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From bargaining solutions to claims rules: *a proportional approach*

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Abstract

Is it important to negotiate on proportions rather than on numbers? To answer this question, we analyze the behavior of well-known bargaining solutions and the claims rules they induce when they are applied to a "proportionally transformed" bargaining set $S^{\mathcal{P}}$– so-called *bargaining-in-proportions set*. The idea of applying bargaining solutions to claims problems was already developed in Dagan and Volij (1993). They apply the bargaining solutions over a bargaining set that is the one defined by the claims and the endowment. A comparison among our results and theirs is provided.

Keywords: Bargaining problem, Claims problem, Proportional, Constrained Equal Awards, Constrained Equal Losses, Nash bargaining solution
JEL classification: C71, D63, D71.

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1. Introduction

In solving distribution problems, in which involved agents have some rights (or claims), "the proportional solution is the most widely used" (Chun (1988)). In the same vein, Young (1994) stated that "in western society, for example, the customary solution would be to split the asset in proportion to the claims." So people, when faced with a problem to share a certain amount, taking into account their claims, are more concerned with the proportion of their claim they receive than in the absolute amount. This is so mainly because a proportional sharing allows them to compare the treatment that each one receives. And the proportional equal treatment is, in our opinion, the best way to express the notion of *fairness*. Under this approach (focusing in the proportion of the claim one receives) people discuss about proportions of the claims that are satisfied: *equal proportions, proportional proportions,* ...

An interesting interpretation of proportionality, when analyzing claims problems, can be found in the following text (Malkevitch, J. (2012), Resolving Bankruptcy Claims, Feature Column from the AMS):

> "A few years ago I developed what appears to be a new viewpoint which leads to the *proportional solution*. Since the amount E is not enough to pay off the bankruptcy, one might adopt the following point of view: Instead of giving the claimants less than they are entitled to now, one can postpone paying them off and wait until the available money E grows, by investing it at the current interest rate until the invested amount plus interest totals the amount being claimed. The judge at this future point in time would pay off each claimant his/her full amount. Using the well-known accounting principle of computing the present value of this future asset we can see what amount of money this approach would yield each claimant today. If one does the algebra involved, one sees that the solution is the same as the proportional solution."

To illustrate our idea, let us consider a typical claims problem (E, c) with two individuals, defined by the endowment, $E = 100$, and respective claims, $c = (80, 120)$. The possible (efficient) agreements are the Pareto boundary points in the set S defined by equations (see Figure 1):

$$x_1 + x_2 \leq E = 100; \quad 0 \leq x_1 \leq c_1 = 80; \quad 0 \leq x_2 \leq c_2 = 120.$$

To solve this claims problem we may apply any of the many proposed rules in the literature (*proportional* rule, P, *constrained equal awards*, CEA, *constrained equal losses*, CEL, *adjusted proportional*, AP, ...; see Thomson (2003) for a survey). For instance, in this example we obtain:

$$P(100, (80, 120)) = (40, 60), \quad CEA(100, (80, 120)) = (50, 50),$$

$$CEL(100,(80,120)) = (30,70), \quad AP(100,(80,120)) = (40,60), \ \ldots$$

An alternative approach is to consider the claims situation as a (*particular*) bargaining problem and apply the usual solutions in this context (*Nash*, *Kalai-Smorodinsky*, *Egalitarian* , *Equal Losses*, ...; see Thomson (1994) for a survey). This is the idea developed in Dagan and Volij (1993) where "each claims (bankruptcy) problem is associated with a bargaining problem and old allocation rules are derived for the former by applying well known bargaining solutions to the latter." Some important results they obtained are:

- *The Nash bargaining solution induces the constrained equal awards rule.*
- *The α-asymmetric Nash bargaining solution (for $\alpha = c$) induces the proportional rule.*
- *The Kalai-Smorodinsky bargaining solution induces the E-truncated proportional rule.*
- *The Kalai-Smorodinsky bargaining solution when the minimal rights are taken as the disagreement point induces the adjusted proportional rule.*

