 3. *Given a bankruptcy problem (E, c) the scalar α^* is:*

$$\alpha^*(E, c) = max\left\{0, \frac{C\,(E - nc_1)}{E\,(C - nc_1)}\right\} \qquad C = \sum_{i=1}^{n} c_i$$

Remark 4. *It is immediate to see that $\alpha^*(E, c)$ is an increasing and concave function of E for fixed claims vector, as shown in Figure 1.*

Now, trying to facilitate the comparison with the main solutions in the literature, we compute our proposal for the next two examples taken from Bosmans and Lauwers (2011).

Example 2. $(E, c) = (1500, (500, 2000, 3500))$.

c_i	$CEA, \varphi_{\min}$	Pin, T, CE	A	RA, MO	P	CEL
500	500	250	214	166.7	125	0
2000	500	625	643	666.7	500	0
3500	500	625	643	666.7	875	1500

with $\alpha^*(E, c) = 0$.

Example 3. $(E, c) = (4500, (500, 2000, 3500))$.

c_i	CEA, CE	Pin	$\varphi_{\min}$	P	RA	A	T	MO	CEL
500	500	500	500	375	333.3	286	250	166.7	0
2000	2000	1625	1500	1500	1333.3	1375	1375	1416.7	1500
3500	2000	2375	2500	2625	2333.3	2857	2875	2916.7	3000

with $\alpha^*(E, c) = \frac{8}{9}$.

Finally, in the following result, we find a precise expression of our solution which gives us an interesting interpretation: this rule assigns the minimal claim to any agent; thus it distributes the remaining estate $E_1 = E - nc_1$ in a proportional way among the other agents. The proof is given in the Appendix.

Proposition 1. *For each* $(E, c) \in \mathcal{B}$, *with* $c > \boldsymbol{0}$,

$$\varphi_{min}(E, c) = \begin{cases} (E/n)\boldsymbol{1} & c_1 \geq E/n \\ \boldsymbol{c}^1 + P(E - nc_1, c - \boldsymbol{c}^1) & \textit{otherwise} \end{cases}$$

where $\boldsymbol{c}^1 = \begin{pmatrix} c_1 \\ ... \\ c_1 \end{pmatrix}_{n\times 1}$ *and* $\boldsymbol{1} = \begin{pmatrix} 1 \\ ... \\ 1 \end{pmatrix}_{n\times 1}$

The condition that splits both cases in Proposition 1 is known in the literature with the name of *sustainable claim* (see Herrero and Villar (2002)). Note that if the smaller claim c_1 is not a sustainable claim, $c_1 > E/n$, then no claim is sustainable. Therefore, the result in Proposition 1 can be stated as:

- If c_1 is *sustainable*, then $\varphi_{min}(E,c) = \mathbf{c}^1 + P(E - nc_1, c - \mathbf{c}^1)$.
- If c_1 is not *sustainable*, then $\varphi_{min}(E,c) = EA(E,c)$.

In Figure 2 we represent the distribution of the *estate*, by depending on E, given by the α_{min}-Egalitarian solution.

