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The Limits of Discrete Time Repeated Games: Some Notes and Comments*

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October 3, 2012

Abstract

This paper studies the limits of discrete time repeated games with public monitoring. We solve and characterize the Abreu, Milgrom and Pearce (1991) problem. We found that for the "bad" ("good") news model the lower (higher) magnitude events suggest cooperation, i.e., zero punishment probability, while the highrt (lower) magnitude events suggest defection, i.e., punishment with probability one. Public correlation is used to connect these two sets of signals and to make the enforceability to bind. The dynamic and limit behavior of the punishment probabilities for variations in r (the discount rate) and Δ (the time interval) are characterized, as well as the limit payoffs for all these scenarios (We also introduce uncertainty in the time domain). The obtained $r \downarrow 0$ limits are to the best of my knowledge, new. The obtained $\Delta \downarrow 0$ limits coincide with Fudenberg and Levine (2007) and Fudenberg and Olszewski (2011), with the exception that we clearly state the precise informational conditions that cause the limit to converge from above, to converge from below or to degenerate.

JEL: C73, D82, D86.

KEYWORDS: Repeated Games, Frequent Monitoring, Random Public Monitoring, Moral Hazard, Stochastic Processes.

1 Introduction

This paper studies the limits of discrete time repeated games with public monitoring. Contrary to the canonical setup, time is not fixed but is rather a

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parameter of the model. We call this situation deterministic monitoring. In search of more efficient results, we also consider the case where the repetitions of the stage game are not known or controlled by the players. Many economic situations with repeated interaction share this feature. In practical terms, it tends to be the rule than the exception. We call it random monitoring.[1][2]

In this paper, we solve and characterize the Abreu, Milgrom and Pearce (1991) problem. We focus on Poissonian events and consider the associated "bad" and "good" news models. We found that for the "bad" ("good") news model the lower (higher) magnitude events must suggest cooperation, i.e., zero punishment probability, while the higher (lower) magnitude events must suggest defection, i.e., punishment with probability one. Public correlation is used to connect these two sets of signals and to make the enforceability constraint to bind in equilibrium. This constraint must be satisfied first, and only after are the payoffs obtained. Consequently, the payoffs and the punishment probabilities are not always perfectly correlated.

The dynamic behavior of the punishment probabilities for variations in r (discount rate) and Δ (time interval) is characterized. We found that as r decreases, punishment tends to decrease and more events suggesting cooperation are added. The limit set of events include all of the possible events. In this case, payoffs and punishment probabilities are perfectly correlated. In the "bad" news model, we obtain a limited fully efficient result, which does not extend to the "good" news model.

However, when we decrease Δ, the result is less clear-cut and does not generalize; the effect on the information structure tends to be adverse. The "good" news model degenerates because the non-occurrence of events becomes infinitely likely for small Δ, independently of the number of events considered. The "bad" news model has an inverse structure, and under some informational conditions, it is possible to sustain limit payoffs above the static Nash equilibrium. As found by Abreu, Milgrom and Pearce (1991), information or monitoring delay has a positive effect on the payoffs; this is true in both models.

We fully characterize the limit payoffs for all of these scenarios. The obtained $r \downarrow 0$ limits are new, to the best of my knowledge. The obtained $\Delta \downarrow 0$ limits coincide with Fudenberg and Levine (2007) and Fudenberg and Olszewski (2011), with the exception that we clearly state the precise informational conditions that make the limit converge from above, converge from below or degenerate.

We also consider the possibility of payoff gains due to random monitoring w.r.t. the deterministic modeling, not only in the limit but also away from it. Surprisingly, deterministic monitoring tends, in general, to be superior, not only in payoff terms but also because it enlarges the spectrum of frequencies of play

[1] This terminology should not be confused with Miyahara and Sekiguchi (2011), where random refers to whether or not an actual monitor exist.

[2] When monitoring is public, the players' commonly observe noisy signals about others' actions. Green and Porter (1984), Porter (1983) and Radner, Myerson and Maskin (1986) are classic examples with this information structure. See Fudenberg and Tirole (1991) and Mailath and Samuelson (2006) for complete surveys in repeated games.

and the discount rates that enforce cooperation. Our intuition is that the benefits of higher discounting are not enough to compensate for the informational losses associated with random monitoring.

Related Literature - After the seminal work of Abreu, Milgrom and Pearce (1991), renewed interest in frequent monitoring has re-emerged, in particular due to Sannikov (2007).[3]

Abreu, Milgrom and Pearce (1991) show that the value of the best strongly symmetric equilibrium degenerates at the limit when the realizations of the public process represent "good" news. The lack of observed signals becomes infinitely likely at the limit (when Δ takes arbitrarily small values). Fudenberg and Levine (2007, 2009) and Sannikov and Skrzypacz (2007) (see also Sannikov and Skrzypacz (2010)) present similar limit results when the public signal is Brownian rather than Poissonian.

Not all of the obtained results point to degeneracy. When the realizations of the public process are interpreted as "bad" news, Abreu, Milgrom and Pearce (1991) show that equilibrium payoffs above the static Nash, although not fully efficient, can be sustained in the limit.[4]

Fudenberg and Olszewski (2011) study a repeated game with stochastic asynchronous monitoring. These authors show that at the limit, synchronous and asynchronous monitoring technologies are equivalent if the signals are exponential. However, when the signals are Brownian, the limit value of the asynchronous games might be lower in some cases. Based on the idea of delayed-responses,[5] Fudenberg, Ishii and Kominers (2012) show that if players wait long enough, after having observed a given signal, then it becomes likely that almost all of the other players have observed the same signal; this way, they construct a folk theorem.

The perfect monitoring case with time uncertainty was studied by Kawamori (2004), who shows that the set of strongly symmetric equilibrium payoffs is larger in this case than in the deterministic case. Although the true discount rate remains unchanged, the players' decisions are based on a "lower discount rate." This result typically fails under public monitoring.

The rest of the paper is organized as follows. Section 2 proposes the model. Section 3 characterizes the best symmetric payoff. Sections 4 and 5 study the punishment probabilities behavior for varying r and Δ, respectively. Section 6 study the limit case. Section 7 discusses non-limit payoffs and concludes. All of the proofs are relegated to an appendix.

[3] In the same spirit, for games in continuous time, see Faingold and Sannikov (2007) and Faingold (2006).

[4] Under Brownian signals, Fudenberg and Levine (2007) and Osório (2011) show that full efficiency can emerge at the limit. The latter assumes that players control the drift of the process, and different action profiles have associated different initial conditions. The former assumes that a deviation increases the volatility of the process. Fudenberg and Levine (2007) also show that if a deviation has the inverse effect on the noise parameter, it is possible to obtain payoffs above the static Nash, but not fully efficient.

[5] See Abreu, Milgrom and Pearce (1991).

	C	D
C	π, π	$-(\pi' - \pi), \pi'$
D	$\pi', -(\pi' - \pi)$	$0, 0$

Table 1: The Prisoners' Dilemma Stage Game Payoffs.

2 The Model

We study the effects of frequent monitoring in Abreu, Milgrom and Pearce's (1991) model. The infinitely repeated prisoners' dilemma payoffs are shown in Table 1.[6]

We assume $\pi' > \pi > 0$ to guarantee that (C, C) returns the best symmetric payoff and that defection is a dominant strategy.

At moments in time $t_0, t_1, t_2, ...$, the players' simultaneously take their actions. In the subsequent period, an imperfect signal about these actions is commonly observed. The signals follow a homogeneous Poisson process with rate parameter $\lambda \in \{\beta, \mu\}$. We have two relevant action profiles: the cooperative profile where $\lambda = \beta$, and the unilateral defection profile where $\lambda = \mu$. In a given time interval Δ, the process returns an event $k \in \{0, 1, ...\}$.

As in Abreu, Milgrom and Pearce (1991), we consider two models, namely,

The "*bad*" *news model* $\mu > \beta$: the occurrence of events in a given time interval is interpreted as a signal of defection. Cooperation has associated a lower arrival intensity.

The "*good*" *news model* $\beta > \mu$: the non-occurrence of events in a given time interval is interpreted as a signal of defection. Cooperation has associated a higher arrival intensity.

2.1 Deterministic Monitoring

Definition 1 *A repeated game is of deterministic monitoring if the interval of time* $\Delta \equiv t_\tau - t_{\tau-1}$ *with* $\tau \geq 1$ *between the repetitions of the stage game is constant and predetermined.*

It corresponds to the canonical repeated games setup parameterized by Δ instead of $\Delta = 1$.

The common discount factor is exponential,[7] i.e., $\delta \ni \delta^\Delta \equiv e^{-r\Delta}$, where $r \in (0, \infty)$ denotes the discount rate.

Probabilities - In a given time interval of length Δ, the probability of observing a particular event of size k, when the profile (C, C) is chosen, is given by

$$p_k \ni \overline{p}_k \equiv \frac{(\beta\Delta)^k}{k!} e^{-\beta\Delta}; \tag{1}$$

[6] We restrict our analysis to the simplest setting to avoid adding unnecessary complexities.

[7] We can consider other discounting functions. The qualitative features of the model remain provided that discounting is convex in time.

otherwise, in the case of unilateral defection, i.e., (D, C) or (C, D), we have

$$q_k \ni \overline{q}_k \equiv \frac{(\mu\Delta)^k}{k!} e^{-\mu\Delta}. \tag{2}$$

In the case of cooperation, the probability of observing one of the events $k \in \{0, 1, ..., K-1\}$ and $k \in \{K+1, ..., \infty\}$, in the time interval Δ is given by

$$\sum\nolimits_{k=0}^{K-1} \overline{p}_k \equiv \Gamma(K, \beta\Delta) / \Gamma(K),$$

and

$$\sum\nolimits_{k=K+1}^{\infty} \overline{p}_k \equiv 1 - \Gamma(K+1, \beta\Delta) / \Gamma(K+1),$$

respectively. In the case of unilateral defection, β is replaced by μ and we have $\sum_{k=0}^{K-1} \overline{q}_k$ and $\sum_{k=K+1}^{\infty} \overline{q}_k$, respectively. $\Gamma(.)$ is the gamma function, $\Gamma(.,.)$ is the (upper) incomplete gamma function and $\Gamma(.,.)/\Gamma(.)$ is called the regularized incomplete gamma function. This process has mean and variance $\lambda\Delta$.

2.2 Random Monitoring

Definition 2 *A repeated game is of random monitoring if the interval of time, $x = t_\tau - t_{\tau-1}$ with $\tau \geq 1$, between repetitions of the stage game is stochastic and of uncertainty duration.*

Random monitoring, in contrast to deterministic, refers to uncertainty in the repetitions of the stage game, while perfect and imperfect monitoring refers to the actions observability, i.e., the signals' informativeness (see Footnote 1).

We assume that $x \sim Exp(1/\Delta)$, i.e., a continuous random variable with $x \in (0, \infty)$.[8] The *i.i.d.* assumption implies that the length of each time interval is independent of the previous and subsequent intervals.

Meaningful comparisons require that the expected time interval length associated with random monitoring, matches the deterministic monitoring frequency Δ, i.e., $E(x) = \Delta < \infty$. Consequently, the discount factor is a random function of time. The expected discount factor is

$$\delta \ni E(\delta^x) \equiv \int_{(0,\infty)} e^{-rx} e^{-\frac{x}{\Delta}} / \Delta dx = 1/(1 + r\Delta),^{9} \tag{3}$$

with $E(\delta^x) > \delta^\Delta$. Note also that $\delta = \left\{\delta^\Delta, E(\delta^x)\right\}$.

[8] The exponential distribution is interesting because of its tractability; in addition, it maximizes the entropy of *random monitoring* for distributions with support $x \in (0, \infty)$.

[9] For $t_\tau - t_{\tau-1}, ..., t_1 - t_0$, we have a sequence of τ *i.i.d.* intervals. We can write

$$E\left(\delta^{t_\tau}\right) = E\left(\delta^{(t_\tau - t_{\tau-1}) + ... + (t_1 - 0)}\right) = \prod\nolimits_{j=1}^{\tau} E(\delta^x) = E(\delta^x)^\tau.$$

Consequently, each payoff is discounted and treated independently of the previous period payoff.

Probabilities - Payoffs are discounted from the end of the interval x, at which point the value of the process is observed. Consequently, we cannot separate discounting from the events distribution, i.e., $E(\delta^x p_k)$. This difference is critical w.r.t. the deterministic setup. Moreover, the sum of all possible events over the entire support does not add to one. To address this issue, we divide $E(\delta^x p_k)$ by $E(\delta^x)$ and define $\widetilde{p}_k \equiv E(\delta^x p_k)/E(\delta^x)$ and $\widetilde{q}_k \equiv E(\delta^x q_k)/E(\delta)$. Now, these are actual probabilities and are associated with a well-defined stochastic process. The probability of the occurrence of the event $k \in \{0, 1, ...\}$ in the case of cooperation and unilateral defection are given by

$$p_k \ni \widetilde{p}_k \equiv (1 + r\Delta) \int_{(0,\infty)} e^{-rx} \frac{1}{\Delta} e^{-\frac{x}{\Delta}} \frac{(\beta x)^k}{k!} e^{-\beta x} dx = \frac{(1 + r\Delta)(\beta\Delta)^k}{(1 + r\Delta + \beta\Delta)^{k+1}}, \quad (4)$$

and

$$q_k \ni \widetilde{q}_k \equiv (1 + r\Delta) \int_{(0,\infty)} e^{-rx} \frac{1}{\Delta} e^{-\frac{x}{\Delta}} \frac{(\mu x)^k}{k!} e^{-\mu x} dx = \frac{(1 + r\Delta)(\mu\Delta)^k}{(1 + r\Delta + \mu\Delta)^{k+1}}, \quad (5)$$

respectively.