We try to do something similar, but considering an alternative model of negotiation, namely a *bargaining-in-proportions approach*: the individuals are concerned about the proportion of their claims they receive. Then, the first step will be to "transform" the claims problem by writing it in terms of proportions: let us call p_i the proportion of the claim c_i that individual i gets. Then, in the previous example, the possible efficient solutions of the claims problem (E, c) are now represented by the Pareto boundary points in the set $S^{\mathcal{P}}$ defined by equations (see Figure 1):

$$p_1c_1 + p_2c_2 \leq 100; \quad 0 \leq p_1 \leq 1; \quad 0 \leq p_2 \leq 1,$$

which for the claims problem $(E, c) = (100, (80, 120))$ is

$$80p_1 + 120p_2 \leq 100; \quad 0 \leq p_1 \leq 1; \quad 0 \leq p_2 \leq 1.$$

Then, if we apply the usual bargaining solutions to this "proportionally transformed" bargaining set, what result we obtain? This is the question we address in the present work. The first thing we need to mention is that the application of bargaining solutions to the set S or to the set $S^{\mathcal{P}}$ provides

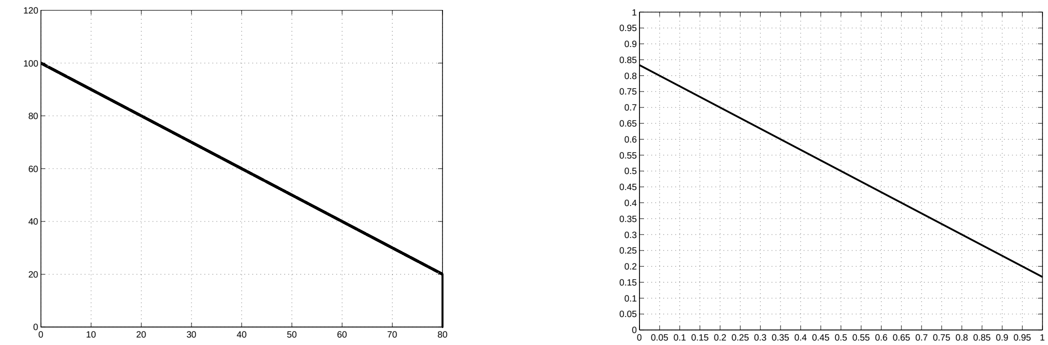

Figure 1: **A claims problem** ($E = 100$, $c = (80, 120)$) and its associated ***bargaining-in-proportions*** **problem**. The main difference is the gradient of the Pareto boundary, which is -1 in the claims problem, and $-8/12$ in the associated *bargaining-in-proportions* problem.

different results, both the amount each agent receives and the meaning of the solution. For instance, if we apply the *Egalitarian* (Kalai, 1977) bargaining solution to the problem S, the obtained result is $x_1 = 50$, $x_2 = 50$, that corresponds to the result given by the CEA rule (which equalizes the amount each individual receives). However, if we apply the same *Egalitarian* solution to the problem $S^{\mathcal{P}}$, we are equalizing proportions and then we obtain the result, $p_1 = p_2 = 0.5$, that induces the *proportional* division of the endowment, relative to claims: $x_1 = 40$, $x_2 = 60$, that is, the *proportional claims rule.*

This fact is (obviously) always true: *the Egalitarian bargaining solution applied to $S^{\mathcal{P}}$ induces the proportional rule in the claims problem.* What if we apply the *Equal Loss* bargaining solution to the problem $S^{\mathcal{P}}$? The idea behind this solution concept (when applied to our particular setting) is clear: now each individual losses the same proportion from its "maximum aspirations" (utopia point, see Section 2). But, as we will see, the induced rule for claims problems does not fulfill one of the basic established requirements: *the biggest the claim, the biggest the amount you receive.* Some other bargaining solutions have no clear intuition. For instance, what does mean the application of the *Kalai-Smorodinsky* (Kalai and Smorodinsky, 1975) solution? In this case, we are applying proportionality on a "proportionally transformed" bargaining set. In order to understand the meaning of this solution concept we need to observe its behavior and obtain the induced rule in claims problems. What about the Nash bargaining solution? We will see that, not surprisingly, in this case the result obtained in Dagan and Volij (1993) remains true and the induced claims rule is the *constrained equal awards.* We also add some new results.

To answer the above mentioned questions, throughout the paper we analyze what happens with some usual bargaining solutions when applied to the problem $S^{\mathcal{P}}$, and observe the induced rule in the original claims problem. Section 2 contains the main definitions on claims and bargaining problems. In Section 3 we define properly our model and obtain the main results about the correspondence between claims problem solutions and bargaining solutions applied to problem $S^{\mathcal{P}}$. Some final comments in Section 4 comparing our results and those in Dagan and Volij (1993) close the paper.

2. Preliminaries

In this section we present the main notions in both claims and bargaining problems, as well as the solution concepts we will use.

