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Some game-theoretic grounds for meeting people half-way*

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Abstract

It is well known that, in distributions problems, fairness rarely leads to a single viewpoint (see, for instance, Young (1994)). In this context, this paper provides interesting bases that support the simple and commonly observed behavior of reaching intermediate agreements when two prominent distribution proposals highlight a discrepancy in sharing resources. Specifically, we formalize such a conflicting situation by associating it with a 'natural' cooperative game, called *bifocal distribution game*, to show that both the Nucleolus (Schmeidler (1969)) and the Shapley value (Shapley (1953a)) agree on recommending the average of the two focal proposals. Furthermore, we analyze the interpretation of the previous result by means of axiomatic arguments.

Keywords: Distribution problems, Cooperative games, Axiomatic analysis, Nucleolus, Shapley value.
JEL Classification Numbers: C71, D63, D71.

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1. Introduction

In the summer of 2010, during an International Meeting on Game Theory, we had a very interesting conversation about the cost allocation that a major company undertook after receiving a detailed report carried out by an economics research group. As a conclusion, the report provided two possible cost distributions and, surprisingly enough, the company's final decision was to distribute the cost according to the average of the two.

This paper aims at providing some new theoretical support for the popular proverb '*Virtue lies in the middle ground*' which gathers the previous behavior, so common in so many and different situations. Particularly, we consider the normative approach to sharing problems which, as superbly expressed by Young (1994), does not boil down to a single formula, but represents a balance between different competing principles.

In this context, we introduce *bifocal distribution problems* by adding, to generic distribution problems, two prominent proposals for solving them. We then model these kinds of problems as transferable utility cooperative games (TU-games, hereinafter) by associating to each coalition the smallest quantity of the 'good' to be distributed that it would receive according to the two proposed allocations. These games, that we call *bifocal distribution games*, are the minimum of two additive games and we show that this specific structure leads to 'solid' grounds of intermediate compromises.

Specifically, our main result states that, although these games are not convex in general, the Shapley value is a Core selection that coincides with the Nucleolus and recommends the *Average value*, that is, the average of the two focal distributions.

Finally, we interpret the *Average value* by means of two different axiomatic characterizations. The first one is based on an adaptation to our context of *Additivity*, the well-known property introduced by Shapley (1953b) to propose his value for TU-games. The main property of our second axiomatic result, which was first studied in the context of bankruptcy problems by O'Neill (1982), is *No Advantageous Merging or Splitting*, and demands a solution immunity to manipulations of regrouping or division of agents.

The paper is organized as follows. Section 2 introduces the main concepts and definitions. Section 3 provides game-theoretic grounds of intermediate compromises. Section 4 presents two axiomatic characterizations of the *Average value.* Section 5 summarizes our conclusions. The Appendices contains the technical proofs.

2. Bifocal distribution problems

We consider situations in which an amount M of a perfectly divisible 'good' should be distributed among a group of agents $N = \{1, .., n\}$ and there exists discrepancy about the way of distributing M, which is represented by two different proposals, specified by the vectors, $x = (x_i)_{i \in N}$ and $y = (y_i)_{i \in N}$. It is supposed that each proposal is supported by a distinct legal or moral Authority so they represent incompatible agents' rights on M since both are considered objective and socially admisible. In our framework the population of agents involved in a problem, although finite, may vary. Formally, there is a set of 'potential' agents indexed by the natural numbers $\mathbb{N}$ and $\mathcal{N}$ denotes the class of non-empty finite subsets of $\mathbb{N}$. For each $N \in \mathcal{N}$, $\mathcal{B}^N$ denotes the class of these problems.

Definition 1. *A* ***bifocal distribution problem*** *is obtained by first specifying a set of agents $N \in \mathcal{N}$, then a triplet $(M, x, y) \in \mathbb{R}_{++} \times \mathbb{R}^n_+ \times \mathbb{R}^n_+$ such that $M = \sum\limits_{i \in N} x_i = \sum\limits_{i \in N} y_i$.*

Note that a wide range of situations that have been analyzed in the economic literature can be modeled in this way. For instance: pure distribution problems, bankruptcy problems, cost sharing problems and TU-games.