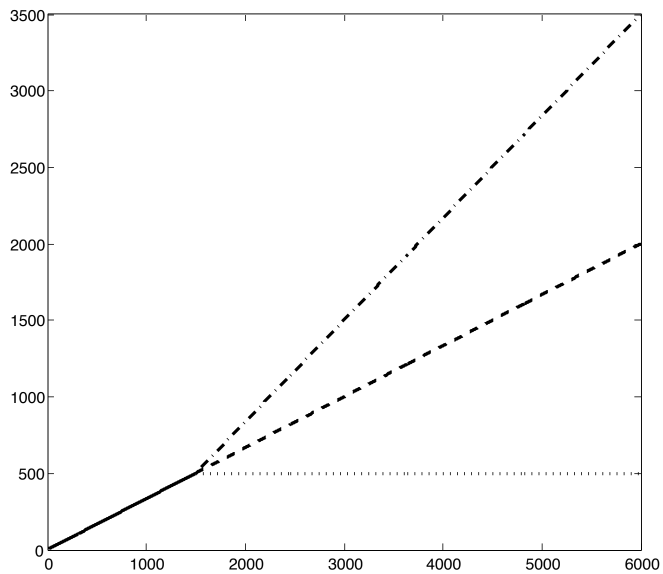

Figure 2: The α_{min}-Egalitarian solution. The horizontal axis represents different levels of the *estate* E, and vertical axis denotes the amount each agent receives according her claims, $c = (500, 2000, 3500)$. The solid black line represents the egalitarian distribution of the estate our proposal obtains when $E \leq 1500$. From this point on, our proposal recommends the pointed-dashed lines for agents 1, 2, 3, from bottom to top, respectively.

4. Axiomatic analysis and comparison with other rules.

In this section we analyse our solution from an axiomatic point of view. First, next table summarizes the axiomatic comparative between the α_{min}-Egalitarian rule and the ones more directly related to it, CEA and P.

	φ_{min}	P	CEA
Order preservation	Yes	Yes	Yes
Resource monotonicity	Yes	Yes	Yes
Super-modularity	Yes	Yes	Yes
Order preservation under claims variations	Yes	Yes	Yes
Invariance under claims truncation	No	No	Yes
Self-duality	No	Yes	No
Midpoint property	No	Yes	No
Limited consistency	Yes	Yes	Yes
Reasonable lower bounds on awards	Yes	No	Yes

In order to check that the α_{min}-Egalitarian solution satisfies, or not, these properties, we formally give their definitions.

Order preservation (Aumann and Maschler (1985)) requires respecting the ordering of the claims: if agent $i's$ claim is at least as large as agent $j's$ claim, she should receive and lose at least as much as agent j does, respectively.

> **Order preservation:** for each $(E, c) \in \mathcal{B}$, and each $i, j \in N$, such that $c_i \geq c_j$, then $\varphi_i(E, c) \geq \varphi_j(E, c)$, and $c_i - \varphi_i(E, c) \geq c_j - \varphi_j(E, c)$.

Resource monotonicity (Curiel et al. (1987), Young (1987)) demands that if the endowment increases, then all individuals should get at least what they received initially.

> **Resource monotonicity:** for each $(E, c) \in \mathcal{B}$ and each $E' \in \mathbb{R}_+$ such that $C > E' > E$, then $\varphi_i(E', c) \geq \varphi_i(E, c)$, for each $i \in N$.

Super-modularity (Dagan et al. (1997)) requires that if the amount to divide increases, given two individuals, the one with the greater claim experiences a larger gain than the other.

> **Super-modularity:** for each $(E, c) \in \mathcal{B}$, all $E' \in \mathbb{R}_+$ and each $i, j \in N$ such that $C > E' > E$ and $c_i \geq c_j$, then $\varphi_i(E', c) - \varphi_i(E, c) \geq \varphi_j(E', c) - \varphi_j(E, c)$.

Reasonable lower bounds on awards (Moreno-Ternero and Villar (2004); Dominguez and Thomson (2006)) ensures that each individual receives at least the minimum of (i) her claim divided by the number of individuals, and (ii) the amount available divided by the number of individuals.

Reasonable lower bounds on awards: for each $(E,c) \in \mathcal{B}$ and each $i \in N$, $\varphi_i(E,c) \geq \frac{\min\{c_i,E\}}{n}$.

Order preservation under claims variations (Thomson (2006)) requires that if the claim of some individual decreases, given two other individuals, the one with the greater claim experiences a larger gain than the other.

Order preservation under claims variations: for each $k \in N$, each pair (E,c) and $(E,c') \in \mathcal{B}$, with[8] $c' = (c'_k, c_{-k})$ and $c'_k < c_k$ and each pair i and $j \in N \setminus k$ with $c_i \leq c_j$, $\varphi_i(E,c') - \varphi_i(E,c) \leq \varphi_j(E,c') - \varphi_j(E,c)$.

Next Proposition, whose proof in given in the Appendix, shows that the α_{min}-Egalitarian rule fulfills the above mentioned properties.