2.1. ***Claims problems and claims rules***

Throughout the paper we consider a set of individuals $N = \{1, 2, ..., n\}$. Each individual is identified by her *claim*, c_i, $i \in N$, on the *endowment* E. A ***claims problem*** appears whenever the endowment is not enough to satisfy all the claims; that is, $\sum_{i=1}^{n} c_i > E$. Without loss of generality, we will order the agents accordingly to their claims: $c_1 \leq c_2 \leq \ldots \leq c_n$. The pair (E, c) represents the claims problem, and we will denote by $\mathcal{B}$ the set of all claims problems. A ***claims rule*** is a single valued function $\varphi : \mathcal{B} \to \mathbb{R}^n_+$ such that: $0 \leq \varphi_i(E, c) \leq c_i, \forall i \in N$ (***non-negativity*** and ***claim-boundedness***); and $\sum_{i=1}^{n} \varphi_i(E, c) = E$ (***efficiency***).

We introduce some of the most important rules in claims problems. The interested reader is referred to the survey by Thomson (2003). The first rule recommends a distribution of the endowment which is proportional to the claims.

Proportional: for each $(E, c) \in \mathcal{B}$ and each $i \in N$, $P_i(E, c) \equiv \lambda c_i$, where $\lambda = \dfrac{E}{\sum_{i \in N} c_i}$.

The *Constrained Equal Awards* rule (Maimonides, 12th century) recommends equal awards to all agents subject to no one receiving more than his claim.

Constrained Equal Awards: for each $(E, c) \in \mathcal{B}$ and each $i \in N$, $CEA_i(E, c) \equiv \min\{c_i, \mu\}$, where μ is such that $\sum_{i \in N} \min\{c_i, \mu\} = E$.

The following rule, discussed by Maimonides (Aumann and Maschler (1985)) chooses the awards vector at which all agents incur equal losses, subject to no one receiving a negative amount.

Constrained Equal Losses: for each $(E, c) \in \mathcal{B}$ and each $i \in N$, $CEL_i(E, c) \equiv \max\{0, c_i - \mu\}$, where μ is such that $\sum_{i \in N} \max\{0, c_i - \mu\} = E$.

Hereinafter, let $c_i^T = \min\{c_i, E\}$, and $\boldsymbol{v} = (v_1, v_1, \ldots, v_n)$ denote the *minimal rights vector*: for each $i \in N$, $v_i = max\{0, E - \sum_{k \neq i} c_k\}$.

Truncated (by the endowment) rules: given a rule φ, the associated *truncated* claims rule φ^T is defined by $\varphi^T(E, c) \equiv \varphi(E, c^T)$.

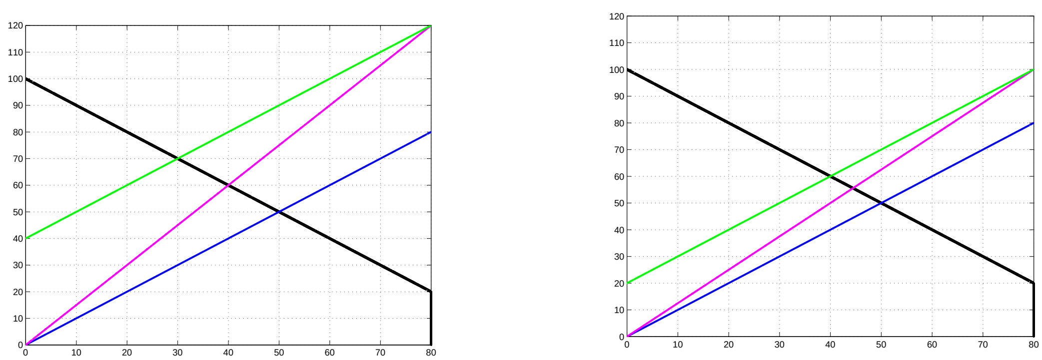

Figure 2: **Claims rules** (CEA (blue), CEL (green), P(pink), on the left hand side) and **truncated claims rules** (CEA (blue), CEL^T (green) and P^T (pink), on the right hand side). Note that CEA is invariant to claims truncation.

The *adjusted proportional* rule (Curiel et al. (1987)), is defined in the following way:

Adjusted proportional: for each $(E, c) \in \mathcal{B}$ and each $i \in N$, $AP_i(E, c) = v_i + \left(E - \sum_{j=1}^n c_j\right) \frac{c_i^T - v_i}{\sum_{j=1}^n (c_j^T - v_j)}$.