In the case of cooperation, the probability of observing one of the events $k \in \{0, 1, ..., K-1\}$ and $k \in \{K+1, ..., \infty\}$ in an interval of expected length Δ is given by

$$\sum\nolimits_{k=0}^{K-1} \widetilde{p}_k = 1 - (\beta\Delta / (1 + r\Delta + \beta\Delta))^K$$

and

$$\sum\nolimits_{k=K+1}^{\infty} \widetilde{p}_k = (\beta\Delta / (1 + r\Delta + \beta\Delta))^{K+1},$$

respectively. In the case of unilateral defection β, is replaced by μ, and we have $\sum_{k=0}^{K-1} \widetilde{q}_k$ and $\sum_{k=K+1}^{\infty} \widetilde{q}_k$, respectively. The process has mean $\lambda\Delta/(1 + r\Delta) < \lambda\Delta$ and variance $\widetilde{\sigma}_\lambda^2 = \lambda\Delta(1 + r\Delta + \lambda\Delta)/(1 + r\Delta)^2$, which is lower than $\lambda\Delta$ if $r\Delta$ is sufficiently large, i.e., $\lambda\Delta < r\Delta(1 + r\Delta)$. The variance decreases (increases) in r (Δ).

Note that in both the deterministic and the random cases, a deviation increases the variance in the "bad" news model but decreases it in the "good" news model. Fudenberg and Levine (2007) have noted the distinction between these two situations as the cause for the latter's Δ limit degeneracy. Throughout the text, we refer to

$$\widetilde{\sigma}_\mu^2 / \widetilde{\sigma}_\beta^2 = \frac{\mu(1 + r\Delta + \mu\Delta)}{\beta(1 + r\Delta + \beta\Delta)},$$

as the variance ratio. This ratio is larger (smaller) than one in the "bad" ("good") news model. In terms of inference, for the "bad" ("good") news model, we want this ratio to be as large (small) as possible. The derivative w.r.t. r is given by

$$-\Delta^2 \mu(\mu - \beta)/\beta(1 + r\Delta + \beta\Delta)^2,$$

and is negative (positive) in the "bad" ("good") news model. The derivative w.r.t. Δ is given by

$$\mu\left(\mu-\beta\right)/\beta\left(1+r\Delta+\beta\Delta\right)^{2},$$

and is positive (negative) in the "bad" ("good") news model. Note that the ratio varies in opposite directions depending on whether we change r or Δ.

We look at the profiles of strategies that form a *perfect public equilibrium (PPE).*[10,11]

3 The Best Symmetric Equilibrium

To accommodate random monitoring, we take into account that discounting cannot be separated from the signals' distribution. Despite this restriction, the expression that characterizes the best symmetric payoff for the prisoners' dilemma of Table 1 has a similar structure to the deterministic monitoring case. However, the actual payoffs can differ substantially.

The players' employ $\alpha_k - grim$ strategies.[12] After the occurrence of a given event k, players coordinate the punishment (perpetual stage game Nash equilibrium) decision on a public random device, which effectively punishes with probability $\alpha_k \in (0,1]$ and forgives otherwise.[13] When monitoring is random (respectively, deterministic), this probability is denoted as $\widetilde{\alpha}$ (respectively, $\overline{\alpha}$).

A critical question is which events should suggest cooperation, which should suggest defection, and which are determined by public randomization?

Let Π denote the set of events that suggest cooperation, with $\Pi = \overline{\Pi}$ in the deterministic and $\Pi = \widetilde{\Pi}$ in the random monitoring cases. Recall that discounting and signals are convolved in the latter case, as in (4) and (5).[14]

The dynamic programming algorithm of Abreu, Pearce and Staccetti (1986, 1990) generalizes straightforwardly. The value of the infinitely repeated game is given by

$$v = (1-\delta)\,\pi + \delta v \sum\nolimits_{k\in\Pi} p_k\left(\Delta\right)\left(1-\alpha_k\right). \tag{6}$$

---

[10] The publicly observed history is $h^{t_k} \equiv \left\{y^{t_0}, y^{t_1}, ..., y^{t_{k-1}}\right\}$ with $h^{t_0} \equiv \varnothing$. Player i has also a private history $h_i^{t_k} \equiv \left\{y^{t_0}, a_i^{t_0}, ..., y^{t_{k-1}}, a_i^{t_{k-1}}\right\}$.

[11] A strategy is public if it depends only on the public histories and not on the private history of player i. Given a public history, a profile of public strategies that induces a Nash equilibrium on the continuation game from that time on is called a PPE.

[12] When signals have a Poisson distribution, $\alpha - grim$ strategies are required to make the enforceability constraint to bind.

[13] The probability α, as well as other functions in this paper, depend on r and Δ (and the other parameters). However, to shorten the notation, we denote them without this dependence.

[14] To be general enough, at this stage, we do not specify the upper and lower bound of summation nor do we specify the monitoring technology. Recall also that $v = \{\overline{v}, \widetilde{v}\}$, $p_k = \{\overline{p}_k, \widetilde{p}_k\}$, $q_k = \{\overline{q}_k, \widetilde{q}_k\}$ and $\delta = \left\{\delta^{\Delta}, E\left(\delta^{x}\right)\right\}$, where the first and second elements of each set refer to the deterministic and random monitoring cases, respectively.

The normalized value of the infinitely repeated game is a convex combination between the immediate cooperative payoff and the continuation value. This structure is enforceable if

$$v \geq (1-\delta)\,\pi' + \delta v \sum\nolimits_{k \in \Pi} q_k\,(\Delta)\,(1-\alpha_k)\,, \tag{7}$$

where $q_k = \{\overline{q}_k, \widetilde{q}_k\}$. The crucial differences between deterministic and random monitoring are in the discount factors and in the signals' distribution. In equilibrium, these differences propagate to α_k and to the events set Π.

Proposition 3 *The best symmetric payoff is given by*

$$v = \pi - (\pi' - \pi)\,/\,(l_K - 1)\,, \tag{8}$$

where

$$l_K = \frac{1 - \left(\sum_{k\in\Pi} q_k + q_K\,(1-\alpha_K)\right)}{1 - \left(\sum_{k\in\Pi} p_k + p_K\,(1-\alpha_K)\right)}, \tag{9}$$

and

$$(0,1] \ni \alpha_K = 1 - \frac{\pi\left(1-\delta\sum_{k\in\Pi} q_k\right) - \pi'\left(1-\delta\sum_{k\in\Pi} p_k\right)}{\delta\,(\pi q_K - \pi' p_K)}. \tag{10}$$

The Lemma is valid for the "bad" and "good" news models and for the deterministic and random monitoring technologies. Optimal behavior requires that in addition to δ, p_k and q_k, the set Π and the vector α vary for each model and each monitoring technology. The derivation of these sets can be found in the proof of Proposition 3. The following scheme resumes them, respectively:

Deterministic "bad" news:

δ^{Δ}, $\overline{p}_k$, $\overline{q}_k$, $\overline{\Pi} = \left\{0, 1, ..., \overline{K}-1\right\}$ and $\overline{\alpha} = \{0, ..., 0, \overline{\alpha}_{\overline{K}}, 1, ...\}$

Random "bad" news:

$E\left(\delta^x\right)$, p_k, q_k, $\widetilde{\Pi} = \left\{0, 1, ..., \widetilde{K}-1\right\}$ and $\widetilde{\alpha} = \left\{0, ..., 0, \widetilde{\alpha}_{\widetilde{K}}, 1, ...\right\}$

Deterministic "good" news:

δ^{Δ}, $\overline{p}_k$, $\overline{q}_k$, $\overline{\Pi} = \left\{\overline{K}+1, ..., \infty\right\}$ and $\overline{\alpha} = \{1, ..., 1, \overline{\alpha}_{\overline{K}}, 0, ...\}$

Random "good" news:

$E\left(\delta^x\right)$, p_k, q_k, $\widetilde{\Pi} = \left\{\widetilde{K}+1, ..., \infty\right\}$ and $\widetilde{\alpha} = \left\{1, ..., 1, \widetilde{\alpha}_{\widetilde{K}}, 0, ...\right\}$

In the "bad" news model, we have $p_k\,(\Delta)\,/q_k\,(\Delta) > p_{k+1}\,(\Delta)\,/q_{k+1}\,(\Delta)$ for all k. The most informative events are the low magnitude events. In other words, the difference between cooperative and defective behavior is easier to infer. Consequently, these events are natural candidates for signaling cooperation. Moreover, if the observation of the event $K-1$ is accepted as suggesting cooperation, then all the other events with smaller magnitude must suggest cooperation, as well. However, if the observation of the event $K+1$ suggests defection, so does all the higher magnitude one. In between, the punishment probability α_K associated with the observation of the event K forces enforceability to bind. Similar reasoning applies to the "good" news model, where $p_{k+1}(\Delta)/q_{k+1}(\Delta) > p_k(\Delta)/q_k(\Delta)$. The larger magnitude events are the events that suggest cooperation.

Definition 4 *We say that a K non-trivial equilibrium exists if $\alpha_K \in (0,1]$ enforces cooperation.*

However, not explicitly expressed, when monitoring is deterministic, the payoff function depends on r but through $\overline{\alpha}_{\overline{K}}$ and K (both are functions of the all model parameters). However, when monitoring is random, the payoffs depend on r through the signals' distribution as well. It is a consequence that discounting and signals are not independent.

The expression $1-\left(\sum_{k\in\Pi} q_k + q_K(1-\alpha_K)\right)$ is the probability of correct punishment, while $1-\left(\sum_{k\in\Pi} p_k + p_K(1-\alpha_K)\right)$ is the probability of mistaken punishment. Then, l_K is the likelihood of the correct detection of defective behavior. The most efficient equilibria are associated with high values of l_K. In other words, we want a large value of l_K, but first we need to guarantee enforceability. E.g., the effect of r or Δ changes the binding enforceable function, determining l_K and consequently the payoffs, but not the other way around. In other words, a binding enforceable condition is the starting point, then, given this restriction, the payoffs are obtained. For that reason, the payoff function is not always smooth or monotonic. This function is continuous, but not differentiable in all of its domain. These issues are discussed in more detail later.

We have the following important continuity property, which does not depend on the monitoring technology.

Corollary 5 *In the "bad" news model, $\alpha_K = 0 \Longleftrightarrow \alpha_{K+1} = 1$ or $\alpha_K = 1 \Longleftrightarrow \alpha_{K-1} = 0$.*

In the "good" news model, $\alpha_K = 1 \Longleftrightarrow \alpha_{K+1} = 0$ or $\alpha_K = 0 \Longleftrightarrow \alpha_{K-1} = 1$.

The idea is that ("bad" news model) monitoring with the set of events $\Pi = \{0,1,...,K-1\}$ suggesting cooperation and punishing the occurrence of an event of magnitude K with probability zero, is the same as monitoring with the set of events $\Pi = \{0,1,...,K\}$ suggesting cooperation and punishing the occurrence of an event of magnitude $K+1$ with probability one. Similar reasoning applies in the "good" news model.

Another consequence ("bad" news model) is as follows: if for a given equilibrium parameterization $\alpha_K \in (0,1]$, then for that K equilibrium parameterization, the α-function (10) must return $\alpha_{K+1} > 1$ and $\alpha_{K-1} < 0$. In other words, "α_{K+1}" cannot enforce cooperation, while "α_{K-1}" enforces cooperation with slack. Similarly, in the "good" news model, if for a given equilibrium parameterization $\alpha_K \in (0,1]$, the function (10) must return $\alpha_{K+1} < 0$ and $\alpha_{K-1} > 1$.

Proposition 6 *In a K non-trivial "bad" ("good") news equilibrium, $\pi q_k - \pi' p_k$ increases (decreases) in $k = 0, ..., K$ (respectively, $k = K, ..., \infty$) and $\pi q_K - \pi' p_K > 0$.*

The results state that the equilibrium value of K must be such that $\pi q_K / \pi' p_K$ is larger than one. Otherwise, the subtraction of one more event to the set of signals suggesting cooperation drives the ratios $\pi q_{K-1}/\pi' p_{K-1}$ and $\pi q_{K+1}/\pi' p_{K+1}$ below one ("bad" and "good" news models respectively).

4 Varying the Discount Rate r

It is known that low discount rates favor the provision of incentives. Here, it is not an exception, formally,

Proposition 7 *In a K non-trivial equilibrium $\partial \alpha_K / \partial r > 0$, where $K > 0$ and $K < \infty$, in the "bad" and "good" news models, respectively.*

The result is valid for any model and monitoring technology. In the "bad" news model, a necessary condition for the existence of some r frequency that supports cooperation is that $\mu > \beta$. This is the case, because we can always find a r sufficiently small and an associated K such that $\mu > \beta$ is sufficiently informative. On the contrary, in the "good" news model, we have additional informational requirements for the existence of a non-trivial equilibrium for some r frequency, $(\beta - \mu) \Delta > \ln (\pi'/\pi)$ for the deterministic case and $\pi (1 + \beta \Delta) > \pi' (1 + \mu \Delta)$ for the random monitoring case (see the proof). In both cases, either the signals are sufficiently informative or the time interval is sufficiently large.

A decrease in r decreases the punishment probability and the equilibrium value of K must move in the $K + 1$ and $K - 1$ directions in the "bad" and "good" news models, respectively. Such is an expansion on the set of events that suggest cooperation, i.e., from $\Pi = \{0, ..., K-1\}$ to $\Pi' = \{0, ..., K\}$ in the "bad" news model and from $\Pi = \{K+1, ..., \infty\}$ to $\Pi' = \{K, ..., \infty\}$ in the "good" news model. Formally,

Corollary 8 *In a non-trivial equilibrium, the set Π weakly increases when r decreases.*

The set increases weakly because, while $\alpha_K \in (0,1]$, the set has the same dimension. As r decreases, the incentives to defection decrease and monitoring relaxes, expanding the set of events that suggest cooperation. Payoffs and incentives are necessarily connected. This relationship is formally stated as follows.