Proposition 2. *The α_{min}-Egalitarian rule fulfills Order preservation, Resource monotonicity, Super-modularity, Reasonable lower bounds on awards, and Order preservation under claims variations.*

Limited consistency states that adding an agent with a zero claim does not change the awards of the individuals already present. Obviously, if $(E,(c_1,c_2,...,c_n))$ is a bankruptcy problem involving n individuals, then $(E,(0,c_1,c_2,...,c_n))$ is a problem with $n+1$ individuals.

Limited consistency: for each $(E,c) \in \mathcal{B}$, for all $i = 1,2,\ldots,n$ $\varphi_i(E,c) = \varphi_i(E,(0,c_1,...,c_n))$.

It is clear, by the way we have defined our consistent extension (see Remark 5), that the α_{min}-Egalitarian rule fulfills this property.

Remark 5. *Note that there is a property our solution fulfills that is not satisfied by the Proportional rule: Reasonable lower bounds on awards. This is the part that the EA division brings to our solution. The drawback is that some properties P fulfilfs are lost. Next we show some of them[9].*

[8]We write (c'_k, c_{-k}) for the claims vector obtained from c by replacing c_k by c'_k.

[9]It must be noticed that the main reason for not satisfying these properties is that EA, taken as a function, does not satisfy them.

Self-Duality implies that a rule recommends the same allocation when dividing awards and losses.

Self-duality: for each $(E,c) \in \mathcal{B}$ and each $i \in N$, $\varphi_i(E,c) = c_i - \varphi_i(L = \sum_{i \in N} c_i - E, c)$.

Midpoint Property ensures to each agent half of her claim when the estate equals half of the aggregate claim.

Midpoint Property: for each $(E,c) \in \mathcal{B}$ and each $i \in N$, if $E = C/2$, then $\varphi_i(E,c) = c_i/2$.

Invariance under claims truncation tells us that the part of a claim that is above the resources should not be taken into account.

Invariance under claims truncation: for each $(E,c) \in \mathcal{B}$ and each $i \in N$, $\varphi_i(E,c) = \varphi_i((E, \min\{c_i, E\}_{i \in N})$.

The following example shows that the α_{min}-Egalitarian rule does not satisfy these properties.

Example 4. *Consider* $(E,c) = (2000, (500, 2000, 3500))$. *Then*

$$\varphi_{min}(E,c) = (500, 666.66, 833.33).$$

$(L,c) = (4000, (500, 2000, 3500))$, *and* $\varphi_{min}(L,c) = (500, 1333.33, 2166.66)$. *So,* $c - \varphi_{min}(L,c) = (0, 727.28, 1272.73) \neq \varphi_{min}(E,c)$, *not satisfying Self-duality.*

Midpoint property implies $\varphi(E,c) = (250, 1000, 1750) \neq \varphi_{min}(E,c)$.

For $(E,c') = (2000, (500, 2000, 2000))$, $\alpha_{min}(E,c') = (500, 750, 750) \neq \varphi_{min}(E,c)$, *not satisfying Invariance under claims truncation.*

Finally, we introduce an operation for bankruptcy rules that will help us to analyse the iterative application of such a rule. We name this operation *Self-composition*, since it is related to the *Self-consistency* property[10] (see

[10]**Self-consistency:** for each $(E,c) \in \mathcal{B}$, each $S \subseteq N$ and each $i \in S$, then $\varphi_i(E,c) = \varphi_i\big(\sum_{k \in S} \varphi_k(E,c), c_{|S}\big)$.

for instance Grahn and Voorneveld (2002)). In particular, *Self-composition* proposes a "recursive" distribution of the resources starting from agent 1. Formally,

Definition 6. ***Self-composition:*** *for each* $(E, c) \in \mathcal{B}$*, and each* m*,* $1 \leq m \leq n$*, then the Self-composition of degree* m *is defined by:*

$$\varphi^m(E,c) = \Big(\varphi_1(E^1,c^1),\ldots,\varphi_{m-1}(E^{m-1},c^{m-1}),\Phi\big(E^m,c^m\big)\Big),$$

where $(E^1, c^1) = (E, c)$ *and*

$$E^m = E^{m-1} - \varphi_{m-1}(E^{m-1}, c^{m-1}); \qquad c^m = (0, \ldots, 0, c_m, \ldots, c_n);$$

$$\Phi\big(E^m, c^m\big) = (\varphi_m(E^m, c^m), \varphi_{m+1}(E^m, c^m), \ldots, \varphi_n(E^m, c^m))$$