We now demand that any *bifocal distribution solution* should provide efficient allocations representing an intermediate agreement between the two focal proposals, for each agent. Moreover, we define the *bifocal distribution solution* that compromises between the two different viewpoints by averaging. Our subsequent analysis provides different theoretic rationales that support the previous ideas. Formally,

Definition 2. *A* ***bifocal distribution solution*** *is a function, $\varphi : \bigcup_{N \in \mathcal{N}} \mathcal{B}^N \to \mathbb{R}^n$, such that for each $N \in \mathcal{N}$ and each bifocal distribution problem $(M, x, y) \in \mathcal{B}^N$,*

(a) $\sum_{i \in N} \varphi_i(M, x, y) = M$ and (*Efficiency*)
(b) for each $i \in N$, $\min\{x_i, y_i\} \leq \varphi_i(M, x, y) \leq \max\{x_i, y_i\}$. (*Boundedness*)

Definition 3. *The* ***Average value*** *is the function $\varphi^{Av} : \bigcup_{N \in \mathcal{N}} \mathcal{B}^N \to \mathbb{R}^n$ such that it associates to each $N \in \mathcal{N}$, each bifocal distribution problem $(M, x, y) \in \mathcal{B}^N$ and each agent $i \in N$, the amount $\varphi_i^{Av}(M, x, y) = (x_i + y_i)/2$.*

3. Bifocal distribution TU-games: characteristics and results

A TU-game involving a set of agents $N \in \mathcal{N}$ can be described as a function V, known as the characteristic function, which associates a real number to each subset of agents, or coalition, S contained in N. Formally, for each $N \in \mathcal{N}$, a **TU-game** is a pair (N, V), where $V : 2^N \to \mathbb{R}$. For each coalition $S \subseteq N$, $V(S)$ is commonly called its worth and denotes the quantity that agents in S can guarantee for themselves if they cooperate. Therefore, it is assumed that $V(\varnothing) = 0$. It is also often supposed that (N, V) is **superadditive**, i.e., for any pair of coalitions $S, T \subset N$ such that $S \cap T = \varnothing$, $V(S \cup T) \geq V(S) + V(T)$, so that there is incentive for the grand coalition forms. Let $\mathcal{G}^N$ denote the family of TU-games with agents set N.

A **solution for TU-games** is a correspondence which for each $N \in \mathcal{N}$ and each $(N, V) \in \mathcal{G}^N$, selects a set of allocations of the worth of the grand coalition among the agents. If a TU-game solution consists of a unique allocation, it is called a **TU-value**.

Given $(N, V) \in \mathcal{G}^N$, for each $i \in N$ and each $S \subset N$, we call the **marginal contribution of agent** i **to coalition** S, denoted by $\Delta_i V(S)$, the amount which his adherence contributes to the value of the coalition, that is, $\Delta_i V(S) = V(SU\{i\}) - V(S)$.

A TU-game is convex if the larger the coalition that an agent joins, the larger his marginal contribution. Formally, $(N, V) \in \mathcal{G}^N$ is **convex** if and only if, for all $i \in N$, $\Delta_i V(S) \leq \Delta_i V(T)$ for all $S \subseteq T \subseteq N \setminus \{i\}$.

We next present three well-known TU-games solution concepts which play a central role in our analysis: the Core (Gillies (1953) and Shapley (1953b)), the Nucleolus (Schmeidler (1969)) and the Shapley value (Shapley (1953b)).

A Core distribution demands that no set of agents can collectively improve it by their own cooperation. Formally, for each $N \in \mathcal{N}$ and each $(N, V) \in \mathcal{G}^N$, the **Core**, $\mathbb{C}$, is the set $\mathbb{C}(N, V) = \left\{ x \in \mathbb{R}^n : \sum\limits_{i \in N} x_i = V(N), \sum\limits_{i \in S} x_i \geq V(S) \ \forall S \subset N. \right\}$

According to the Shapley value, the worth of the grand coalition is distributed assuming that all orders of agents' arrivals to the grand coalition are equally probable and in each order, each agent gets his marginal contribution to the coalition that he joins. Formally, for each $N \in \mathcal{N}$ and each $(N, V) \in \mathcal{G}^N$, the **Shapley value**, γ^{Sh}, associates to each $i \in N$, the amount $\gamma_i^{Sh}(N, V) = \sum\limits_{S \subseteq N \setminus \{i\}} [(s!(n - s - 1)!)/n!] \Delta_i V(S)$.