Proposition 9 *In a deterministic K non-trivial equilibrium $\partial v/\partial r < 0 \iff \partial \alpha_K/\partial r > 0$.*

In the random "bad" news model, a deviation increases the variance σ^2_λ, which improves inference. As r decreases, this inference effect improves even more. At the same time, a low discount effect favors the provision of incentives. In fact, there are no trade-offs, and we should expect not only that the payoffs improve but also we should expect a fully efficient result at the limit. In the deterministic case, the information structure is not affected by variations in r.

In the random "good" news model, a deviation decreases the variance favoring inference. At the same time, a decrease in r tends to strengthen this effect. On the other hand, a low discount rate favors the provision of incentives. Consequently, the payoffs improve for lower values of r. However, we should not expect a limit efficient result because mistaken punishments do occur on the equilibrium path.

Proposition 7 states that $\partial \alpha_K/\partial r < 0$ cannot occur in a K equilibrium. Then, $\partial \alpha_K/\partial r > 0$ while enforceability holds, i.e., while r is sufficiently small. When enforceability fails, Proposition 6 states that $\pi q_K - \pi' p_K > 0$. In the "bad" news model, as r increases, we subtract events (the monitoring tightens), and there must be a point when enforceability fails, i.e., we must have $\partial \alpha_{K-1}/\partial r < 0$ and, because of Corollary 5 $\alpha_K = 1 \iff \alpha_{K-1} = 0$, it must be the case that $\pi q_{K-1} - \pi' p_{K-1} < 0$. Similarly, in the "good" news model at a certain point $\partial \alpha_{K+1}/\partial r < 0$, because by Corollary 5 $\alpha_K = 1 \iff \alpha_{K+1} = 0$ we have $\pi q_{K+1} - \pi' p_{K+1} < 0$. Otherwise, we partially or totally overlap the r interval where K is an equilibrium but with a small set Π, this is not optimal behavior. The idea is that α_k is increasing in r, and when this is no longer the case, a non-trivial equilibrium fails to be enforceable. The remainder of the argument follows in the proof of the following result.

Proposition 10 *In the "bad" news model, the value of K that satisfies $\pi q_K - \pi' p_K > 0$ and $\pi q_{K-1} - \pi' p_{K-1} < 0$ is the last (lowest) K that enforces cooperation.*

In the "good" news model, the value of K that satisfies $\pi q_K - \pi' p_K > 0$ and $\pi q_{K+1} - \pi' p_{K+1} < 0$ is the last (largest) K that enforces cooperation.

The result is valid for large r and Δ. As it is for larger r, for large Δ, we must also have $\partial \alpha_K/\partial \Delta > 0$, otherwise enforceability fails. This way we know the punishment probability slope for large Δ. Consequently, it is enough to study the behavior of the punishment probability for small Δ and then link Proposition 10 together. This result is important for this reason, but it also establishes explicit conditions on K for a non-limit and non-trivial equilibrium.

5 Varying the Time Interval Δ

Proposition 10 is valid for a sufficiently large Δ or r. Consequently, in the last enforceable neighborhood, we must have $\partial \alpha_K/\partial \Delta > 0$ for sufficiently large Δ,

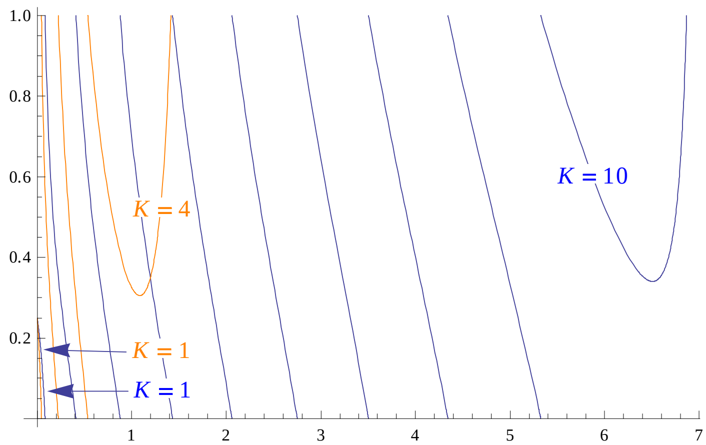

Figure 1: Deterministic (blue/dark) and random (orange/grey) "bad" news model punishment probability α as a function of Δ.

and $\partial\alpha_{K-1}/\partial\Delta < 0$ in its neighborhood. Now, we want to know what happens when Δ is small and establish a bridge between these two. The following result characterizes the behavior of the punishment probability with respect to Δ.

Proposition 11 *Suppose that $\pi(\mu + r) > \pi'(\beta + r)$. In a K non-trivial "bad" news equilibrium, $\partial\alpha_K/\partial\Delta < 0$ for small Δ and $\partial\alpha_K/\partial\Delta > 0$ for large Δ, with $K > 0$.*[15]

In a K non-trivial "good" news equilibrium, $\partial\alpha_0/\partial\Delta < 0$ for small $\Delta > 0$ and $\partial\alpha_K/\partial\Delta > 0$ for large Δ, with $K \geq 0$ and $K < \infty$.

In the "bad" news model, a decrease in Δ reduces the variances ratio in the unit direction, which corresponds to an informational loss. For small Δ, better information is obtained with fewer but more precise events, i.e., $\Pi = \{0, 1\}$. The information degradation must bound the payoffs below the efficient frontier, even when the discounting incentives are maximal. For large Δ, signals become more informative and more events are considered, but the discounting incentives become weaker. This process lasts while the informational gains are stronger. Afterwards, the set of events shrinks until the point that the incentives to cooperate vanish due to low discounting. Figure 1 illustrates some of these arguments.

As we can see, for small (large) Δ, the punishment probability is negatively (positively) sloped as stated in Proposition 11 (and Proposition 10). The same

[15] If the sign

$$-\beta\left(\pi'(\beta + r)\right)^2 - \mu\left(\pi(\mu + r)\right)^2 + \pi\pi'\left(\left(r^2 + \beta\mu\right)(\beta + \mu) + 2r\left(\beta^2 + \mu^2\right)\right),$$

is positive then $\overline{\alpha}_1 \downarrow r(\pi' - \pi)/(\pi\mu - \pi'\beta)$ as $\Delta \downarrow 0$ and $\overline{\alpha}_1$ is monotonic increases in $\Delta \in (0, \Delta_a)$. This case is special in the sense that exist alternating intervals of time where enforceability fails while in others no.

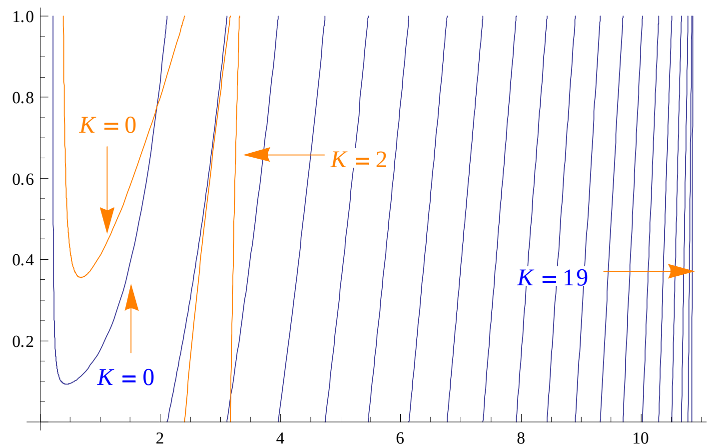

Figure 2: Deterministic (blue) and random (orange) "good" news model punishment probability α as a function of Δ.

occurs in the "good" news model, see Figure 2 below.

In the "good" news model, we always have $\partial\alpha_K/\partial\Delta > 0$; the exception is when $K = 0$ (a small Δ, below which enforceability fails) where $\partial\alpha_0/\partial\Delta < 0$. Consequently, α_0 has a U-shape with the minimum above zero. Intuitively, for a small $\Delta \downarrow 0$, it becomes impossible to statistically distinguish cooperation from defection. Still, for small $\Delta > 0$, non-trivial equilibrium payoffs are possible via the addition of events. The information quality of each signal decays; to compensate, we add events suggesting cooperation. Incentives are provided with more signals but of lower informational quality. In some sense the monitoring relaxes when Δ is small. As Δ increases, at a certain point the information degradation is so important than the incentives via discounting are not sufficient and the equilibrium degenerates. The same conclusion is obtained by noticing that the variance ratio increases in the unit direction when Δ gets small. Figure 2 illustrates these arguments and the intuition around Proposition 11.

In terms of payoffs, the higher the number of signals, the better. In terms of the provision of incentives, the number of events is determined endogenously by the model, and more events do not necessarily lead to better incentives. These two objectives are correlated, but not perfectly. Consequently, Proposition 9 does not generalize. We might observe the payoffs increasing while the punishment probability decreases or increases, and vice versa.

Corollary 12 *In a non-trivial "bad" news equilibrium, while $\partial\alpha_K/\partial\Delta < 0$, the set Π weakly decreases when Δ decreases and while $\partial\alpha_K/\partial\Delta > 0$, the set Π weakly increases when Δ decreases.*

In a non-trivial "good" news equilibrium, the set Π weakly increases when Δ decreases.

In the "bad" news model, increases in Δ initially cause an expansion in the set until the transition point, where it shrinks again. For the smallest values of Δ, we have $K = 1$, i.e., monitoring becomes strict to compensate for the signal's low information. At this point, the discounting incentives for cooperation are strong. As Δ increases, information becomes more reliable, and consequently more events suggesting cooperation are added. At the same time, the incentives via discounting weaken. At a certain point, these incentives are so weak that monitoring needs to tighten again. Intuitively, the compensation is made with fewer but more precise events.

In a $K = 0$ "good" news equilibrium, in spite of the change in the derivative sign, the set remains constant and equal to $\Pi = \{1, ..., \infty\}$. Otherwise, the set decreases in Δ, i.e., a reduction in the set of events suggesting cooperation. The argument is similar; as the set of events shrinks, we trade informational quality for quantity to balance the weaker discounting incentives.

6 Limit Payoffs

6.1 The Limit $r \downarrow 0$

Placing together Proposition 9 and Corollary 8, in equilibrium, as the players become more patient, the number of events suggesting cooperation weakly increases. This is the case when the lowest punishment probability $\alpha_K = 0$ is reached and an additional event is required (the $K+1$ equilibrium in the "bad" news model and the $K-1$ equilibrium in the "good" news model). Consequently, the set of signals suggesting cooperation Π enlarges.

We do not have the functional form that establishes the relationship between K and r (the same occurs with Δ); nonetheless, we know the sign of its "derivative", which is enough to proceed. To take the most from the available information, in the limit, i.e., $r \downarrow 0$, we must have $K \uparrow \infty$ for the "bad" news model, and $K \downarrow 0$ for the "good" news model. Actually, it is more correct to state $K = 0$, rather than $K \downarrow 0$.

Proposition 13 *In the "bad" news model,* $r \downarrow 0 \Longrightarrow v \uparrow \pi$.
In the "good" news model,

$$r \downarrow 0 \Longrightarrow \begin{cases} \overline{v} \uparrow \pi - \frac{e^{-\beta\Delta}}{e^{-\mu\Delta} - e^{-\beta\Delta}} (\pi' - \pi) & \Delta > \frac{\ln(\pi'/\pi)}{\beta - \mu} \\ \widetilde{v} \uparrow \pi - \frac{1+\mu\Delta}{(\beta-\mu)\Delta} (\pi' - \pi) & \Delta > \frac{\pi' - \pi}{\beta\pi - \mu\pi'} \end{cases} . \tag{11}$$

These limits are new to the best of my knowledge. Clearly, the result obtained for the "bad" news model suggests the existence of a limit folk-theorem. The result is possible because the distribution of the public signals is of unbounded support (see Milgrom 1977) and the variance ratio improves (constant) with reductions in r for the random (deterministic) monitoring case. However, because a reduction in the variance after a deviation is not as statistically informative as an increase (see Fudenberg and Levine (2007)), in the "good" news

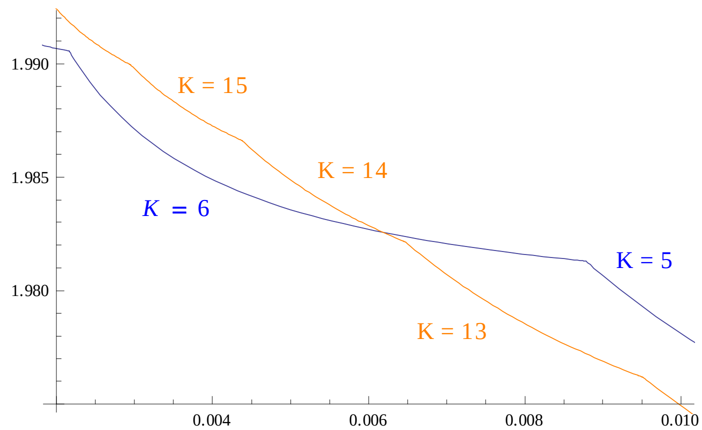

Figure 3: "Bad" news model: payoffs $\overline{v}$ (blue) and $\widetilde{v}$ (orange) as a function of r. ($\mu = 2, \beta = 1, \Delta = 1, \pi = 2, \pi' = 3$)

model, we are bounded away from a fully efficient result. In spite of that fact, if the signals are sufficiently informative, i.e., $\beta >> \mu$, we can become arbitrarily close. In both models, convergence occurs from below and the best payoffs are obtained at the limit.

In the "good" news model $\beta > \mu$, when $r \downarrow 0$ we must have $\overline{v}$ converging above $\widetilde{v}$. To see it, note that $\overline{v} > \widetilde{v}$ for $r \downarrow 0$ is equivalent to looking at $l_{\overline{K}} > l_{\widetilde{K}}$ at the limit, i.e.,

$$e^{-\mu\Delta}/e^{-\beta\Delta} > (1 + \beta\Delta) / (1 + \mu\Delta)\,.$$

Note that the RHS is equivalent to an expansion of the LHS of Δ around zero. Consequently, the RHS is a lower bound to the LHS, not only for small but for all Δ. The sense is that the "good" news result extends to all enforceable r. If this is the case, then random monitoring brings no improvement to the "good" news model.