For instance, the *Self-composition* of degree 2 for some rule, φ^2 is obtained in the following way: first, agent 1 receives the amount recommended for her by $\varphi(E, c)$; then we solve the new problem in which the *estate* is reduced in the amount given to agent 1, and this agent has no claim anymore. That is,

$$\varphi^2(E,c) = \Big(\varphi_1(E,c),\Phi\big(E-\varphi_1(E,c),(0,c_2,\ldots,c_n)\big)\Big) =$$

$$= \Big(\varphi_1(E,c),\Phi\big(E^2,c^2\big)\Big) = \Big(\varphi_1(E,c),\varphi_2\big(E^2,c^2\big),\varphi_3\big(E^2,c^2\big),\ldots,\varphi_n\big(E^2,c^2\big)\Big).$$

It is immediate to observe that if a rule is *Self-consistent*, then the *Self-composition* of any degree coincides with the own function (in some sense, it is *idempotent*); i.e., if φ satisfies *Self-consistency*, then

$$\forall (E,c) \in \mathcal{B}, \quad \forall m \qquad \varphi^m(E,c) = \varphi(E,c).$$

Next result, which can be straightforwardly obtained from Proposition 1, shows that if we compute the *Self-composition* of degree n (the number of agents) of the α_{min}-Egalitarian rule, we obtain the CEA rule.

Theorem 1. *The Self-composition of degree n of the $\alpha_{min}-$Egalitarian rule retrieves the CEA rule, where n is the number of agents.*

The $\alpha_{min}-$Egalitarian rule does not satisfy self-consistency (otherwise, self-composition could not retrieve the CEA solution). But it satisfies a weaker version that we call *backwards consistency.* This condition requires that if the agent with largest claim leaves with his part, none of the other agents takes advantage.

Definition 7. ***Backwards Consistency:*** *for each* $(E, c) \in \mathcal{B}$,

$$\varphi(E, c) = \big((\varphi(E - \varphi_n(E, c), (c_1, c_2, \dots, c_{n-1})), \varphi_n(E, c)\big)$$

It is obvious that Self-consistency implies Backwards-consistency, but the converse is not true as shows the following result in which we prove that the $\alpha_{min}-$Egalitarian rule satisfies this property. The proof is given in the Appendix.

Proposition 3. *The α_{min}-Egalitarian rule satisfies Backwards-consistency.*

5. Lorenz dominance.

As we have already mentioned, among the solutions analysed in Bosmans and Lauwers (2011), only CEA dominates the α_{min}-Egalitarian solution. Next result shows the Lorenz relationships between our solution and the ones on that paper.

Proposition 4.

a) The α_{min}-Egalitarian solution Lorenz dominates P and CEL.

b) There is no Lorenz domination between the α_{min}-Egalitarian solution and CE, Pin, RA, MO, T, and A rules.

Part b), with respect to CE and Pin is directly obtained from examples 2 and 3. Moreover, example 3 shows a bankruptcy problem in which the α_{min}-Egalitarian solution Lorenz dominates RA,MO, T and A. Next example shows a case in which these solutions are not Lorenz dominated by the α_{min}-Egalitarian solution.

Example 5. *Let* $(E, c) = (20, (2, 20, 40))$. *Then,*

c_i	$\varphi_{\min}$	$RA = MO$	A	T
2	2	0.66	0.96	1.9
20	6.5	9.66	9.52	9.5
40	11.5	9.66	9.52	9.5

Proof of part a) is given in the Appendix.

6. Final comments.

In this paper we have proposed the convex combination of two important and well-known ways of solving distribution problems: the *Proportional* and the *Equal Awards*. Moreover, we have analysed the properties of this new rule and defined a recursive process, *Self-composition*, which allows us to recover the *Constrained Equal Awards* rule, by using our solution.