The *adjusted proportional* rule is a generalization of the *contested garment principle* (Babylonian Talmud), which is defined for claims problems involving just two people.

Contested Garment rule: for each $(E, c) \in \mathcal{B}$, $c \in \mathbb{R}^2$, $CG_i(E, c) = \left(\frac{E + c_1^T - c_2^T}{2}, \frac{E + c_2^T - c_1^T}{2}\right)$.

2.2. ***Bargaining problems and bargaining solutions***

We consider bargaining problems (S, d), such that $S \subseteq \mathbb{R}^n_+$ is a convex and comprehensive set.[1] Furthermore, since we analyze claims problems, throughout our approach we assume the disagreement point at $\mathbf{0}$, so that, $(S, \mathbf{0})$. Given a bargaining problem its Pareto boundary is defined by

$$\partial_P(S) = \{x^* \in S : y_i > x_i \quad \forall i \quad \Rightarrow y \notin S\}.$$

The ***ideal*** point of a bargaining problem $(S, \mathbf{0})$ represents the maximum amount that each agent can achieve in such a problem. Formally, this point $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in S$ is defined so that for each $i \in N$, a_i is the maximum in S of function $\pi_i : \mathbb{R}^n \to \mathbb{R}$, $\pi_i(x) = x_i$.

Now, we introduce some of the main bargaining solutions. The first one (Nash, 1950) selects the point maximizing the product of utility gains from the disagreement point in the Pareto boundary of S.

Nash bargaining solution: $N(S, \mathbf{0})$ is the point that maximizes in $\partial_P(S)$ the function $u(x) = \prod_{i=1}^n x_i$.

The next solution (Kalai, 1977) selects the maximal point of S at which all agents' utility gains are equal.

Egalitarian bargaining solution: $E(S, \mathbf{0})$ is the point in $\partial_P(S)$ that intersects the line throughout $\mathbf{0}$ with gradient 1.

The following solution (Kalai and Smorodinsky, 1975) selects the point in $\partial_P(S)$ at which the agents' gains are proportional to their *ideal* situation.

Kalai-Smorodinsky bargaining solution: $KS(S, \mathbf{0})$ is the point in $\partial_P(S)$ that intersects the line throughout $\boldsymbol{a}$ and $\mathbf{0}$.

[1] S is comprehensive in $\mathbb{R}^n_+$ if $x \in S$, $0 \le y \le x$, implies $y \in S$.

The Equal Loss solution (Chun, 1988) selects the maximal feasible point at which the losses from the ideal point of all agents are equal.

Equal Loss bargaining solution: $EL(S, \mathbf{0})$ is the point in $\partial_P(S)$ that intersects the line throughout $\boldsymbol{a}$ with gradient 1.

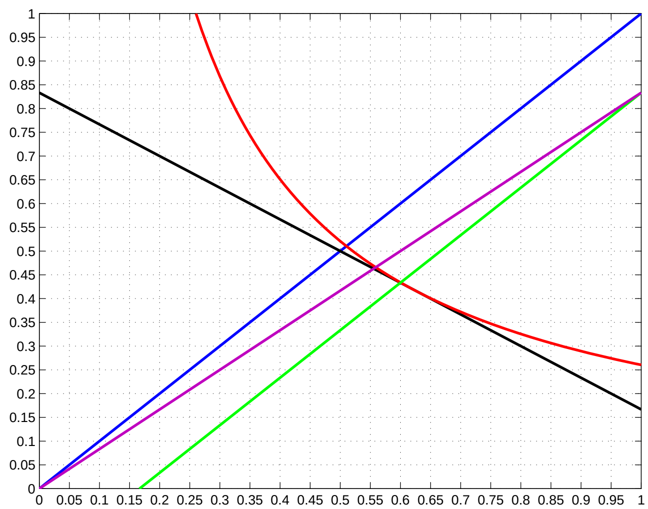

Figure 3: **Main bargaining solutions:** N (red), KS (purple), E (blue) and EL (green).

Finally, we use the next extended notions of the Nash bargaining solution.

Nash α-asymmetric bargaining solution, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ (Harsányi and Selten (1972) and Roth (1979)): $AN^{\alpha}(S, \mathbf{0})$ is the point that maximizes in $\partial_P(S)$ the function $u(x) = \prod_{i=1}^{n}(x_i)^{\alpha_i}$.

