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The Lottery Blotto Game

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The Lottery Blotto Game[*]

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Abstract

In this paper we relax the Colonel Blotto game assumption that for a given battle the player who allocates the higher measure of resources wins that battle. We assume that for a given battle, the Colonel who allocates the higher measure of resources is more likely to win that battle. We have a simpler model for which we are able to compute all Nash equilibria in pure strategies for any valuations profile that players might have. Something that is not possible for the original Blotto game.

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KEYWORDS: Colonel Blotto game; lottery contest function.

1 Introduction

In the Colonel Blotto game two players must allocate a fixed amount of resources over n battles, see Borel (1921) and Gross and Wagner (1950). For a given battle $j \in \{1, ..., n\}$, the player $i \in \{A, B\}$ who allocates the higher measure of resources $x_{i,j}$, wins that battle. In this paper we assume that for a given battle, the Colonel who allocates the higher measure of resources is more likely to win that battle.

In practical terms the allocation of more resources in a given issue does not always guarantee a sure win. Exogenous factors and stochastic shocks may force a different outcome, as in electoral competitions, conflict resolution or even in a battlefield. However, the positive relation between resources allocation and winning likelihood cannot be broken. A natural candidate to solve this lottery is the contest success function $p_{i,j} = x_{i,j}/ \left(x_{i,j} + x_{-i,j} \right)$.

With this assumption we have a simpler model for which we are able to compute all Nash equilibria in pure strategies for any valuations profile that players might have. Something that is not possible for the original Blotto game,

except for a set of particular cases for which we know the solution: symmetric valuations, see Gross and Wagner (1950), symmetric valuations and asymmetric endowments, see Roberson (2006), and the case where $n = 2$, see Macdonell and Mastronardi (2010).[1] We allow for valuations heterogeneity within the same battle, e.g., $v_{i,j} \neq v_{-i,j}$ for all j. An open questions in the original Blotto game, which can be easily accommodated in the present model. In reality two different individual may value the same object differently.

The drawback of the proposed formulation is that solutions do not have a tractable close form. However, such does not presents as a major limitation. In the context of applied work, it is preferred an approximated model that deliver correct predictions than no model at all. This is the main reason of the present paper.

In this paper we formalize the lottery approach to the Colonel Blotto game and discuss some differences with respect to the original game.

2   Model and Results

Let $v_{i,j} \in (0, \infty)$ be player's i valuation for the battle j, and $x_{i,j}$ the amount of resources allocated by player i to the battle j. Player $i \in \{A, B\}$ objective is to maximize the expected payoff $\pi_i = \sum_{j=1}^{n} p_{i,j} v_{i,j}$, with respect to $\{x_{i,j}\}_{j=1}^{n}$, and subject to resources constraint $\sum_{j=1}^{n} x_{i,j} = R_i$, where $R_i \in (0, \infty)$ is the resources endowment of individual i. The resulting $2n + 2$ first order conditions are

$$x_{-i,j} v_{i,j} / (x_{i,j} + x_{-i,j})^2 = \lambda_i,$$

and $\sum_{j=1}^{n} x_{i,j} = R_i$ for all i, j. The following result presents the set of equations that are necessary for the Nash solution.

Proposition 1 *Let $n \geq 2$, the real valued set $\left\{x_{A,j}^*, x_{B,j}^*\right\}_{j=1}^{n}$ that solves the system of $2n$ equations*

$$x_{i,j}^* = p_{i,j}^* p_{-i,j}^* v_{i,j} R_i / \sum_{j=1}^{n} p_{i,j}^* p_{-i,j}^* v_{i,j}, \tag{1}$$

for all i, j, is a Nash equilibrium.[2]

With the exception of some particular symmetric cases, the solution $x_{i,j}^*$ is a $2n + 1$ degree polynomial, with $2n - 1$ different roots, which have to be computed numerically. Not always we have a unique real valued solution, in some cases we might have more than one. For the valuations profile in the following example we have three Nash equilibria.

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[1] See Kovenock and Roberson (2012) and Roberson (2011) for a survey and multiple variations of the original game that have been presented in the literature.