Note that the α_{min}-Egalitarian solution can be also understood as a kind of *"Constrained Proportional"* rule in the sense that it can be used to ensure a minimum amount to any agent. Suppose that a small amount $\tilde{c} < c_1$ must be received by each agent[11]. What remains of the *estate*, if any, is shared proportionally among all agents. Then, given a bankruptcy problem (E, c) this distribution can be obtained by using the α_{min}-Egalitarian rule in the following way:

$$\varphi(E, c) := \varphi_{min}(E + \tilde{c}, c^*) \qquad c^* = (c_0 = \tilde{c}, c_1, \ldots, c_n)$$

where only the last n-components of the α_{min}-Egalitarian rule are considered. This interpretation can be used, as we have mentioned in the Introduction, to obtain the distribution of seats in Spanish Parliament among districts. The Spanish system guarantees two seats to any district. The other seats are distributed to districts proportional to the population. Then, by applying the α_{min}-Egalitarian rule with $\tilde{c} = 2$ we obtain the actual distribution of seats.

Finally, if we return to our example about student admissions, it is interesting to compare the result given by all the mentioned solutions, the

[11]Such situations can be found, for instance, in the distribution of a heritage; or the State's guarantee of a minimum retirement pension; fixing a minimal fishing quota, or milk quota; ...

α_{min}-Egalitarian, and the α_{min}-Egalitarian with a minimum of guaranteed admissions to each group $\tilde{c} = 2$.

group	*applications*	*CEA*	φ_{min}	$Pin = CE = T$	*RA*
graduates	5	5	5	2	2
over 25	9	9	5	4	3
vocational	486	43	25	47	47
baccalaureate	1500	43	65	47	48

group	*applications*	*MO*	$\varphi_{min} : \tilde{c} = 2$	$P = A$	*CEL*
graduates	5	1	2	0	0
over 25	9	3	2	0	0
vocational	486	48	25	25	0
baccalaureate	1500	48	71	75	100

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Appendix

A1: Proof of Remark 1

For each $(E, c) \in \mathcal{B}$ and given an agent $i \neq 1 \in N$,

$$(\varphi_{min}(E,c))_i = (1-\alpha^*)\frac{E}{n} + \alpha^* \frac{c_i E}{C} =$$

$$= c_1 - \alpha^* \frac{c_1 E}{C} + \alpha^* \frac{c_i E}{C} =$$

$$= c_i + \left(\frac{\alpha^* E}{C} - 1 \right) (c_i - c_1) \leq c_i$$

■

A2: Proof of Proposition 1

Given a bankruptcy problem $(E,c) \in \mathcal{B}$, it is clear that whenever $c_1 \geq E/n$ then $\alpha^*(E,c) = 0$ and $\varphi_{min}(E,c) = CEA(E,c) = E/n$.

Suppose now that $c_1 < E/n$. Then, for each $i \in N$, see Remark 3,

$$\begin{aligned}
(\varphi_{min}(E,c))_i &= \alpha^* P_i(E,c) + (1-\alpha^*) EA_i(E,c) = \\
&= \frac{C(E-nc_1)}{E(C-nc_1)} \frac{Ec_i}{\sum_{j=1}^n c_j} + \left(1 - \frac{C(E-nc_1)}{E(C-nc_1)}\right) \frac{E}{n} = \\
&= \frac{E-nc_1}{C-nc_1} c_i + \frac{c_1(C-E)}{C-nc_1} = \\
&= c_1 + (E-nc_1)\frac{c_i - c_1}{C-nc_1} = c_1 + P_i\left(E-nc_1, c-\mathbf{c}^1\right).
\end{aligned}$$