To introduce the next definition we need additional notation. For each $N \in \mathcal{N}$ and each $(N, V) \in \mathcal{G}^N$, $I(N, V) = \{x \in \mathbb{R}^n : \sum_{i \in N} x_i = V(N), x_i \geq V(\{i\}) \ \forall i \in N\}$ is the set of imputations. For each $x \in \mathbb{R}^n$ and each coalition $S \subseteq N$, $e(x, S) = V(S) - \sum\limits_{i \in S} x_i$ is the excess of coalition S in reference to x and represents a measure of dissatisfaction of such a coalition. The vector $e(x) = \{e(x, S)\}_{S \subseteq N}$ provides the excesses of all the coalitions in reference to x. Given $x \in \mathbb{R}^n$, $\theta(x)$ is the vector that results from x by permuting coordinates in decreasing order, $\theta_1(x) \geq \theta_2(x) \geq ... \geq \theta_n(x)$. Finally, $\leq_L$ stands for the lexicographic order, that is, given $x, y \in \mathbb{R}^n$, $x \leq_L y$ if there is $k \in N$ such that for all $j \leq k$, $x_j = y_j$ and $x_{k+1} \leq y_{k+1}$.

The Nucleolus looks for an individually rational distribution of the worth of the grand coalition in which the maximum dissatisfaction is minimized. Formally, for each $N \in \mathcal{N}$ and each $(N, V) \in \mathcal{G}^N$, the **Nucleolus**, γ^{Nu}, is the vector $\gamma^{Nu}(N, V) = x \in I(N, V)$ such that $\theta(e(x)) \leq_L \theta(e(y))$ for all $y \in I(N, V)$.

We define the game corresponding to a *bifocal distribution problem* by associating to each coalition the smallest quantity of the 'good' that it would receive according to the two focal solutions.

Definition 4. *Given $N \in \mathcal{N}$ and $B = (M, x, y) \in \mathcal{B}^N$, the corresponding* ***bifocal distribution game*** *is the TU-game (N, V^B), summarizes by V^B, which associates to each coalition $S \subseteq N$, the real value $V^B(S) = \min\left\{\sum_{i \in S} x_i, \sum_{i \in S} y_i\right\}$.*

Note that, for each $N \in \mathcal{N}$ and each $B = (M, x, y) \in \mathcal{B}^N$, the *bifocal distribution game*, V^B, has a non-empty Core, since both x and y belong to it. The next proposition provides a necessary condition for a proposal to be in the Core of a *bifocal distribution game*. This condition coincides with the 'natural' *Boundedness* requirement (see Definition 2). Therefore, this result shows that the agents' behavior regarding *bifocal distribution problems* has, albeit unconsciously, strong theoretic support. Nevertheless, it is easy to check that this condition is not sufficient for a proposal to be in the Core.

Proposition 1. *Given $N \in \mathcal{N}$ and $B = (M, x, y) \in \mathcal{B}^N$, if $z \in \mathbb{C}(V^B)$ then, for all $i \in N$, $\min\{x_i, y_i\} \leq z_i \leq \max\{x_i, y_i\}$.*

Proof. See Appendix 1.

It is easy to check that *bifocal distribution games* are not, in general, convex games. Therefore, it cannot be a guarantee that the Shapley value belongs to the Core. However, as shown in our main result, the Shapley value is a Core selection and coincides with the Nucleolus for any *bifocal distribution game*. The concept of PS-games, introduced by Kar et al. (2009), is used to justify this coincidence.

A PS-game is a TU-game in which for each player $i \in N$, the sum of i's marginal contribution to any pair of coalitions T, T^* such that $T \cup T^* = N \backslash \{i\}$ and $T \cap T^* = \varnothing$ is a player specific constant. Formally, a TU-game (N, V) is a **PS-game** if for each $i \in N$, there exists $k_i \in \mathbb{R}$ such that, for all $T \subseteq N \diagdown \{i\}$, $\Delta_i V(T) + \Delta_i V(N \diagdown [T \cup \{i\}]) = k_i$.