However, at the limit $r \downarrow 0$ and for the "bad" news model, we cannot formally show whether $\overline{v}$ is above $\widetilde{v}$ or the other way around. The reason that we cannot show this relationship is because the limit $K \uparrow \infty$ is non-trivial. Figure 3 shows the existence of a crossing point for small r, in which random monitoring is superior on the left neighborhood. We are then tempted to think that at the limit, random monitoring is payoff superior, but notice that the number of signals required to sustain cooperation is much lower with deterministic monitoring. Intuitively, in the limit and with a same number of signals "$K \uparrow \infty$" deterministic monitoring must converge to efficiency above random monitoring.

Clearly, the figure shows the existence of a discounting interval where random monitoring leads to payoff improvements.[16]

[16] However, not shown in the picture, random monitoring tends to reduce the $r-$*discounting* interval that sustains cooperation for a large r.

Away from the limit, the comparisons between $\overline{v}$ and $\widetilde{v}$ are difficult to formalize because, typically, $\overline{K} \neq \widetilde{K}$ for equal parameterization.

6.2 Limit Time Frequency

In the "bad" news model, for a given K, as Δ decreases, the punishment probability increases; consequently, v must decrease as well, because of a higher likelihood of mistaken punishment on the equilibrium path. However, this relationship is not perfect. In other words, as Δ gets small, the signals become less informative and the detection technology becomes stricter. In extreme cases, i.e., $(r+\beta)/(r+\mu) > \pi/\pi'$, for a small Δ (even before the limit), the equilibrium degenerates. In spite of it, for large values of Δ, we can move arbitrarily close to a full efficient enforceable equilibrium if the signals are sufficiently informative.

The "good" news model degenerates because of the low information content from the public signals for a low Δ. The discounting incentives are not sufficient to compensate for this effect.

Proposition 14 *In the "bad" news model,*

$$\Delta \downarrow 0 \Longrightarrow \begin{cases} v \downarrow \pi - \frac{\beta}{\mu-\beta}(\pi'-\pi) & \frac{\pi}{\pi'} \geq \frac{2r+\beta}{2r+\mu} \geq \frac{r+\beta}{r+\mu} \\ v \uparrow \pi - \frac{\beta}{\mu-\beta}(\pi'-\pi) & \frac{2r+\beta}{2r+\mu} > \frac{\pi}{\pi'} \geq \frac{r+\beta}{r+\mu} \\ v \downarrow 0 & \text{otherwise} \end{cases} . \tag{12}$$

In the "good" news model, $\Delta \downarrow 0 \Longrightarrow v \downarrow 0$.

These limits coincide with Fudenberg and Levine (2007) and Fudenberg and Olszewski (2011), with the exception that we clearly state the exact informational conditions that cause the limit to converge from above, to converge from below or to degenerate.

One important conclusion for the "bad" news model is that for any Δ, neither $K=0$ nor $K=2$, can enforce limit cooperation. The latter may enforce cooperation for a large Δ, even if the equilibrium degenerates at the limit. Limit degeneracy is guaranteed when $(r+\beta)/(r+\mu) > \pi/\pi'$. In spite of that fact, for large values of Δ, we might be able to sustain cooperation providing that at least $\pi/\pi' \geq \beta/\mu$.[17] Limit payoffs above the static Nash are only possible if a $\alpha_1 \in (0,1]$ exists.

The "good" news model degenerates, even before the limit $\Delta \downarrow 0$. The information degradation for a small Δ is the key aspect.

Proposition 15 *In the "bad" news model,*

$$\Delta \downarrow 0 \Longrightarrow \begin{cases} \widetilde{v} \text{ converges above } \overline{v} & \pi/\pi' \geq (2r+\beta)/(2r+\mu) \\ \overline{v} \text{ converges above } \widetilde{v} & \pi/\pi' < (2r+\beta)/(2r+\mu) \end{cases}$$

In the "good" news model, $\Delta \downarrow 0 \Longrightarrow \overline{v}$ *is above* $\widetilde{v}$.

[17] Even if $\pi/\pi' \geq (2r+\beta)/(2r+\mu)$ we can observe discontinuities in the Δ domain. In other words, there might exist Δ intervals in which cooperation is enforceable alternating with intervals where enforceability fails.

In the "bad" news model, random monitoring is typically superior at the limit and its neighborhood. In fact, away from the limit, deterministic monitoring tends to be superior.

In the "good" news model, random monitoring does not produce any efficiency gain, not only in the limit but also away from it. This result is even the case when we consider the possibility of enlarging the frequencies of play that support cooperation. See Figures 4 and 5 below.

7 Non-limit payoffs: Discussion

Under perfect random monitoring we can enforce the same payoffs as in the deterministic setup with higher discount rates. The players' decisions are based on a larger discount factor (the expected discount factor). The effect is similar; as if the players had become "more patient" (see Kawamori (2004)).

Surprisingly, when monitoring is imperfect, the result is less clear cut, in particular, because the uncertainty regarding the time repetitions of the stage game adversely affect the informational content of the public signals. Before any other considerations, we define how random monitoring can improve the payoffs w.r.t. the canonical deterministic case.

Definition 16 *Given the same parameterization, the best symmetric equilibrium $\widetilde{v}$ is larger than the best symmetric equilibrium $\overline{v}$ when:*
(i) $\widetilde{\alpha}_{\widetilde{K}} \in (0,1]$, $\overline{\alpha}_{\overline{K}} \in (0,1]$ *and* $\widetilde{v} > \overline{v}$.
(ii) $\widetilde{\alpha}_{\widetilde{K}} \in (0,1]$ *and* $\overline{\alpha}_{\overline{K}} \notin (0,1]$.

In part (i), both monitoring technologies enforce cooperation. In this case, both $\widetilde{v}$ and $\overline{v}$ are, at least, weakly above zero. Consequently, the random monitoring returns higher payoffs if $\widetilde{v} \geq \overline{v}$, or, equivalently, if $\widetilde{l}_{\widetilde{K}} \geq \overline{l}_{\overline{K}}$. Part (ii) states when cooperation is enforced exclusively with random monitoring. Consequently, we have a gain equal to $\widetilde{v}$ because $\overline{v} = 0$. In this case, random monitoring expands the spectrum of the frequencies of play that sustain cooperation. Outside of the Definition 16, either deterministic monitoring leads to higher payoffs or no monitoring technology can improve on the static Nash.

Random monitoring in the "bad" and "good" news models presents tractable structures of interest in the applied work. However, payoff comparisons with the deterministic setting are tricky, in particular, because we cannot establish most relationships in close form. Definition 16 is general and ambiguous w.r.t. a particular model. For that reason, Figures 4 and 5, attempt to elucidate these issues.

Figure 4 illustrates the payoffs from the "bad" news deterministic and random monitoring models. For a small Δ, the payoffs decay when Δ decreases, but random monitoring is superior in the sense of part (i) of Definition 16. For a large Δ, we observe the opposite: the payoffs are higher with deterministic monitoring and the enforceable space grows larger.

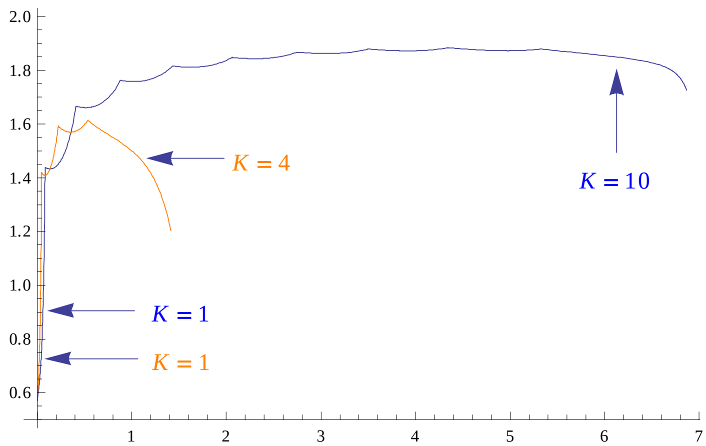

Figure 4: Deterministic (blue) and random (orange) "bad" news model payoffs $\overline{v}$ and $\widetilde{v}$ as a function of Δ. ($r = 0.1, \pi = 2, \pi' = 3, \beta = 1, \mu = 1.7$)

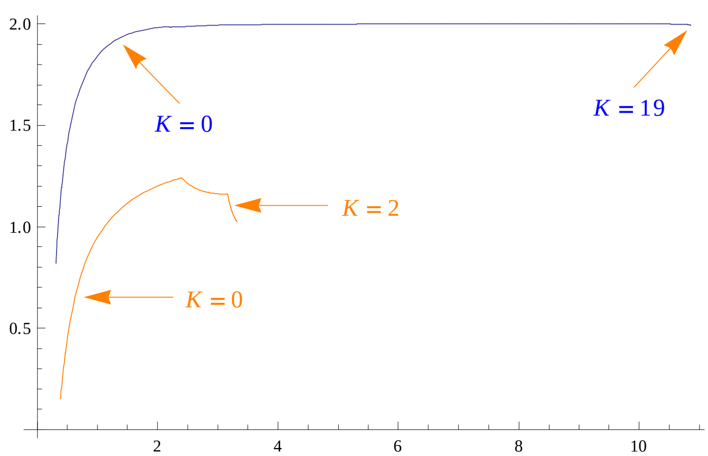

Figure 5: Deterministic (blue) and random (orange) "good" news model payoffs $\overline{v}$ and $\widetilde{v}$ as a function of Δ. ($r = 0.1, \pi = 2, \pi' = 3, \beta = 3, \mu = 1$)

In the deterministic monitoring case, the payoffs increase in Δ for $\overline{K} = 1$, they present a u-shape for $\overline{K} = 2, ..., 8$, and they are concave decreasing in Δ for $\overline{K} = 9$. The largest payoff is achieved at $\alpha_7 = 0$ or $\alpha_8 = 1$, i.e., the coordinate $(\Delta, \overline{v}) = (4.35, 1.88)$. In the random monitoring case, the payoffs increase in Δ for $\widetilde{K} = 1$, they present a u-shape for $\widetilde{K} = 2, 3$, and they present a concave decreasing in Δ for $\widetilde{K} = 4$. The largest payoff is achieved at $\widetilde{\alpha}_3 = 0$ or $\widetilde{\alpha}_4 = 1$, i.e., the coordinate $(\Delta, \widetilde{v}) = (0.55, 1.61)$.

The same conclusion holds for the "good" news model. Here, we add events of increasing magnitude as Δ decreases. Figure 5 provides an illustration. In the deterministic monitoring case, the payoffs increase in Δ for $\overline{K} = 0$, they present a u-shape for $\overline{K} = 1, ..., 13$, and they are convex decreasing in Δ for $\overline{K} = 14, ..., 19$. The largest payoff is achieved at $\alpha_9 = 1$ or $\alpha_{10} = 0$, i.e., the coordinate $(\Delta, \overline{v}) = (8.43, 1.99)$. In the random monitoring case, the payoffs increase in Δ for $\widetilde{K} = 0$ and are convex decreasing in Δ for $\widetilde{K} = 1, 2$. The largest payoff is achieved at $\widetilde{\alpha}_0 = 1$ or $\widetilde{\alpha}_1 = 0$, i.e., the coordinate $(\Delta, \widetilde{v}) = (2.24, 1.24)$. The conditions of Definition 16 fail and deterministic monitoring is superior in all dimensions.

The results presented in these figures tend to be general and these patterns repeat for other parameterizations. Consistent with Abreu, Milgrom and Pearce (1991), delay allows for larger payoffs: this is true independently of the monitoring structure.

APPENDIX: PROOF OF COROLLARIES AND PROPOSITIONS

Proof of Propositions 3. Consider the general case. Under cooperation we have expression (6) and the incentives to defection are removed if inequality (7) is satisfied. Replace (6) with (7) to obtain

$$v \geq \frac{(1-\delta)(\pi' - \pi)}{\delta \sum_{k \in \Pi} (p_k - q_k)(1 - \alpha_k)}. \tag{13}$$

Our objective is to

$$\max_{\overline{\alpha} = \{\overline{\alpha}_0, \overline{\alpha}_1, ...\}} (6) \text{ subject to (13) and } \overline{\alpha}_k \in (0, 1].$$

The best symmetric non-trivial equilibrium $v > 0$ requires inequality (13) binds. Because $p_k > 0$, the more elements there are in Π and/or the smaller α_k is, the larger the value (6).

Consider first the "*bad" news model*, i.e., $\mu > \beta > 0$. In this case, we have $p_k > q_k$ for a small k and $p_k < q_k$ for a large k.[18] In the former (latter),

[18] E.g., in the deterministic "*bad" news model*, the value k such that $\overline{p}_k(\Delta) > \overline{q}_k(\Delta)$ and $\overline{p}_{k+1}(\Delta) \leq \overline{q}_{k+1}(\Delta)$, is bounded by

$$(\mu - \beta)\Delta / \ln(\mu/\beta) - 1 \leq \overline{k}^* < (\mu - \beta)\Delta / \ln(\mu/\beta).$$

the occurrence of k events is more likely if there is cooperation (defection). Our concern is not with a particular event but instead is with the aggregate. Starting from $k=0$ with $p_k - q_k > 0$, the denominator of (13) increases and the RHS of the inequality decreases as we add events. Next (for some $k > k^*$), i.e., when $p_k - q_k < 0$, the addition of a new signal decreases the denominator and consequently increases the RHS of (13). If we continue adding events, for some $k = K$, the inequality (13) fails. Note that at $k = k^*$, the RHS of (13) reaches its minimum; consequently, we must have $K \geq k^*$. Because $p_k > p_{k+1}$, if we take some $k \geq k^*$ satisfying $k < K$, as a signal of cooperation, then it must be the case that $\alpha_k = 0$. The relative measure is the relevant one, i.e., $p_k/q_k > p_{k+1}/q_{k+1}$ for all k. Because (13) holds if $\Pi = \{0, 1, ..., K-1\}$ and fails if $\Pi = \{0, 1, ..., K\}$ then we must have $\alpha_K \in (0, 1]$ and $\alpha_k = 1$ for all $k > K$. The argument holds independently of the monitoring technology, with $K \equiv \overline{K}$ and $K \equiv \widetilde{K}$ for the deterministic and random cases, respectively.