Given a point $\boldsymbol{r} = (r_1, r_2, \ldots, r_n)$ such that $\boldsymbol{r} \geq x$, for each $x \in$, ***Nash from a reference point*** bargaining solution, $N^r(S, \mathbf{0})$, is the point that maximizes in $\partial_P(S)$ the function $u(x) = \prod_{i=1}^{n}(r_i - x_i)$.

3. Correspondence between bargaining solutions and claims rules

3.1. ***The model***

The *bargaining-in-proportions* problem $(S^{\mathcal{P}}, \mathbf{0})$ associated to a claims problem (E, c) is defined by considering the part (*proportion*) of the claim that each agent is willing to disclaim. So, if we name p_i the proportion of her claim that individual i receives, the feasible claims set can be written as:

$$S^{\mathcal{P}} = \{\boldsymbol{p} = (p_1, p_2, \ldots, p_n) : p_i \in [0,1], \sum_{i=1}^{n} c_i p_i = E\}.$$

In this framework, we call ***utopia*** point to the *ideal point* in the transformed problem: $\boldsymbol{u} = (u_1, u_2, \ldots, u_n) \in S^{\mathcal{P}}$ such that for each $i \in N$, u_i is the maximum in $S^{\mathcal{P}}$ of function $\pi_i : \mathbb{R}^n \to \mathbb{R}$, $\pi_i(x) = x_i$, i.e., $u_i = \min\{1, E/c_i\}$. Furthermore, the ***maximum*** point[2] corresponds to the unitary vector that represents the maximum proportion of her claim that an individual may expect to obtain before knowing the precise endowment E, that is, $\boldsymbol{1} = (1, 1, \ldots, 1) \in \mathbb{R}^n$.

3.2. ***An illustration***

Consider the three person claims problem

$$(E, c) = (100, (20, 70, 110)).$$

We first compute the aforementioned claims rules in this example:

- $CEA(E, c) = (20, 40, 40)$.
- $CEL(E, c) = (0, 30, 70)$.
- $P(E, c) = (10, 35, 55)$.
- $CEL^T(E, c) = CEL(100, (20, 70, 100)) = (0, 35, 65)$.
- $P^T(E, c) = P(100, (20, 70, 100)) = (200/19, 700/19, 1000/19)$.
- $AP(E, c) = (10, 35, 55)$.

[2] It must be noticed that this point corresponds to the claims vector in the original claims problem (E, c), divided by the claim.

Now we obtain the associated *bargaining-in-proportions* set which is defined by

$$S^{\mathcal{P}} = \{(p_1, p_2, p_3) \in \mathbb{R}^3 : p_i \in [0,1],\ 20p_1 + 70p_2 + 110p_3 = 100\}.$$

The *utopia point* of this problem is $(1, 1, 10/11)$. Now, if we compute the bargaining solutions in $(S^{\mathcal{P}}, \mathbf{0})$ we obtain:

- $N(S^{\mathcal{P}}, \mathbf{0}) = (1, 4/7, 4/11)$ that induces the proposal $(20, 40, 40)$.
- $E(S^{\mathcal{P}}, \mathbf{0}) = (1/3, 1/3, 1/3)$ that induces the proposal $(10, 35, 55)$.
- $KS(S^{\mathcal{P}}, \mathbf{0}) = (110/209, 110/209, 100/209)$ that induces the proposal $(200/19, 700/19, 1000/19)$.
- $EL(S^{\mathcal{P}}, \mathbf{0}) = (121/220, 121/220, 101/220)$ that induces the proposal $(11, 77/2, 101/2)$.
- $AN^c(S^{\mathcal{P}}, \mathbf{0}) = (1/3, 1/3, 1/3)$ that induces the proposal $(10, 35, 55)$.
- $N^1(S^{\mathcal{P}}, \mathbf{0}) = (0, 3/7, 7/11)$ that induces the proposal $(0, 30, 70)$.
- $N^u(S^{\mathcal{P}}, \mathbf{0}) = (0, 1/2, 13/22)$ that induces the proposal $(0, 35, 65)$.

Finally, it is important to note two facts. On the one hand, there are some correspondences between both groups of solutions in this example (the obvious question to be analyzed is if this is true in general):

1. The result of the Nash bargaining solution corresponds with the one given by the CEA rule.
2. Both the Nash asymmetric (with $\alpha = c$) and the Egalitarian bargaining solutions induce a result that corresponds with the Proportional and Adjusted Proportional rules.
3. The result obtained with the Nash bargaining solution from point *1* corresponds with the one provided by the CEL rule.
4. The Nash bargaining solution from point u gives a result that corresponds with the one given by the truncated CEL rule.
5. The result obtained with the Kalai-Smorodinsky bargaining solution corresponds with the one given by truncated proportional rule, P^T.