The following result shows that *bifocal distribution games* are PS-games in which each k_i is the sum of the recommendations made for agent i by the two focal proposals.

Proposition 2. *Given $N \in \mathcal{N}$ and $B = (M, x, y) \in \mathcal{B}^N$, the associated bifocal distribution game, V^B, is a PS-game such that for all $i \in N$ and for all coalition $T \subseteq N \diagdown \{i\}$, $\Delta_i V^B(T) + \Delta_i V^B(N \diagdown [T \cup \{i\}]) = x_i + y_i$.*

Proof. See Appendix 2.

Note that this proposition reinforces the subclass of PS-games by identifying within it a broad range of real situations modeled as TU-games. Up to now, only different games underlying queueing problems had been identified as PS-games (see Kar et al. (2009)).

Our main result provides 'solid' grounds for selecting the average of the two focal viewpoints from among all the intermediate compromises when facing *bifocal distribution problems*, ratifying the practice regularly observed in these situations.

Theorem 3.1. ***'Virtue lies in the middle ground'.*** *For each $N \in \mathcal{N}$ and each bifocal distribution game, V^B, with $B = (M, x, y) \in \mathcal{B}^N$, the Shapley value and the Nucleolus coincide with the Average value, that is, $\gamma^{Sh}(V^B) = \gamma^{Nu}(V^B) = \varphi^{Av}(M, x, y)) = (x + y)/2$.*

Proof. See Appendix 3.

Remark 1. *The previous theorem can be easily replicated for the modification of our model in which the two prominent proposals are distribution solutions rather than problem data. Thus, it could be used as foundation to new solutions concept for certain classes of distribution problems. By way of example, let us consider a class of TU-games $\mathcal{G}_C^N \subset \mathcal{G}^N$, for each $N \in \mathcal{N}$, in which fairness is represented by the TU-values γ^1 and γ^2. Then, a bifocal TU-game problem in this class is defined by a triplet $((N, V_C), \gamma^1(N, V_C), \gamma^2(N, V_C))$ where $(N, V_C) \in \mathcal{G}_C^N$, and Theorem 3.1 provides the new solution concept $\gamma^* = (\gamma^1 + \gamma^2)/2$. A pertinent application of this idea to bankruptcy problems, where two significant viewpoints arise as any distribution can be observed by focusing either on gains or losses, can be found in Gadea-Blanco et al. (2010).*

Unfortunately, we show through the next example that the previous result cannot be extended for distribution problems with more than two focal proposals.

Example 1. *Let us consider $N = \{1, 2, 3\}$ and the problem of distributing $M = 60$ with the following three focal proposals on the table: $x = (10, 25, 25)$, $y = (0, 27.5, 32.5)$ and $z = (10, 22.5, 27.5)$. Let the associated TU-game be defined by $V^{(M,x,y,z)}(S) = \min\left\{\sum_{i\in S} x_i, \sum_{i\in S} y_i, \sum_{i\in S} z_i\right\}$ for each $S \subseteq N$. It is easy to verify that $V^{(M,x,y,z)}(\{1\}) = 0$, $V^{(M,x,y,z)}(\{2\}) = 22.5$, $V^{(M,x,y,z)}(\{3\}) = 25$, $V^{(M,x,y,z)}(\{1, 2\}) = 27.5$, $V^{(M,x,y,z)}(\{1, 3\}) = 32.5$, $V^{(M,x,y,z)}(\{2, 3\}) = 50$ and $V^{(M,x,y,z)}(\{1, 2, 3\}) = 60$. Moreover, $\gamma^{Sh}(V^{(M,x,y,z)}) = (5.42, 25.42, 29.16)$, $\gamma^{Nu}(V^{(M,x,y,z)}) = (6.25, 25, 28.75)$ and $(x + y + z)/3 = (6 + (2/3), 25, 28 + (1/3))$. Therefore, the Nucleolus and the Shapley value do not coincide, and neither of them corresponds to the average of the three focal allocations.*

Somehow, the previous example limits our main result. However, the following two facts, which reinforce its applicability, should be noted. On the one hand, with the aim of being operative, societies establish mechanisms for reducing the number of proposals in controversial situations. On the other hand, once a society specifies some equity principles, the number of acceptable proposals is greatly reduced and they could lead to the natural form of bipolarity.