■

A3: Proof of Proposition 2

In order to check this result, note that for each $(E,c) \in \mathcal{B}$, if $c_1 \geq \frac{E}{n}$, then the φ_{min} distributes the estate as the EA rule, which satisfies all properties. Otherwise,

$$\varphi_{min}(E,c) = \mathbf{c}^1 + P(E-nc_1, c-\mathbf{c}^1).$$

That is, each agent receives the smallest claim c_1 and the remaining estate $E_1 = E - nc_1$ is distributed in a proportional way among the other agents. Then, *Order Preservation* is obvious. With respect to *Resource monotonicity* the only unclear case is whenever

$$c_1 < \frac{E'}{n} \qquad \textit{and} \qquad c_1 \geq \frac{E}{n}.$$

Then,

$$\varphi_{min}(E,c) = \frac{E}{n}\,, \qquad \varphi_{min}(E',c) = \mathbf{c}^1 + P\left(E'_1, c-\mathbf{c}^1\right).$$

and the property is fulfilled. A similar reasoning can be made with *Supermodularity*. Finally, *Reasonable lower bounds on awards* is satisfied, since

$$(\varphi_{min}(E,c))_i \geq min\left\{\frac{E}{n}, c_1 + P_i\left(E_1, c-c^1\right)\right\} \geq \frac{min\{c_i, E\}}{n}.$$

Finally, in order to prove that our solution fulfills *Order preservation under claims variations* consider two bankruptcy problems (E, c), $(E, c') \in \mathcal{B}$, such that $c' = (c'_k, c_{-k})$, $c'_k < c_k$, and consider $i, j \in N \setminus k$ with $c_i \leq c_j$. We have the following possibilities:

(1.) If $c_1 \geq c'_1 \geq \frac{E}{n}$, then the α_{min} distributes the estate as the CEA rule, which satisfies *Order preservation under claims truncation.*

(2.) If $c_1 \geq \frac{E}{n} > c'_1$, then $k = 1$ and

$$(\varphi_{min})_i(E, c) = \frac{E}{n} \qquad (\varphi_{min})_i(E, c') = c'_1 + \frac{E - nc'_1}{\sum\limits_{i \in N \setminus 1} (c_i - c'_1)}(c_i - c'_1).$$

So, for each pair $i, j \in N \setminus 1$ with $c_i \leq c_j$,

$$[(\varphi_{min})_i(E, c') - (\varphi_{min})_i(E, c) \leq (\varphi_{min})_j(E, c') - (\varphi_{min})_j(E, c)] \Leftrightarrow$$

$$\Leftrightarrow \left[c'_1 + \frac{E - nc'_1}{\sum\limits_{i \in N \setminus 1} (c_i - c'_1)}(c_i - c'_1) - \frac{E}{n} \leq c'_1 + \frac{E - nc'_1}{\sum\limits_{i \in N \setminus 1} (c_j - c'_1)}(c_j - c'_1) - \frac{E}{n} \right] \Leftrightarrow$$

$$\Leftrightarrow [c_i - c'_1 \leq c_j - c'_1] \Leftrightarrow c_i \leq c_j.$$

(3.) If $c_1 \leq \frac{E}{n}$, then

$$(\varphi_{min})_i(E, c) = c_1 + \frac{E - nc_1}{\sum\limits_{i \in N \setminus 1} (c_i - c_1)}(c_i - c_1)$$

(3.1.) If $k = 1$, for each pair $i, j \in N \setminus 1$ with $c_i \leq c_j$,

$$[(\varphi_{min})_i(E, c') - (\varphi_{min})_i(E, c) \leq (\varphi_{min})_j(E, c') - (\varphi_{min})_j(E, c)] \Leftrightarrow$$

$$\Leftrightarrow \left[c'_1 + \frac{E - nc'_1}{\sum\limits_{i \in N \setminus 1} (c_i - c'_1)}(c_i - c'_1) - c_1 - \frac{E - nc_1}{\sum\limits_{i \in N \setminus 1} (c_i - c_1)}(c_i - c_1) \leq \right.$$

$$\left. \leq c'_1 + \frac{E - nc'_1}{\sum\limits_{i \in N \setminus 1} (c_j - c'_1)}(c_j - c'_1) - c_1 - \frac{E - nc_1}{\sum\limits_{i \in N \setminus 1} (c_j - c_1)}(c_j - c_1) \right] \Leftrightarrow$$