In the "good" news model, i.e., $\beta > \mu > 0$, the argument is reversed. In this case, we have $p_k < q_k$ for a small k, while $p_k > q_k$ for a large k.[19] In the former (latter), the observation of k signals is more likely if there is defection (cooperation). To obtain the aggregate of events that lead (13) to hold with equality, starting from a k that is sufficiently large, i.e., when $p_k - q_k > 0$, the addition of a new event (the event $k-1$) increases the denominator on the RHS of (13) and consequently decreases the inequality. Below a certain k^*, $p_k - q_k < 0$, the denominator of (13) decreases and the RHS of (13) increases as we add signals. If we continue adding signals, at a certain point, $k = K$, and the inequality (13) fails. Observe that now $p_k/q_k < p_{k+1}/q_{k+1}$ for all k. Consequently, (13) holds for $\Pi = \{K+1, ..., \infty\}$ and fails for $\Pi = \{K, ..., \infty\}$ with $0 \leq K \leq k^*$. Then, in equilibrium, we must have $\alpha_K \in (0, 1]$, $\alpha_k = 0$ for all $k > \overline{K}$, and $\overline{\alpha}_k = 1$ for all $k < \overline{K}$.

The computation of $\overline{v}$, $\widetilde{v}$, $\overline{\alpha}_{\overline{K}}$ and $\widetilde{\alpha}_{\widetilde{K}}$ follows standard steps, but with the obtained optimal behavior incorporated. For the deterministic *"bad" ("good") news model*, we have $\Pi = \overline{\Pi} \equiv \left\{0, 1, ..., \overline{K}-1\right\}$ and $\alpha = \overline{\alpha} \equiv \{0, ..., 0, \overline{\alpha}_{\overline{K}}, 1, ...\}$ (respectively, $\Pi = \overline{\Pi} \equiv \left\{\overline{K}+1, ..., \infty\right\}$ and $\alpha = \overline{\alpha} \equiv \{1, ..., 1, \overline{\alpha}_{\overline{K}}, 0, ...\}$), while in the random case $\Pi = \widetilde{\Pi} \equiv \left\{0, 1, ..., \widetilde{K}-1\right\}$ and $\alpha = \widetilde{\alpha} \equiv \left\{0, ..., 0, \widetilde{\alpha}_{\widetilde{K}}, 1, ...\right\}$ (respectively, $\Pi = \widetilde{\Pi} \equiv \left\{\widetilde{K}+1, ..., \infty\right\}$ and $\alpha = \widetilde{\alpha} \equiv \left\{1, ..., 1, \widetilde{\alpha}_{\widetilde{K}}, 0, ...\right\}$). Consequently, for the general case, (6) and (7) are replaced by

$$v = (1-\delta)\,\pi + \delta v \left(\sum\nolimits_{k \in \Pi} p_k + p_K\,(1-\alpha_K)\right),$$

and

$$v \geq (1-\delta)\,\pi' + \delta v \left(\sum\nolimits_{k \in \Pi} q_k + q_K\,(1-\alpha_K)\right),$$

[19] E.g., in the deterministic "bad" news model, the value k such that $\overline{p}_{k+1} > \overline{q}_{k+1}$ and $\overline{p}_k \leq \overline{q}_k$ is bounded by

$$(\beta-\mu)\,\Delta/\ln(\beta/\mu) - 1 \leq \overline{k}^* < (\beta-\mu)\,\Delta/\ln(\beta/\mu).$$

respectively. The solution of this system of two equations (with the latter holding with equality) and two unknown returns, (8) with (9) and (10).

Alternatively to (8), we can replace the α function into (9) and write

$$v=\frac{(1-\delta)(\pi q_K-\pi' p_K)}{\left(1-\delta\sum_{k\in\Pi}p_k\right)q_K-\left(1-\delta\sum_{k\in\Pi}q_k\right)p_K}. \tag{14}$$

On some occasions, this formulation is more convenient. ■

Proof of Corollary 5. In the "bad" news model, after some algebra, both $\alpha_K=0$ and $\alpha_{K+1}=1$ lead to

$$\pi\left(1-\delta\sum_{k=0}^{K}q_k\right)-\pi'\left(1-\delta\sum_{k=0}^{K}p_k\right)=0,$$

while for the "good" news model, both $\alpha_K=1$ and $\alpha_{K+1}=0$ lead to

$$\pi\left(1-\delta\sum_{k=K+1}^{\infty}q_k\right)-\pi'\left(1-\delta\sum_{k=K+1}^{\infty}p_k\right)=0.$$

■

Proof of Proposition 6. Note that $\alpha_K=1$ implies

$$\pi\left(1-\delta\sum_{k\in\Pi}q_k\right)-\pi'\left(1-\delta\sum_{k\in\Pi}p_k\right)=0, \tag{15}$$

i.e., the event K is fully considered to be a signal of defection. On the other hand, $\alpha_K=0$ implies that

$$\pi\left(1-\delta\sum_{k\in\Pi}q_k\right)-\pi'\left(1-\delta\sum_{k\in\Pi}p_k\right)=\delta\left(\pi q_K-\pi' p_K\right).$$

From $\alpha_K=0$ to $\alpha_K=1$, we have subtracted (in the reverse direction, we have added) the event K from the pool of events suggesting cooperation. We want to know the sign of $\pi q_K-\pi' p_K$ on the RHS of the last equality. Optimality requires the enforceability constraint

$$\delta\geq\frac{\pi'-\pi}{\left(\pi'\sum_{k\in\Pi}p_k-\pi\sum_{k\in\Pi}q_k\right)-(\pi q_K-\pi' p_K)(1-\alpha_K)},$$

to bind, but with the largest possible set of events Π. Suppose that $\pi q_K>\pi' p_K$ and enforceability binds, then we cannot add more signals without losing enforceability (i.e., a decrease in α_K increases the RHS ratio). On the other hand, an increase in α_K decreases the RHS ratio but "reduces" the set of events. We do not want enforceability to hold with slack and with a small set of signals. Suppose the opposite, i.e., $\pi q_K<\pi' p_K$ and enforceability binds, then a decrease in α_K decreases the RHS ratio and we could add events until we obtain the largest set of signals $\Pi=\{0,...,\infty\}$. Enforceability would improve with the addition of no informative events, which is impossible. In the opposite direction, if $\pi q_K < \pi' p_K$ and we increase α_K, we would increase the RHS. Consequently, $\pi q_K > \pi' p_K$ in a K non-trivial equilibrium.

Now, we show the first part of the result. We first focus on the deterministic case; similar arguments can be generalized in a straightforward manner to random monitoring. We want to show that $\pi \overline{q}_{\overline{K}} - \pi' \overline{p}_{\overline{K}} > \pi \overline{q}_{\overline{K} \mp n} - \pi' \overline{p}_{\overline{K} \mp n}$. In the "bad" news model, we have

$$\pi \overline{q}_{\overline{K}} \left(1 - \frac{\overline{K}!}{(\overline{K} - n)!} (\mu \Delta)^{-n} \right) > \pi' \overline{p}_{\overline{K}} \left(1 - \frac{\overline{K}!}{(\overline{K} - n)!} (\beta \Delta)^{-n} \right),$$

and because $\mu > \beta$, the term in brackets in the LHS is larger than the one in the RHS. Then, because $\pi \overline{q}_{\overline{K}} - \pi' \overline{p}_{\overline{K}} > 0$, the result holds for any $n = 1, ..., \overline{K}$. In the "good" news model we have

$$\pi \overline{q}_{\overline{K}} \left(1 - \frac{\overline{K}!}{(\overline{K} + n)!} (\mu \Delta)^{n} \right) > \pi' \overline{p}_{\overline{K}} \left(1 - \frac{\overline{K}!}{(\overline{K} + n)!} (\beta \Delta)^{n} \right);$$

because $\pi \overline{q}_{\overline{K}} - \pi' \overline{p}_{\overline{K}} > 0$ and $\beta > \mu$, the term in brackets in the LHS is larger than in the RHS and $\pi q_k - \pi' p_k$ must decrease in $k = K, ..., \infty$, for any $n = 1, ..., \infty$. ■

Proof of Proposition 7. We provide a partial proof; the remaining argument follows in the proof of Proposition 10. Note that at $\alpha_K = 1$, we have the equality (15). W.l.o.g., we differentiate α_K w.r.t. $z = r$ and evaluate it at $\alpha_K = 1$ (the same can be done at $\alpha_K = 0$). Our goal is to find the sign of $\partial \overline{\alpha}_{\overline{K}} / \partial r$. Because $\pi q_K > \pi' p_K$ by Proposition 6), after some algebra, we obtain the expression that determines the sign of this derivative,

$$\pi \left(\frac{\partial \delta}{\partial z} \sum_{k \in \Pi} q_k + \delta \sum_{k \in \Pi} \frac{\partial q_k}{\partial z} \right) - \pi' \left(\frac{\partial \delta}{\partial z} \sum_{k \in \Pi} p_k + \delta \sum_{k \in \Pi} \frac{\partial p_k}{\partial z} \right). \tag{16}$$

(i) Deterministic "bad" news model: we have $\partial \delta^{\Delta} / \partial r = -\Delta \delta^{\Delta}$, $\sum_{k=0}^{\overline{K}-1} \partial \overline{q}_k / \partial r = 0$ and $\sum_{k=0}^{\overline{K}-1} \partial \overline{p}_k / \partial r = 0$, and after some algebra, we obtain $\pi' \sum_{k=0}^{\overline{K}-1} \overline{p}_k - \pi \sum_{k=0}^{\overline{K}-1} \overline{q}_k > 0$, i.e., $\partial \overline{\alpha}_{\overline{K}} / \partial r > 0$. Note that we cannot have an equilibrium with $\overline{K} = 0$.

(ii) Random "bad" news model: we have $\partial E(\delta^x) / \partial r = -\Delta E(\delta^x)^2$, $\sum_{k=0}^{\widetilde{K}-1} \partial \widetilde{q}_k / \partial r = \Delta \widetilde{K} \widetilde{q}_{\widetilde{K}} / (1 + r\Delta)$ and $\sum_{k=0}^{\widetilde{K}-1} \partial \widetilde{p}_k / \partial r = \Delta \widetilde{K} \widetilde{p}_{\widetilde{K}} / (1 + r\Delta)$, and after some algebra, we obtain

$$\pi' \sum_{k=0}^{\widetilde{K}-1} \widetilde{p}_k - \pi \sum_{k=0}^{\widetilde{K}-1} \widetilde{q}_k + \widetilde{K} \left(\pi \widetilde{q}_{\widetilde{K}} - \pi' \widetilde{p}_{\widetilde{K}} \right) > 0,$$

i.e., $\partial \widetilde{\alpha}_{\widetilde{K}} / \partial r > 0$. Again, we cannot have an equilibrium with $\overline{K} = 0$.

(iii) Deterministic "good" news model: we have $\sum_{k=\overline{K}+1}^{\infty} \partial \overline{q}_k / \partial r = 0$ and $\sum_{k=\overline{K}+1}^{\infty} \partial \overline{p}_k / \partial r = 0$, and after some algebra we obtain $\pi' \sum_{k=\overline{K}+1}^{\infty} \overline{p}_k - \pi \sum_{k=\overline{K}+1}^{\infty} \overline{q}_k >$

0, i.e., $\partial\overline{\alpha}_{\overline{K}}/\partial r > 0$. A minimum requirement for the existence of a non-trivial payoff is at least $\overline{\alpha}_0 \geq 0$ or $\pi\overline{q}_0 > \pi'\overline{p}_0$, i.e., when $(\beta - \mu)\Delta > \ln(\pi'/\pi)$. Note also that $\overline{K} \uparrow \infty$ cannot be an equilibrium.

(iv) Random "good" news model: in this case, the sign of the derivative is not clear. We follow a different approach. Note that $\widetilde{\alpha}_0 = 0$ has three roots in r, i.e., $r = -(1+\beta\Delta)/\Delta$, $r = -(1+\mu\Delta)/\Delta$, and $r = 0$. Moreover, $\widetilde{\alpha}_0$ has two asymptotes when $\pi\widetilde{q}_0 = \pi'\widetilde{p}_0$, i.e., at $r = -1/\Delta$ and $r_a = (\pi(1+\beta\Delta) - \pi'(1+\mu\Delta))/\Delta(\pi' - \pi)$, which is positive if $\pi(1+\beta\Delta) > \pi'(1+\mu\Delta)$. In the latter case, because $\widetilde{\alpha}_0$ is monotonic in $r \in (0, r_a)$ with $\widetilde{\alpha}_0 \uparrow \infty$ when $r \uparrow r_a$, it must be the case that $\widetilde{\alpha}_0 = 1$ for some $r \in (0, r_a)$; consequently, $\partial\widetilde{\alpha}_0/\partial r > 0$ in equilibrium.

Similar to all cases is that $\partial\alpha_k/\partial r > 0$ for $k \geq 0$ while enforceability holds, in which case the sign is reversed (see Proposition 10 and its proof to complete the argument). ■

Corollary 17 *In a K non-trivial equilibrium*

$$\frac{q_K}{p_K} \geq \frac{1 - \delta\sum_{k\in\Pi} q_k}{1 - \delta\sum_{k\in\Pi} p_k} \geq \frac{\pi'}{\pi} > 1.$$

If, in a K non-trivial equilibrium, $\pi q_K - \pi' p_K > 0$ and $\alpha \in (0,1]$, then the numerator in (10) must be non-negative and smaller than the denominator. In other words, we have

$$\pi\left(1 - \delta\sum\nolimits_{k\in\Pi} q_k\right) - \pi'\left(1 - \delta\sum\nolimits_{k\in\Pi} p_k\right) \in [0, \delta(\pi q_K - \pi' p_K)).$$

When we switch between a K equilibrium to the following or from the subsequent, it is because the numerator has reached one of its bounds.