On the other hand, it must be noticed that the result obtained in applying the Equal Loss bargaining solution is somewhat strange. At first sight, it does not correspond with any of the rules we have introduced (neither any of the known claims rules). In the rest of this section we analyze formally these correspondences and the case of the Equal Loss bargaining solution.

3.3. ***Nash bargaining solution***

Our first result states that the Nash bargaining solution $N(S^{\mathcal{P}}, \mathbf{0})$ corresponds with the Constrained Equal Awards rule.

Proposition 1. *The Nash bargaining solution applied to* $(S^{\mathcal{P}}, \mathbf{0})$ *induces* $CEA(E, c)$.

Proof. Let (E, c) be a claims problem and $(S^{\mathcal{P}}, \mathbf{0})$ its associated *bargaining-in-proportions* problem. We proceed by rounds until all p_i come lower than the unit.

First round: The associated Nash bargaining Lagrangian is

$$L = \prod_{j=1}^{n} p_j + \lambda \left(E - \sum_{j=1}^{n} p_j c_j \right),$$

where we do not impose $0 \leq p_j \leq 1$. The first order conditions are

$$\frac{\partial L}{\partial p_i} = \prod_{j=1}^{n} p_j / p_i - \lambda c_i,$$

for each $i \in N$, and

$$\frac{\partial L}{\partial \lambda} = E - \sum_{j=1}^{n} p_j c_j.$$

Then, after some algebra, we obtain $p_i c_i = p_j c_j$ for each $j \neq i$. Consequently, after replacing in the endowment constrain, we obtain

$$n p_i c_i = E \text{ and } p_i = \frac{E}{n c_i} \quad \forall i.$$

If for each i we have $p_i \leq 1$, we stop and

$$x_i = p_i c_i = \frac{E}{n} \text{ for each } i \in N$$

is the induced solution in the claims problem.

Otherwise, for each values of $p_i > 1$, we set $p_i = 1$. Let S_1 be the set of individuals i such that $p_i = 1$ in the first optimization round, that is, $p_i c_i = c_i$ ($i's$ claim is fully satisfied, and individual i cannot receive more than her claim). Let n_1 be the cardinality of this set, $n_1 = \#(S_1)$. Then, we proceed one more time.

Second round: Rewrite the associated Lagrangian with the conditions $p_i = 1$ for each $i \in S_1$ imposed, i.e.,

$$L = \prod_{j \notin S_1} p_j + \lambda \left(E - \sum_{j \in S_1} c_j - \sum_{j \notin S_1} p_j c_j \right),$$

where we do not impose $0 \leq p_j \leq 1$ for $j \notin S_1$. After some algebra, the first order conditions return

$$\sum_{j \in S_1} c_j - \sum_{j \notin S_1}^{n} p_i c_i = E \Rightarrow \sum_{j \in S_1} c_j + (n - n_1)\, p_i c_i = E$$

i.e.

$$p_i = \frac{E - \sum_{j \in S_1} c_j}{(n - n_1)\, c_i} \quad \forall i \in S_1.$$

If for each $i \notin S_1$ we have $p_i \leq 1$, we stop and the induced solution in the claims problem is

$$x_i = p_i c_i = \min \left\{ c_i, \frac{E - \sum_{j \in S_1} c_j}{n - n_1} \right\} \text{ for each } i \in N.$$

Otherwise, for each values of $p_i > 1$, set $p_i = 1$. Let S_2 be the set of individuals i such that $p_i > 1$ in the second optimization, i.e., $p_i c_i = c_i$. Let n_2 the cardinality of this set, $n_2 = \#(S_2)$. Then, we proceed one more time.

Third round: Rewrite the associated Lagrangian with these conditions $p_i = 1$ for each $i \in S_1 \cup S_2$ imposed, i.e.,

$$L = \prod_{j \notin S_1 \cup S_2} p_j + \lambda \left(E - \sum_{j \in S_1 \cup S_2} c_j - \sum_{j \notin S_1 \cup S_2} p_j c_j \right),$$

where we do not impose $0 \leq p_j \leq 1$ for $i \notin S_1 \cup S_2$. After some algebra, the first order conditions return

$$\sum_{j \in S_1 \cup S_2} c_j - \sum_{j \notin S_1 \cup S_2} p_i c_i = E \Rightarrow$$

$$\Rightarrow \sum\nolimits_{j\in S_1\cup S_2} c_j + (n-(n_1+n_2))\, p_i c_i = E$$

and

$$p_i = \frac{E - \sum_{j\in S_1\cup S_2} c_j}{(n-(n_1+n_2))\, c_i} \quad \forall i.$$