$$\Leftrightarrow \left[\frac{E - nc'_1}{\sum\limits_{i \in N \setminus 1} (c_i - c'_1)}(c_i - c'_1) - \frac{E - nc_1}{\sum\limits_{i \in N \setminus 1} (c_i - c_1)}(c_i - c_1) \leq \right.$$

$$\left. \frac{E - nc'_1}{\sum\limits_{i \in N \setminus 1} (c_j - c'_1)}(c_j - c'_1) - \frac{E - nc_1}{\sum\limits_{i \in N \setminus 1} (c_j - c_1)}(c_j - c_1) \right] \Leftrightarrow$$

$$\Leftrightarrow \left[\frac{E - nc_1}{\sum\limits_{i \in N \setminus 1} (c_j - c_1)}(c_j - c_i) \leq \frac{E - nc'_1}{\sum\limits_{i \in N \setminus 1} (c_j - c'_1)}(c_j - c_i) \right] \Leftrightarrow c'_1 \leq c_1.$$

(3.2.) If $k \neq 1$, then

$$(\varphi_{min})_i(E, c) = c_1 + \frac{E - nc_1}{\sum\limits_{i \in N \setminus 1} (c_i - c_1)}(c_i - c_1)$$

$$(\varphi_{min})_j(E, c) = c_1 + \frac{E - nc_1}{\sum\limits_{i \in N \setminus 1} (c_i - c_1)}(c_j - c_1),$$

and the property is fulfilled.

■

A4: Proof of Proposition 3

Consider a bankruptcy problem $(E, c) \in \mathcal{B}$.

(1.) If $c_1 \leq \dfrac{E}{n}$, and we name $(x_1, x_2, \ldots, x_n) = \varphi_{min}(E, c)$

$$x_i = c_1 + \frac{c_i - c_1}{C - c_1}(E - nc_1); \qquad C = \sum_{i=1}^{n} c_i;$$

$$E' = E - x_n = (n-1)c_1 + (E - nc_1) - \frac{c_n - c_1}{C - nc_1}(E - nc_1);$$

$$c' = (c_1, c_2, \ldots, c_n - 1); \qquad C' = C - c_n; \qquad c_1 \leq \frac{E'}{n-1}.$$

Then,

$$(\varphi_{min})_i(E',c') = c_1 + \frac{c_i - c_1}{C' - c_1}(E' - (n-1)c_1), \qquad i = 1,2,\ldots,n-1,$$

which coincides with x_i.
(2.) If $c_1 > \dfrac{E}{n}$, then $\varphi_{min}(E,c) = EA(E,c) = \dfrac{E}{n}$ and the property is fulfilled.

■

A5: Proof of Proposition 4

a) For each $(E,c) \in \mathcal{B}$ and each $i \in N$, it follows from Bosmans and Lauwers (2011) that φ_{min} Lorenz dominates CEL. In order to prove that it also dominates the proportional rule P, some notation will help. Given a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ we define the partial sums vector:

$$\mathbf{z}_x = (x_1, x_1 + x_2, \ldots, x_1 + x_2 + ... + x_n)$$

Then, $\mathbf{x} \succ_L \mathbf{y} \quad \Leftrightarrow \quad \mathbf{x} \neq \mathbf{y}$ and $(\mathbf{z}_x)_i \geq (\mathbf{z}_y)_i$. Now denote:

$$\mathbf{x} = EA(E,c) \qquad \mathbf{y} = P(E,c)$$

We know that $\mathbf{x} \succ_L \mathbf{y}$, so $(\mathbf{z}_x)_i \geq (\mathbf{z}_y)_i$. For each $\alpha \in [0,1]$,

$$\alpha\,(\mathbf{z}_y)_i + (1-\alpha)\,(\mathbf{z}_x)_i \geq \alpha\,(\mathbf{z}_y)_i + (1-\alpha)\,(\mathbf{z}_y)_i = (\mathbf{z}_y)_i\,.$$

We conclude that $\left(\mathbf{z}_{\varphi_{min}}(E,c)\right)_i \geq (\mathbf{z}_y)_i$ and then $\varphi_{min}(E,c) \succ_L P(E,c)$.

■