Proof of Proposition 9. Suppose that monitoring is deterministic, then

$$\frac{\partial\overline{\alpha}_{\overline{K}}}{\partial r} = \frac{(\pi' - \pi)\Delta}{\delta^{\Delta}(\pi\overline{q}_{\overline{K}} - \pi'\overline{p}_{\overline{K}})} > 0,$$

this derivative is positive because $\pi\overline{q}_{\overline{K}} > \pi'\overline{p}_{\overline{K}}$ by Proposition 6. However, the derivative of (14) gives

$$\frac{\partial\overline{v}}{\partial r} = \frac{(\pi' - \pi)\left(\left(1 - \sum_{k\in\overline{\Pi}}\overline{p}_k\right)\overline{q}_{\overline{K}} - \left(1 - \sum_{k\in\overline{\Pi}}\overline{q}_k\right)\overline{p}_{\overline{K}}\right)\frac{\partial\overline{\alpha}_{\overline{K}}}{\partial r}}{\left(\sum_{k\in\overline{\Pi}}(\overline{p}_k - \overline{q}_k) + (1 - \overline{\alpha}_{\overline{K}})(\overline{p}_{\overline{K}} - \overline{q}_{\overline{K}})\right)^2}.$$

Then, $\partial\overline{\alpha}_{\overline{K}}/\partial r$ determines the sign of $\partial\overline{v}/\partial r$, which is negative if

$$\left(1 - \sum\nolimits_{k\in\overline{\Pi}}\overline{q}_k\right)\overline{p}_{\overline{K}} > \left(1 - \sum\nolimits_{k\in\overline{\Pi}}\overline{p}_k\right)\overline{q}_{\overline{K}}. \tag{17}$$

The last two inequalities together with Corollary 17 impose

$$\frac{1 - \sum_{k\in\overline{\Pi}}\overline{q}_k}{1 - \sum_{k\in\overline{\Pi}}\overline{p}_k} > \frac{\overline{q}_{\overline{K}}}{\overline{p}_{\overline{K}}} \geq \frac{1 - \delta^{\Delta}\sum_{k\in\overline{\Pi}}\overline{q}_k}{1 - \delta^{\Delta}\sum_{k\in\overline{\Pi}}\overline{p}_k} \geq \frac{\pi'}{\pi}.$$

In addition, because $\delta^{\Delta} \in (0,1)$, we have

$$\frac{1 - \sum_{k\in\overline{\Pi}} \overline{q}_k}{1 - \sum_{k\in\overline{\Pi}} \overline{p}_k} > \frac{1 - \delta^{\Delta} \sum_{k\in\overline{\Pi}} \overline{q}_k}{1 - \delta^{\Delta} \sum_{k\in\overline{\Pi}} \overline{p}_k}.$$

and inequality (17) is guaranteed.

In the random monitoring case, these derivatives are more complex. We show it by contradiction. Suppose instead that $\partial\widetilde{v}/\partial r < 0$ with $\partial\widetilde{\alpha}_{\widetilde{K}}/\partial r < 0$. Then, a decrease in r increases the punishment probability. However, because $\partial\widetilde{v}/\partial r < 0$, a decrease in r should increase the payoffs, which is possible if we are adding more events suggesting cooperation. Consequently, for the "bad" news model, at $\widetilde{\alpha}_{\widetilde{K}} = 1$ we would have $\widetilde{\alpha}_{\widetilde{K}+1} = 0$, while in the "good" news model $\widetilde{\alpha}_{\widetilde{K}} = 1$ we would have $\widetilde{\alpha}_{\widetilde{K}-1} = 0$, contradicting Corollary 5. The cases $\partial\widetilde{v}/\partial r > 0$ with $\partial\widetilde{\alpha}_{\widetilde{K}}/\partial r > 0$ and $\partial\widetilde{v}/\partial r > 0$ with $\partial\widetilde{\alpha}_{\widetilde{K}}/\partial r < 0$ lead to similar contradictions. ■

Proof of Proposition 10. Proposition 6 establishes that $\pi q_K > \pi' p_K$ in a K equilibrium. Moreover, non-trivial payoffs are obtained while $\partial\alpha_K/\partial r > 0$ (by Proposition 7) and enforceability fails when this sign is reversed. Moreover, increasing r with $\Delta > 0$ fixed and increasing Δ with $r > 0$ fixed must lead to the same terminal condition. Consequently, the result is valid for variations in any primitive of the model, including Δ.

Deterministic "bad" news model: suppose that $\overline{\alpha}_k \in (0,1]$ is increasing in Δ or r for some $k \geq 1$. We want to show that a $\overline{K}-1$ exists such that $\overline{\alpha}_{\overline{K}-1}$ is decreasing in Δ or r. At $\overline{\alpha}_{\overline{K}} = 0$, we have the binding condition, written in the convenient format,

$$\begin{aligned} & \pi\left(1 - \delta^{\Delta} \sum\nolimits_{k=0}^{\overline{K}-2} \overline{q}_k\right) - \pi'\left(1 - \delta^{\Delta} \sum\nolimits_{k=0}^{\overline{K}-2} \overline{p}_k\right) \\ = \; & \delta^{\Delta}\left(\pi\left(\overline{q}_{\overline{K}} + \overline{q}_{\overline{K}-1}\right) - \pi'\left(\overline{p}_{\overline{K}} + \overline{p}_{\overline{K}-1}\right)\right). \end{aligned}$$

By Corollary 5 at $\overline{\alpha}_{\overline{K}} = 1$, we have $\overline{\alpha}_{\overline{K}-1} = 0$. If $\overline{\alpha}_{\overline{K}-1} > \overline{\alpha}_{\overline{K}} = 0$ at the point where $\overline{\alpha}_{\overline{K}} = 0$, we must have $\partial\alpha_{K-1}/\partial r < 0$ or $\partial\alpha_{K-1}/\partial\Delta < 0$. Consequently, we plug the $\overline{\alpha}_{\overline{K}} = 0$ binding condition into $\overline{\alpha}_{\overline{K}-1}$ to obtain, after some rearrangement

$$\overline{\alpha}_{\overline{K}-1} = -\left(\pi\overline{q}_{\overline{K}} - \pi'\overline{p}_{\overline{K}}\right) / \left(\pi\overline{q}_{\overline{K}-1} - \pi'\overline{p}_{\overline{K}-1}\right) > 0,$$

Because $\pi\overline{q}_{\overline{K}} > \pi'\overline{p}_{\overline{K}}$ in a $\overline{K}$ equilibrium, it must be the case that $\pi\overline{q}_{\overline{K}-1} < \pi'\overline{p}_{\overline{K}-1}$. After some algebra, we obtain

$$\left(\frac{\beta}{\mu}\right)^{\overline{K}} e^{-(\beta-\mu)\Delta} < \frac{\pi}{\pi'} < \left(\frac{\beta}{\mu}\right)^{\overline{K}-1} e^{-(\beta-\mu)\Delta} \qquad (18)$$

The value of $\overline{K}$ that satisfies the previous inequality is the last (lowest) $\overline{K}$ that enforces cooperation.

Random "good" news model: we want to show that a $\widetilde{K}+1$ exists such that $\widetilde{\alpha}_{\widetilde{K}+1}$ is decreasing in Δ. At $\widetilde{\alpha}_{\widetilde{K}}=0$, we have the binding condition, written in the convenient format,

$$\begin{aligned} & \pi\left(1-E\left(\delta^{x}\right)\sum\nolimits_{k=\widetilde{K}+2}^{\infty}\widetilde{q}_{k}\right)-\pi'\left(1-E\left(\delta^{x}\right)\sum\nolimits_{k=\widetilde{K}+2}^{\infty}\widetilde{p}_{k}\right) \\ = \quad & E\left(\delta^{x}\right)\left(\pi\left(\widetilde{q}_{\widetilde{K}}+\widetilde{q}_{\widetilde{K}+1}\right)-\pi'\left(\widetilde{p}_{\widetilde{K}}+\widetilde{p}_{\widetilde{K}+1}\right)\right), \end{aligned}$$

By Corollary 5 at $\widetilde{\alpha}_{\widetilde{K}}=1$, we have $\widetilde{\alpha}_{\widetilde{K}+1}=0$. Now we want to show that $\widetilde{\alpha}_{\widetilde{K}+1}>\widetilde{\alpha}_{\widetilde{K}}=0$ at the point where $\widetilde{\alpha}_{\widetilde{K}}=0$. To show it, we plug the $\widetilde{\alpha}_{\widetilde{K}}=0$ condition into $\widetilde{\alpha}_{\widetilde{K}+1}$, to obtain, after some rearrangement

$$\widetilde{\alpha}_{\widetilde{K}+1}=-\left(\pi\widetilde{q}_{\widetilde{K}}-\pi'\widetilde{p}_{\widetilde{K}}\right)/\left(\pi\widetilde{q}_{\widetilde{K}+1}-\pi'\widetilde{p}_{\widetilde{K}+1}\right)>0,$$

where the denominator must be negative by Proposition 6. After some algebra, we obtain

$$\left(\frac{\beta}{\mu}\frac{1+r\Delta+\mu\Delta}{1+r\Delta+\beta\Delta}\right)^{\widetilde{K}}<\frac{\pi}{\pi'}\frac{1+r\Delta+\beta\Delta}{1+r\Delta+\mu\Delta}<\left(\frac{\beta}{\mu}\frac{1+r\Delta+\mu\Delta}{1+r\Delta+\beta\Delta}\right)^{\widetilde{K}+1}.$$

The value of $\widetilde{K}$ that satisfies this condition is the last (largest) $\widetilde{K}$ that enforces cooperation.

The random "bad" and the deterministic "good" news model, follow the same arguments, respectively, $\pi\widetilde{q}_{\widetilde{K}}>\pi'\widetilde{p}_{\widetilde{K}}$ and $\pi\widetilde{q}_{\widetilde{K}-1}<\pi'\widetilde{p}_{\widetilde{K}-1}$, and $\pi\overline{q}_{\overline{K}}>\pi'\overline{p}_{\overline{K}}$ and $\pi\overline{q}_{\overline{K}+1}<\pi'\overline{p}_{\overline{K}+1}$, deliver conditions

$$\left(\frac{\beta}{\mu}\frac{1+r\Delta+\mu\Delta}{1+r\Delta+\beta\Delta}\right)^{\widetilde{K}}<\frac{\pi}{\pi'}\frac{1+r\Delta+\beta\Delta}{1+r\Delta+\mu\Delta}<\left(\frac{\beta}{\mu}\frac{1+r\Delta+\mu\Delta}{1+r\Delta+\beta\Delta}\right)^{\widetilde{K}-1},$$

and

$$\left(\frac{\beta}{\mu}\right)^{\overline{K}}e^{-(\beta-\mu)\Delta}<\frac{\pi}{\pi'}<\left(\frac{\beta}{\mu}\right)^{\overline{K}+1}e^{-(\beta-\mu)\Delta}.$$

To gain some intuition, consider the deterministic "bad" news model $\mu>\beta$, and inequality (18). A change to a lower k due to an increase in r increases $(\beta/\mu)^{k}$ and consequently enforceability must fail at a certain point. However, a change to a lower k due to an increase in Δ increases $(\beta/\mu)^{k}$ but, at same time, allows Δ to keep increasing via a decrease in $e^{-(\beta-\mu)\Delta}$. Because, at a certain point enforceability fails, it must be the case that the negative effect in the discount factor (via k) is stronger than the informational gain obtained with a large Δ (increase in the variance ratio). The intuition is similar for the "good" news model; the difference is that $\beta>\mu$ and the transition is made to a higher k. ■

Proof of Proposition 11. Deterministic "bad" news model: we start by looking at the lowest k equilibrium and then generalize it to a larger k via Proposition 10. Note that

$$\overline{\alpha}_{0}=1+\left(\pi'-\pi\right)/\delta\left(\pi\overline{q}_{0}-\pi'\overline{p}_{0}\right),$$

is larger than the unit under the equilibrium condition $\pi\overline{q}_0 > \pi'\overline{p}_0$. Otherwise, it might be in $(0,1]$, but with $\pi\overline{q}_0 < \pi'\overline{p}_0$, which does not satisfy Proposition 6. Consequently, we cannot have an equilibrium with $\overline{K} = 0$. Without loss of generality, note that

$$\overline{\alpha}_1 = 1 - \left(\pi\left(1 - \delta e^{-\mu\Delta}\right) - \pi'\left(1 - \delta e^{-\beta\Delta}\right)\right)/\delta\left(\pi\overline{q}_1 - \pi'\overline{p}_1\right),$$

has an asymptote when $\pi\overline{q}_1 = \pi'\overline{p}_1$, i.e., at $\Delta_a = \ln\left(\pi\mu/\pi'\beta\right)/\left(\mu - \beta\right)$, which is positive if $\pi\mu > \pi'\beta$. Then, $\overline{\alpha}_1 \downarrow -\infty$ as $\Delta \uparrow \Delta_a$. Differentiate $\overline{\alpha}_1$ w.r.t. Δ and take the limit $\Delta \downarrow 0$ to obtain the sign relevant expression

$$-\beta\left(\pi'\left(\beta + r\right)\right)^2 - \mu\left(\pi\left(\mu + r\right)\right)^2 + \pi\pi'\left(\left(r^2 + \beta\mu\right)\left(\beta + \mu\right) + 2r\left(\beta^2 + \mu^2\right)\right).$$

If the sign is negative, then $\overline{\alpha}_1 \uparrow r\left(\pi' - \pi\right)/\left(\pi\mu - \pi'\beta\right)$ as $\Delta \downarrow 0$, otherwise $\overline{\alpha}_1 \downarrow r\left(\pi' - \pi\right)/\left(\pi\mu - \pi'\beta\right)$ as $\Delta \downarrow 0$, and where $r\left(\pi' - \pi\right)/\left(\pi\mu - \pi'\beta\right) < 1$ if $\pi\left(\mu + r\right) > \pi'\left(\beta + r\right)$, which implies $\pi\mu > \pi'\beta$. In the latter case, $\overline{\alpha}_1$ is monotonically increasing in $\Delta \in (0, \Delta_a)$, and if $\overline{\alpha}_1 = \overline{\alpha}_2$ is above the unit, then there is an interval (Δ_1, Δ_2) with a trivial equilibrium where $\overline{\alpha}_1(\Delta_1) = 1$ and $\overline{\alpha}_2(\Delta_2) = 1$. In the former case, $\overline{\alpha}_1$ decreases in $\Delta \in (0, \Delta_a)$, and assuming that it is monotonic, then $\overline{\alpha}_1$ must hit zero at some point in this interval, i.e., $\partial\overline{\alpha}_1/\partial\Delta < 0$. In both cases, with $\pi\left(\mu + r\right) > \pi'\left(\beta + r\right)$, following the Corollary 5 and while enforceability holds, we must have $\partial\overline{\alpha}_k/\partial\Delta < 0$ for some $k \geq 1$.For large Δ by Proposition 10 we must have a shift to $\partial\overline{\alpha}_k/\partial\Delta > 0$ for some $k \geq 1$. Consequently, a Δ interval and an associated k equilibrium exist in which $\overline{\alpha}_k > 0$ has an U-shape and $\overline{\alpha}_k = 1$ in the extreme points. Finally, note that when $\pi\left(\mu + r\right) < \pi'\left(\beta + r\right)$, there is no enforceable equilibrium for a small $\Delta \downarrow 0$, but it might exist for larger Δ. This equilibrium is discontinuous, alternating between enforceable and non-enforceable Δ intervals.