If for each $i \notin S_1 \cup S_2$ we have $p_i \leq 1$, we stop and

$$x_i = p_i c_i = \min\left\{c_i, \left(E - \sum\nolimits_{j\in S_1\cup S_2} c_j\right) / (n - n_1 - n_2)\right\} \text{ for each } i \in N.$$

Otherwise we proceed one more time. The process stops after at most $m \leq n$ rounds, since there is at least one individual that does not obtain his claim. Then, for each $i \in N$,

$$x_i = p_i c_i = \min\left\{c_i, \frac{E - \sum_{j\in \cup_{k=1}^{m-1} S_k} c_j}{n - \sum_{k=1}^{m-1} n_k}\right\}$$

which is exactly the sharing given by the CEA rule. ■

Similarly, we obtain that the CEA rule is also induced by the Nash bargaining solution applied to the problem $(S^{\mathcal{P}}, \boldsymbol{c}^{-1})$, where (with some abuse of notation)

$$\boldsymbol{c}^{-1} = \left(\frac{1}{c_1}, \frac{1}{c_2}, \ldots, \frac{1}{c_n}\right).$$

Proposition 2. *The Nash bargaining solution with disagreement point* $\boldsymbol{c}^{-1}$ *induces* $CEA(E, c)$.

The next proposition tells us that the Nash bargaining solution (i) from the *maximum point* corresponds to Constrained Equal Losses rule and (ii) from the *ideal point* retrieves the Truncated Constrained Equal Losses rule.

Proposition 3. *Let us consider the Nash bargaining solution from a reference point* $\boldsymbol{r}$, N^r. *Then,*

a) if $\boldsymbol{r} = \boldsymbol{1}$, $N^r(S^{\mathcal{P}}, \mathbf{0})$ *induces* $CEL(E, c)$;

b) if $\boldsymbol{r} = \boldsymbol{u}$, $N^r(S^{\mathcal{P}}, \mathbf{0})$ *induces* $CEL^T(E, c)$.

The proof runs parallel to the one in Proposition 1 and we omit it. If we now can consider asymmetric Nash bargaining solutions, when the choice of weights equals the value of the claim we retrieve the Proportional rule for the claims problem.

Proposition 4. *The asymmetric Nash bargaining solution with* $\alpha_i = c_i$ *applied to* $(S^{\mathcal{P}}, \mathbf{0})$ *induces* $P(E, c)$.

Proof. Let (E, c) be a claims problem and $(S^{\mathcal{P}}, \mathbf{0})$ its associated bargaining problem from a proportional approach. We follow a similar reasoning as in the proof of Proposition 1, but now the problem is

$$\max_{p} \prod\nolimits_{i=1}^{n} (p_i)^{c_i}$$

subject to

$$\sum\nolimits_{i=1}^{n} p_i c_i = E; \qquad 0 \leq p_i \leq 1, \text{ for each } i.$$

where c_i is the claim of the individual i. The solution to this problem is

$$p_i = \frac{E}{\sum_{i=1}^{n} c_i} \qquad \forall i,$$

so, $0 < p_i < 1$ for each i, and at the maximum point there is no corner solution. Then all solutions are interior and $x_i = p_i c_i = c_i E / \sum_{i=1}^{n} c_i$, which coincides with the proportional rule. ■

3.4. ***Egalitarian solution***

Proposition 5, which proof can be obtained straightforwardly, states that the Egalitarian bargaining solution corresponds with the Proportional rule.

Proposition 5. *The Egalitarian bargaining solution applied to* $(S^{\mathcal{P}}, \mathbf{0})$ *induces* $P(E, c)$.

3.5. ***Kalai-Smorodinsky bargaining solution***

We obtain that the Kalai-Smorodinsky bargaining solution corresponds with the Truncated Proportional rule.

Proposition 6. *The Kalai-Smorodinsky bargaining solution applied to the problem* $(S^{\mathcal{P}}, \mathbf{0})$ *induces* $P^T(E, c)$.