[2] In alternative to (1) we can fix a battle, say $j = 1$, and write the $2(n-1)$ first order conditions $x_{-i,1} v_{i,1} / (x_{i,1} + x_{-i,1})^2 = x_{-i,j} v_{i,j} / (x_{i,j} + x_{-i,j})^2$, for all $i, j \neq 1$, plus the resources constraints.

Example 2 (Non-Uniqueness) *Let $R_A = 2$ and $R_B = 1$, and consider the following valuations profile*

$v_{A,1}$	$v_{A,2}$	$v_{A,3}$		$v_{B,1}$	$v_{B,2}$	$v_{B,3}$
0.950	0.030	0.020		0.150	0.100	0.750

In this case we have three Nash equilibria that vary in function of the protection that each player distribute among the battles

$x_{A,1}$	$x_{A,2}$	$x_{A,3}$	$\boldsymbol{\pi}_A$	$x_{B,1}$	$x_{B,2}$	$x_{B,3}$	$\boldsymbol{\pi}_B$
1.720	0.241	0.039	**0.909**	0.107	0.317	0.576	**0.768**
1.943	0.053	0.004	**0.768**	0.478	0.272	0.250	**0.851**
0.759	0.420	0.821	**0.982**	0.004	0.043	0.953	**0.413**

Surprisingly, in the second equilibrium the player with less resources obtains a larger expected payoff, typically this is not the case.

The example also shows how the players' payoffs depend on the way resources are allocated.

Note that equilibrium uniqueness in the original Blotto game has been shown by Roberson (2006) for the symmetric case. However, we do not know if such result holds true for asymmetric valuation profiles like the one in the previous example, it is still an open question.

The payoff function does not have an simple expression, however it is continuous and differentiable both in the valuations and endowments. Something that is not the case in the original Blotto game, where the payoff function suffer discontinuities for some degrees of asymmetry in the endowments, see Roberson (2006) and Macdonell and Mastronardi (2010).

To make clear the difference with the Blotto game, consider the following example.

Example 3 (Difference with the original Blotto game) *Let $R_A = R_B = 1$ and consider the following valuations profile*

$v_{A,1}$	$v_{A,2}$	$v_{A,3}$		$v_{B,1}$	$v_{B,2}$	$v_{B,3}$
0.950	0.030	0.020		0.550	0.300	0.150

In the Nash equilibrium we have

$x_{A,1}$	$x_{A,2}$	$x_{A,3}$	$\boldsymbol{\pi}_A$	$x_{B,1}$	$x_{B,2}$	$x_{B,3}$	$\boldsymbol{\pi}_B$
0.986	0.008	0.006	**0.522**	0.820	0.110	0.070	**0.668**

Note the difference between the equilibrium obtained where all battles receive a measurable amount of resources and the standard Blotto game equilibrium, which is known for this case because $v_{i,1} > \sum_{j\neq1}^{n} v_{i,j}$ for all i. In such equilibrium both players allocate all their forces to battle $j = 1$. Payoffs are equal to $1/2$ for both players, strictly less than the expected ones obtained here.

For equal endowed players, note the relatively large payoff difference in favour of the individual B whose preferences are more homogeneous than individual those of A.

In the original Blotto game with asymmetric endowments, the weak player may send zero forces to a battle, here all fronts receive a positive and deterministic measure of resources, providing that valuations are bounded.

Our formulation is general and covers the existent results in the literature. Consider the following special case of Proposition 1 due to Friedman (1958), see also Robson (2005). If $v_{i,j} = v_{-i,j}$ for all j, then players' winning probabilities become proportional to their resources, nothing else is relevant, in other words $p_{i,j}^* = R_i/\left(R_i + R_{-i}\right)$ and $p_{-i,j}^* = R_{-i}/\left(R_{-i} + R_i\right)$ for all j. After canceling the terms in (1), we have the following result.

Corollary 4 *Let $n \geq 2$ and $v_{i,j} = v_{-i,j}$ for all j, in the Nash equilibrium*

$$x_{i,j}^* = v_{i,j} R_i / \sum_{j=1}^{n} v_{i,j},$$

for all i, j.

Another difference with respect to the Blotto game is that with positive probability a player can win all battles even if he has a lower endowment. In the original Blotto game equally endowed players can never win in all fronts.

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