Random "bad" news model: the proof is similar to the deterministic case. Again, we cannot have an equilibrium with $\widetilde{K} = 0$ because $\widetilde{\alpha}_0 > 1$. Note that $\widetilde{\alpha}_1$ has asymptotes at $\pi\widetilde{q}_1 = \pi'\widetilde{p}_1$, i.e., when

$$\mu\pi/\beta\pi' = \left(\left(1 + r\Delta + \mu\Delta\right)/\left(1 + r\Delta + \beta\Delta\right)\right)^2.$$

If $\mu\pi > \beta\pi'$ we have one negative and one positive root Δ_a. Then, $\widetilde{\alpha}_1 \downarrow -\infty$ as $\Delta \uparrow \Delta_a$ and $\widetilde{\alpha}_1 \uparrow r\left(\pi' - \pi\right)/\left(\pi\mu - \pi'\beta\right)$ as $\Delta \downarrow 0$.[20] Where $r\left(\pi' - \pi\right)/\left(\pi\mu - \pi'\beta\right) < 1$ if $\pi\left(\mu + r\right) > \pi'\left(\beta + r\right)$ which is implied by $\pi\mu > \pi'\beta$. Consequently, $\overline{\alpha}_1$ must decrease monotonically in $\Delta \in (0, \Delta_a)$ and reach zero at some point in this interval, implying $\partial\widetilde{\alpha}_1/\partial\Delta < 0$. When $\pi\left(\mu + r\right) < \pi'\left(\beta + r\right)$, there is no enforceable equilibrium for a small Δ, but they might exist for larger Δ. For $\pi\left(\mu + r\right) > \pi'\left(\beta + r\right)$, following Corollary 5 and while enforceability holds, we must have $\partial\widetilde{\alpha}_k/\partial\Delta < 0$ for some $k \geq 1$. Then, for large Δ by Proposition 10, we must have $\partial\widetilde{\alpha}_k/\partial\Delta > 0$ for some $k \geq 1$. Consequently, a Δ interval and an

[20] We also have the case where

$$-\beta\left(\pi'\left(\beta + r\right)\right)^2 - \mu\left(\pi\left(\mu + r\right)\right)^2 + \pi\pi'\left(\left(r^2 + \beta\mu\right)\left(\beta + \mu\right) + 2r\left(\beta^2 + \mu^2\right)\right).$$

is positive and $\overline{\alpha}_1 \downarrow r\left(\pi' - \pi\right)/\left(\pi\mu - \pi'\beta\right)$ as $\Delta \downarrow 0$.

associated k equilibrium exist in which $\widetilde{\alpha}_k > 0$ has an U-shape and $\widetilde{\alpha}_k = 1$ in the extreme points.

Deterministic "good" news model: Note that $\overline{\alpha}_0$ is given by

$$\overline{\alpha}_0 = \left(1 - e^{-r\Delta}\right)\left(\pi' - \pi\right)/e^{-r\Delta}\left(\pi e^{-\mu\Delta} - \pi' e^{-\beta\Delta}\right).$$

Moreover, $\overline{\alpha}_0 = 0$ has a root at $\Delta = 0$ from below, i.e., $\partial\overline{\alpha}_0/\partial\Delta = -r$ at $\Delta \downarrow 0$. Consequently, $\overline{\alpha}_0 \notin (0, 1]$ for very small $\Delta > 0$ and is always above zero for all $\Delta > 0$. Now, and away from the limit, we want to show that at $\overline{\alpha}_0 = 1$ we have $\partial\overline{\alpha}_0/\partial\Delta < 0$ for small Δ and $\partial\overline{\alpha}_0/\partial\Delta > 0$ for large Δ. Note that $\overline{\alpha}_0$ has an asymptote when $\pi'\overline{p}_0 = \pi\overline{q}_0$, i.e., at $\Delta_a = \ln\left(\pi'/\pi\right)/\left(\beta - \mu\right) > 0$, because $\pi' > \pi$ and $\beta > \mu$ always. Then, $\overline{\alpha}_0 \downarrow -\infty$ if $\Delta \uparrow \Delta_a$ while $\overline{\alpha}_0 \uparrow \infty$ if $\Delta \downarrow \Delta_a$. Now, take $\Delta \uparrow \infty$ and observe that $\overline{\alpha}_0 \uparrow \infty$. Because $\widetilde{\alpha}_0$ is continuous and differentiable in $\Delta \in (\Delta_a, \infty)$, then $\overline{\alpha}_0$ must be convex with a U-shape in (Δ_a, ∞) and have a minimum in this interval. If at this minimum $\overline{\alpha}_0 < 1$, then there are two values of $\Delta > \Delta_a$ such that $\overline{\alpha}_0 = 1$. Consequently, for a sufficiently small $\Delta > \Delta_a$, we have $\partial\overline{\alpha}_0/\partial\Delta < 0$, while for a sufficiently large Δ, we have $\partial\overline{\alpha}_0/\partial\Delta > 0$. Following Corollary 5 and while enforceability holds (Proposition 10), $\partial\overline{\alpha}_k/\partial\Delta > 0$ for $k > 0$. Note that if the minimum of $\overline{\alpha}_0$ in $\Delta \in (\Delta_a, \infty)$ is larger than one, then enforceability fails for all Δ.

Random "good" news model: the proof is similar to the deterministic case. Note that $\widetilde{\alpha}_0 = 0$ has three roots at $\Delta = 0$, $\Delta = -1/\left(r + \beta\right)$, and $\Delta = -1/\left(r + \mu\right)$. Consequently, $\widetilde{\alpha}_0 \neq 0$ for $\Delta \in (0, \infty)$. We want to show that at $\widetilde{\alpha}_0 = 1$ we have $\partial\widetilde{\alpha}_0/\partial\Delta < 0$ and $\partial\widetilde{\alpha}_0/\partial\Delta > 0$ for small and large Δ, respectively. Note that $\widetilde{\alpha}_0$ has an asymptote when $\pi'\widetilde{p}_0 = \pi\widetilde{q}_0$, i.e., at $\Delta_a = \left(\pi' - \pi\right)/\left(\pi\left(r + \beta\right) - \pi'\left(r + \mu\right)\right)$, where $\Delta_a > 0$ for $\pi\left(r + \beta\right) > \pi'\left(r + \mu\right)$. The more informative the signal structure is, the lower the monitoring intensity that enforces cooperation. Then, $\widetilde{\alpha}_0 \downarrow -\infty$ if $\Delta \uparrow \Delta_a$ while $\widetilde{\alpha}_0 \uparrow \infty$ if $\Delta \downarrow \Delta_a$. Now, take $\Delta \uparrow \infty$ and observe that $\widetilde{\alpha}_0 \uparrow \infty$, providing that $\pi\left(r + \beta\right) > \pi'\left(r + \mu\right)$. Because $\widetilde{\alpha}_0$ is continuous and differentiable in $\Delta \in (\Delta_a, \infty)$, then $\widetilde{\alpha}_0$ is convex U-shaped in (Δ_a, ∞) and has a minimum in this interval. If, at this minimum, $\widetilde{\alpha}_0 < 1$, then there must be two values of $\Delta > \Delta_a$ such that $\widetilde{\alpha}_0 = 1$. For a sufficiently small $\Delta > \Delta_a$, we have $\partial\widetilde{\alpha}_0/\partial\Delta < 0$, while for a sufficiently large Δ, we have $\partial\widetilde{\alpha}_0/\partial\Delta > 0$. In the latter case, following Corollary 5 and while enforceability holds (Proposition 10) $\partial\widetilde{\alpha}_k/\partial\Delta > 0$ for $k > 0$. If the minimum of $\widetilde{\alpha}_0$ in $\Delta \in (\Delta_a, \infty)$ is larger than one, then enforceability fails for all Δ. ■

Proof of Proposition 13. The difficulty here is that we have two equations and three unknowns, i.e., v, α_K and the associated optimal value of K, which depends on r. Corollary 8 tell us how K varies with r. Proposition 9 describes how v varies with r. Corollary 5 tells us how events are connected. Altogether, we know that more events improve payoffs.

Consider the "bad" news model. Note that $K \uparrow \infty$ implies $\sum_{k\in\Pi} p_k \uparrow 1$ and $\sum_{k\in\Pi} q_k \uparrow 1$, while $p_K \downarrow 0$ and $q_K \downarrow 0$. Recall that $\alpha_K = 0 \Longleftrightarrow \alpha_{K+1} = 1$. A transition to a higher K occurs when r decreases. Suppose that $r \downarrow 0$ implies $K \uparrow \infty$, then we must have $\alpha_K \downarrow 0$. In any case, because $\alpha_K \in (0, 1]$, in the limit, $\sum_{k=0}^{K-1} p_k$ and $\sum_{k=0}^{K-1} q_k$ are equivalent to $\sum_{k=0}^{K} p_k$ and $\sum_{k=0}^{K} q_k$, respectively. Then, in the limit, we can ignore the terms $p_K(\Delta)(1-\alpha_K)$ and $q_K(\Delta)(1-\overline{\alpha}_{\overline{K}})$ from l_K.

The adjusted likelihood ratio in the deterministic "bad" news model is given by

$$l'_{\overline{K}} \equiv \frac{1-\sum_{k=0}^{\overline{K}} q_k}{1-\sum_{k=0}^{\overline{K}} p_k} = \frac{1-\frac{\Gamma(\overline{K}+1,\mu\Delta)}{\Gamma(\overline{K}+1)}}{1-\frac{\Gamma(\overline{K}+1,\beta\Delta)}{\Gamma(\overline{K}+1)}},$$

Note that $\overline{K}$ takes discrete values; however, because $\sum_{k=0}^{\overline{K}} p_k$ and $\sum_{k=0}^{\overline{K}} q_k$ are expressed as regularized incomplete gamma functions with derivatives that converge suitably fast at infinity, we can compute the asymptotic expansion of the ratio of two gamma functions. Consequently, the first order expansion of the adjusted likelihood ratio about ∞ is given by[21]

$$\frac{e^{1+\ln\frac{1}{\overline{K}}+O\left(\frac{1}{\overline{K}}\right)^2}(\Delta\mu)^{\overline{K}}\left(e^{-\mu\Delta}\left(\frac{1}{\overline{K}}\right)^{3/2}+O\left(\frac{1}{\overline{K}}\right)^{5/2}\right)+O\left(\frac{1}{\overline{K}}\right)}{e^{1+\ln\frac{1}{\overline{K}}+O\left(\frac{1}{\overline{K}}\right)^2}(\Delta\beta)^{\overline{K}}\left(e^{-\beta\Delta}\left(\frac{1}{\overline{K}}\right)^{3/2}+O\left(\frac{1}{\overline{K}}\right)^{5/2}\right)+O\left(\frac{1}{\overline{K}}\right)}.$$

Then, for $\overline{K}\uparrow\infty$ and because $\mu>\beta$, we obtain $l_{\overline{K}}\uparrow\infty$. Note that the distinct elements are $(\mu/\beta)^{\overline{K}} e^{-(\mu-\beta)\Delta}$, which converge to ∞.

Consider now the random monitoring case. The adjusted likelihood ratio is

$$l'_{\widetilde{K}} \equiv \left(\frac{\Delta\mu}{1+\Delta r+\Delta\mu}\Big/\frac{\Delta\beta}{1+\Delta r+\Delta\beta}\right)^{\widetilde{K}+1}.$$

Because, $\partial(x/(1+x))/\partial x>0$ and $\mu>\beta$, the ratio inside brackets is larger than one and at the limit we have $l'_{\widetilde{K}}\uparrow\infty$.