Proof. Let (E, c) be a claims problem and $(S^{\mathcal{P}}, \mathbf{0})$ its associated bargaining problem from a proportional approach. If $c_i \leq E$, for each $i \in N$, then $\boldsymbol{a} = \boldsymbol{1}$ and $KS(S^{\mathcal{P}}, \mathbf{0}) = E(S^{\mathcal{P}}, \mathbf{0})$, and from Proposition 5 we know that it induces $P(E, c)$ which, in this case, coincides with $P^T(E, c)$. If, on the contrary, there is some $k \in N$ such that $c_k > E$ (and then $c_r > E$, for each $r > k$), then $\boldsymbol{a} = (1, \ldots, 1, E/c_k, \ldots, E/c_n)$. In this case, the Kalai-Smorodinsky solution implies $p_i = p_1$, for $i < k$, and $p_j = (E/c_j)p_1$, for $j \geq k$. This coincides with the result of applying the egalitarian bargaining solution to the problem (E, c^T), that induces $P^T(E, c)$. ■

3.6. ***Equal Loss bargaining solution***

We obtain, as show in the following example, that the rule induced by the Equal Loss bargaining solution does not fulfill desirable properties. In particular, we observe that the individual with smaller claim gets the higher part of the endowment.

Example 1. *Consider* $(E, c) = (100, (100, 200))$. *Then,* $u = (1, 0.5)$*, and the Equal Loss bargaining solution is* $EL(S^{\mathcal{P}}, \mathbf{0}) = (4/6, 1/6)$*, which induces the claims proposal* $x^{EL} = (200/3, 100/3)$. *This proposal violates the order preservation property (the biggest the claim, the biggest the amount you receive), a condition that all claims rules fulfill (as far as we know).*

3.7. ***Contested Garment and Adjusted Proportional rules***

The *adjusted proportional* rule (Curiel et al. (1987)) is retrieved in our context through the Kalai-Smorodinsky bargaining solution. First, from the *minimal rights* vector $\boldsymbol{v}$, we define the reference vector $\boldsymbol{w}$ by $w_i = v_i/c_i$.

Proposition 7. *The Kalai-Smorodinsky bargaining solution applied to the problem* $(S^{\mathcal{P}}, \boldsymbol{w})$ *induces* $AP(E, c)$.

Finally, the AP rule is a generalization of the *contested garment principle* (Babylonian Talmud). This particular case, with only two individuals, can also be obtained throughout the Nash bargaining solution from point $\boldsymbol{w}$.

Proposition 8. *Let us consider the case* $n = 2$. *The Nash bargaining solution with disagreement point* $\boldsymbol{w}$ *induces* $CG(E, c)$.

Proof. Let (E, c) be a claims problem and $(S^{\mathcal{P}}, \mathbf{0})$ its associated bargaining problem from a proportional approach. It is easy to check that the solution for the Nash optimization problem is

$$p_1 = \frac{E - (v_2 - v_1)}{2c_1} \quad p_2 = \frac{E + (v_2 - v_1)}{2c_2},$$

and then, the induced solution in the claims problem $x_1 = c_1 p_1$, $x_2 = c_2 p_2$ coincides with $CG(E, (c_1, c_2))$. ■

Therefore, the claims rule induced by the Nash bargaining solution with disagreement point $\boldsymbol{w}$ induces a claims rule that may also be considered an extension of the *Contested Garment* rule.

4. Final remarks

Some of the results we obtain are the same as the ones obtained in Dagan and Volij (1993). The difference is in the set we apply the bargaining solutions: we focus in proportions while they work directly in the amounts individuals get. It must be noticed that it is not surprisingly that the Nash and Kalai bargaining solutions induce in our model the same results as shown in the paper by Dagan and Volij (1993). This fact is due to the *scale invariance* property.

Scale invariance: for each $i \in N$, $\lambda_i F_i(S, \mathbf{0}) = F_i(\lambda \circ S, \mathbf{0})$; where F denotes a general bargaining solution, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $x = (x_1, x_2, \ldots, x_n) \in S$, and $\lambda \circ x = (\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n)$.

The transformation we made in the claims problem to obtain our *proportionally* transformed bargaining problem is an scale invariance one and, as both Nash and Kalai-Smorodinsky bargaining solutions satisfy scale invariance, then the results about these bargaining solutions are the same as in the Dagan and Volij (1993) case.

Our main interest in this work has been to provide a *new scenario* where the relevant notions about what the involved individuals discuss are the proportions of their claims that are (or are not) satisfied. We show that, in general, the obtained result is no the same as the one we obtain by bargain about absolute amounts. This approach may allow to define new rules in the claims problem by using well known bargaining solutions: an example should be the use of the *Egalitarian* bargaining solution with disagreement point $\boldsymbol{w}$.

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