Consider the "good" news model. Now, at the limit we have $K\downarrow 0$. We begin with the deterministic monitoring case. Before the limit $K\downarrow 0$, $\sum_{k\in\overline{\Pi}}\overline{p}_k(\Delta)=1-e^{-\beta\Delta}$ and $\sum_{k\in\overline{\Pi}}\overline{q}_k(\Delta)=1-e^{-\mu\Delta}$ stabilize. Replace $\overline{\alpha}_{\overline{K}}$ into $l_{\overline{K}}$ to obtain

$$l_{\overline{K}}=\frac{\frac{\Gamma(\overline{K},\mu\Delta)}{\Gamma(\overline{K})}-\frac{(\mu\Delta)^{\overline{K}}}{\overline{K}!}e^{-\mu\Delta}\frac{\pi\left(1-\delta^\Delta\left(1-\frac{\Gamma(\overline{K},\mu\Delta)}{\Gamma(\overline{K})}\right)\right)-\pi'\left(1-\delta^\Delta\left(1-\frac{\Gamma(\overline{K},\beta\Delta)}{\Gamma(\overline{K})}\right)\right)}{\delta^\Delta\left(\pi\frac{(\mu\Delta)^{\overline{K}}}{\overline{K}!}e^{-\mu\Delta}-\pi'\frac{(\beta\Delta)^{\overline{K}}}{\overline{K}!}e^{-\beta\Delta}\right)}}{\frac{\Gamma(\overline{K},\beta\Delta)}{\Gamma(\overline{K})}-\frac{(\beta\Delta)^{\overline{K}}}{\overline{K}!}e^{-\beta\Delta}\frac{\pi\left(1-\delta^\Delta\left(1-\frac{\Gamma(\overline{K},\mu\Delta)}{\Gamma(\overline{K})}\right)\right)-\pi'\left(1-\delta^\Delta\left(1-\frac{\Gamma(\overline{K},\beta\Delta)}{\Gamma(\overline{K})}\right)\right)}{\delta^\Delta\left(\pi\frac{(\mu\Delta)^{\overline{K}}}{\overline{K}!}e^{-\mu\Delta}-\pi'\frac{(\beta\Delta)^{\overline{K}}}{\overline{K}!}e^{-\beta\Delta}\right)}}.$$

Now, let $\overline{K}=0$; after some algebra, we obtain $e^{-\mu\Delta}/e^{-\beta\Delta}$. Consequently, while $\widetilde{\alpha}_0\in(0,1]$, $\widetilde{v}$ is constant and does not depend on r.

[21] This result is obtained using Mathematica®.

Consider now the random monitoring case. After substituting $\widetilde{\alpha}_{\widetilde{K}}$ into $l_{\widetilde{K}}$ we obtain

$$l_{\widetilde{K}}=\frac{1-\left(\frac{\mu\Delta}{1+r\Delta+\mu\Delta}\right)^{\widetilde{K}+1}-\frac{(1+r\Delta)(\mu\Delta)^{\widetilde{K}}}{(1+r\Delta+\mu\Delta)^{\widetilde{K}+1}}\frac{\pi\left(1-\frac{\left(\frac{\mu\Delta}{1+r\Delta+\mu\Delta}\right)^{\widetilde{K}+1}}{1+r\Delta}\right)-\pi'\left(1-\frac{\left(\frac{\beta\Delta}{1+r\Delta+\beta\Delta}\right)^{\widetilde{K}+1}}{1+r\Delta}\right)}{\frac{1}{1+r\Delta}\left(\pi\frac{(1+r\Delta)(\mu\Delta)^{\widetilde{K}}}{(1+r\Delta+\mu\Delta)^{\widetilde{K}+1}}-\pi'\frac{(1+r\Delta)(\beta\Delta)^{\widetilde{K}}}{(1+r\Delta+\beta\Delta)^{\widetilde{K}+1}}\right)}}{1-\left(\frac{\beta\Delta}{1+r\Delta+\beta\Delta}\right)^{\widetilde{K}+1}-\frac{(1+r\Delta)(\beta\Delta)^{\widetilde{K}}}{(1+r\Delta+\beta\Delta)^{\widetilde{K}+1}}\frac{\pi\left(1-\frac{\left(\frac{\mu\Delta}{1+r\Delta+\mu\Delta}\right)^{\widetilde{K}+1}}{1+r\Delta}\right)-\pi'\left(1-\frac{\left(\frac{\beta\Delta}{1+r\Delta+\beta\Delta}\right)^{\widetilde{K}+1}}{1+r\Delta}\right)}{\frac{1}{1+r\Delta}\left(\pi\frac{(1+r\Delta)(\mu\Delta)^{\widetilde{K}}}{(1+r\Delta+\mu\Delta)^{\widetilde{K}+1}}-\pi'\frac{(1+r\Delta)(\beta\Delta)^{\widetilde{K}}}{(1+r\Delta+\beta\Delta)^{\widetilde{K}+1}}\right)}}.$$

Let $\widetilde{K}=0$; after some algebra, we obtain

$$l_{\widetilde{K}}=\frac{1+r\Delta+\beta\Delta}{1+r\Delta+\mu\Delta}.$$

Finally, for $r=0$ the result is obtained. ■

Proof of Proposition 14. *"Bad" news deterministic* - We need to show that when $\Delta\downarrow 0$, we have $\overline{\alpha}_{\overline{K}}\uparrow\infty$ for the general $\overline{K}>1$, while $\overline{\alpha}_1\longrightarrow r\left(\pi'-\pi\right)/\mu\pi-\beta\pi'$ and $\overline{K}=0$ cannot be an equilibrium. Because by Proposition 11 and Corollary 12 $\overline{\alpha}_{\overline{K}+1}>\overline{\alpha}_{\overline{K}}$ for $\overline{K}\geq 1$, it is enough to show that $\overline{K}=2$ cannot be part of a $\overline{K}$ limit equilibrium. Expand the numerator and the denominator around zero to obtain

$$\overline{\alpha}_2=1-\frac{-\left(\pi'-\pi\right)r\Delta+O\left(\Delta\right)^2}{O\left(\Delta\right)^2}.$$

Because $O\left(\Delta\right)^2$ goes to zero faster than Δ, then $\overline{\alpha}_2\uparrow\infty$. Consequently, $\overline{\alpha}_2=1$ must occur (when it occurs) at some $\Delta>0$. Now, we will focus on $\overline{\alpha}_1$. A similar expansion for $\overline{\alpha}_1$ returns

$$\overline{\alpha}_1=1-\frac{\left(\pi\left(r+\mu\right)-\pi'\left(r+\beta\right)\right)\Delta+O\left(\Delta\right)^2}{\left(\pi\mu-\pi'\beta\right)\Delta+O\left(\Delta\right)^2}.$$

In the limit, we obtain $r\left(\pi'-\pi\right)/\left(\pi\mu-\pi'\beta\right)$. This number is in $(0,1]$ if $\pi\mu>\pi'\beta$ and $\pi\left(\mu+r\right)\geq\pi'\left(\beta+r\right)$, respectively. The latter implies the former. In addition, we have

$$\overline{\alpha}_0=1-\frac{\pi-\pi'}{e^{-r\Delta}\left(\pi e^{-\mu\Delta}-\pi' e^{-\beta\Delta}\right)},$$

where the denominator must be strictly negative for $\overline{\alpha}_0\in(0,1]$ contradicting Proposition 6. The limit trivially goes to zero. Moreover,

$$\frac{\partial\overline{\alpha}_0}{\partial\Delta}=\left(\pi'-\pi\right)\frac{\pi e^{-\mu\Delta}\left(r+\mu\right)-\pi' e^{-\beta\Delta}\left(r+\beta\right)}{e^{-r\Delta}\left(\pi e^{-\mu\Delta}-\pi' e^{-\beta\Delta}\right)^2},$$

and the limit $\Delta\downarrow 0$ is equal to $\left(\pi\left(r+\mu\right)-\pi'\left(r+\beta\right)\right)/\left(\pi'-\pi\right)$, which is non-negative because $\pi\left(\mu+r\right)\geq\pi'\left(\beta+r\right)$. Consequently, in the limit, we must have $\overline{K} > 0$ and $\overline{K} < 2$, i.e., $\overline{K} = 1$. Then, for $\pi(\mu + r) \geq \pi'(\beta + r)$ and $\overline{K} = 1$, to take the limit of (14), i.e.,

$$\overline{v} = \frac{\left(1 - e^{-r\Delta}\right)\left(\pi e^{-\mu\Delta}\mu\Delta - \pi' e^{-\beta\Delta}\beta\Delta\right)}{\left(1 - e^{-r\Delta}e^{-\beta\Delta}\right)e^{-\mu\Delta}\mu\Delta - \left(1 - e^{-r\Delta}e^{-\mu\Delta}\right)e^{-\beta\Delta}\beta\Delta},$$

we expand it around zero to obtain

$$\overline{v} \approx \frac{\pi\mu - \pi'\beta}{\mu - \beta} + O(\Delta),$$

and the result is obtained. Moreover, the limit of

$$\lim_{\Delta\downarrow 0}\frac{\partial\overline{v}}{\partial\Delta} = \beta\mu\frac{\pi(2r + \mu) - \pi'(2r + \beta)}{2r(\mu - \beta)},$$

is positive if $\pi(2r + \mu) > \pi'(2r + \beta)$ and negative otherwise. In the former case, $\overline{v}$ converges from above, otherwise it converges from below, as in (12).

"Bad" news random - Similarly for a small Δ, $\widetilde{K} = 0$ cannot enforce cooperation, and limit payoffs above the static Nash are possible if a

$$\widetilde{\alpha}_1 = 1 - \frac{\pi(r + \mu)(1 + r\Delta + \beta\Delta) - \pi'(r + \beta)(1 + r\Delta + \mu\Delta)}{\pi\mu\frac{1 + r\Delta + \beta\Delta}{1 + r\Delta + \mu\Delta} - \pi'\beta\frac{(1 + r\Delta + \mu\Delta)}{(1 + r\Delta + \beta\Delta)}}$$

exists belonging to $(0, 1]$. The limit of $\widetilde{\alpha}_1$ is equal to $r(\pi' - \pi)/(\mu\pi - \beta\pi') \in (0, 1]$ because $\pi(\mu + r) \geq \pi'(\beta + r)$. Notice that for $r > 0, \partial((r + \beta)/(r + \mu))/\partial r > 0$. Consequently, for any parameterization satisfying these conditions, for small Δ, in equilibrium $\widetilde{K} = 1$ and $\widetilde{v}$ converge to $(\mu\pi - \beta\pi')/(\mu - \beta)$. The argument is just as explained above, an expansion around $\Delta = 0$. Note that

$$\lim_{\Delta\downarrow 0}\frac{\partial\widetilde{v}}{\partial\Delta} = \beta\mu\frac{\pi(2r + \mu) - \pi'(2r + \beta)}{r(\mu - \beta)},$$

which is positive if $\pi(2r + \mu) > \pi'(2r + \beta)$ and negative otherwise. Then

$$\begin{cases} \uparrow \frac{\mu\pi - \beta\pi'}{\mu - \beta} & \frac{2r + \beta}{2r + \mu} > \frac{\pi}{\pi'} \geq \frac{r + \beta}{r + \mu} \\ \downarrow \frac{\mu\pi - \beta\pi'}{\mu - \beta} & \frac{\pi}{\pi'} \geq \frac{2r + \beta}{2r + \mu} \geq \frac{r + \beta}{r + \mu} \end{cases},$$

which is equal to (12).

Otherwise, i.e., for $\pi(\mu + r) < \pi'(\beta + r)$, we have in the limit $\alpha_1 > 1$. Consequently, for any parameterization satisfying this condition, for a small Δ, in equilibrium $v \downarrow 0$. Moreover, there might not exist a real $\Delta_2 > 0$ that solves $\alpha_2 = 1$, below which the equilibrium degenerates. In this case we cannot enforce cooperation for any Δ.

"Good" news deterministic - We know from Proposition (11) that for a small Δ we must have $K = 0$. To take the limit of

$$\overline{v} = \frac{\left(1 - e^{-r\Delta}\right)\left(\pi e^{-\mu\Delta} - \pi' e^{-\beta\Delta}\right)}{\left(1 - e^{-r\Delta}\left(1 - e^{-\beta\Delta}\right)\right)e^{-\mu\Delta} - \left(1 - e^{-r\Delta}\left(1 - e^{-\mu\Delta}\right)\right)e^{-\beta\Delta}},$$

at $\overline{K}=0$, we expand around zero to obtain

$$\overline{v}\approx -\frac{\pi'-\pi}{(\beta-\mu)\,\Delta}+\frac{\pi'+\pi}{2}+O\left(\Delta\right),$$

which trivially goes to $-\infty$.

"Good" news random - Similarly, take the limit of

$$\widetilde{v}=\frac{\pi\left(1+r\Delta+\beta\Delta\right)-\pi'\left(1+r\Delta+\mu\Delta\right)}{\Delta\left(\beta-\mu\right)},$$

at $\widetilde{K}=0$ to trivially obtain $-\infty$ since $\pi<\pi'$. Consequently, degeneracy occurs even before the limit. ■

Proof of Proposition 15. "Bad" news model: to take the limit of the ratio $\widetilde{v}/\overline{v}$ for $K=1$, we expand it around $\Delta=0$ to obtain

$$\frac{\widetilde{v}}{\overline{v}}\approx 1+\beta\mu\Delta\frac{\pi\left(2r+\mu\right)-\pi'\left(2r+\beta\right)}{2r\left(\mu\pi-\beta\pi'\right)}+O\left(\Delta\right)^{2}.$$

In the limit, this quantity is larger than one if $\pi\left(2r+\mu\right)>\pi'\left(2r+\beta\right)$ and smaller otherwise.

"Good" news model: to take the limit of the ratio $\widetilde{v}/\overline{v}$ for $K=1$, we expand it around $\Delta=0$ to obtain

$$\frac{\widetilde{v}}{\overline{v}}\approx r\Delta^{2}\frac{\beta-\mu}{\pi'-\pi}+O\left(\Delta\right)^{3},$$

which trivially goes to 0. Assuming that $\widetilde{v}$ and $\overline{v}$ do not cross, the result is obtained. ■

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