4. Axiomatic arguments of the Average value

We present four axioms for *bifocal distribution solutions*: *Anonymity*, *P-Impartiality*, *Additivity* and *No Advantageous Merging or Splitting.* They can have relevance in practice and provide two alternative interpretations of the *Average value.*

Anonymity is a standard fairness principle in many distribution problems. It states that the identity of the agents should be irrelevant.
Anonymity: for each $N \in \mathcal{N}$, each bifocal distribution problem $(M, x, y) \in \mathcal{B}^N$ and each permutation π[1], $\varphi(M, \pi(x), \pi(y)) = \pi(\varphi(M, x, y))$.

A-Impartiality comes from the fact that the proposals representing the discrepancy about how to carry out the distribution of a resource are equally valid. It requires that the solution of a problem sholud not depend on which legal or moral Authority supports which proposal. A similar property can be found in the context of meta-bargaining problems (see, for instance, Naeve-Steinweg Naeve-Steinweg (1999)).
A-Impartiality: for each $N \in \mathcal{N}$ and each bifocal distribution problem $(M, x, y) \in \mathcal{B}^N$, $\varphi(M, x, y) = \varphi(M, y, x)$.

Additivity was first proposed, for TU-games, by Shapley (1953b) and from then on, its fulfilment has been demanded in a huge family of allocations problems analyzed from a cooperative perspective (see Moretti and Patrone (2008)). This property pertains to situations in which the amount to divide comes in two parts, each one with its corresponding pair of proposals. It states that first dividing the first part and then the second part should yield the same recommendation as consolidating the two parts and dividing the sum at once.
Additivity: for each $N \in \mathcal{N}$ and each pair of bifocal distribution problems in $\mathcal{B}^N$, $B^1 = \left(M^1, x^1, y^1\right)$ and $B^2 = \left(M^2, x^2, y^2\right)$, $\varphi\left(M^1 + M^2, x^1 + x^2, y^1 + y^2\right) = \varphi\left(M^1, x^1, y^1\right) + \varphi\left(M^2, x^2, y^2\right)$.

No Advantageous Merging or Splitting considers the possibility that a group of agents consolidates and appears as a single agent, or conversely, that an agent splits as several agents. It states that such consolidation or splitting should not be beneficial. This property was introduced, for bankruptcy problems, by O'Neill (1982) and has been considered afterwards by various authors (see Thomson (2003)).

[1] A permutation is a bijection applying N to itself. In this paper, and abusing notation, for any vector $x \in \mathbb{R}^n$, $\pi(x)$ will denote the vector obtained by applying permutation π to its components. That is, the ith component of $\pi(x)$ is x_j whenever $j = \pi(i)$. Similar reasoning considerations apply for $\pi(\varphi(M, x, y))$.

No Advantageous Merging or Splitting: for each $P, Q \in \mathcal{N}$, each $(M, x, y) \in \mathcal{B}^P$ and each $(M, x', y') \in \mathcal{B}^Q$, if $P \subset Q$ and there is $i \in P$ such that $x'_i = x_i + \sum_{j \in Q \backslash P} x_j$ and for each $j \in P \backslash \{i\}$, $x'_j = x_j$, then $\varphi_i (M, x', y') = \varphi_i (M, x, y) + \sum_{j \in Q \backslash P} \varphi_j (M, x, y)$.

Next result characterizes the *Average value* adding *A-Impartiality* to the set of properties that Shapley introduced to identify his TU-value (Shapley (1953b)).[2]

Theorem 4.1. *A bifocal distribution solution, φ, satisfies Anonymity, A-Impartiality and Additivity if and only if, for each $N \in \mathcal{N}$ and each $(M, x, y) \in \mathcal{B}^N$, $\varphi (M, x, y) = \varphi^{Av} (M, x, y)$.*

Proof. See Appendix 5.

Our second axiomatic argument of the *Average value* corresponds to one of the theoretic bases that has provided support to the so commonly principle of proportionality, which has been applied to different classes of distribution problems.

Theorem 4.2. *A bifocal distribution solution, φ, satisfies A-Impartiality and No Advantageous Merging or Splitting if and only if, for each $N \in \mathcal{N}$ and each $(M, x, y) \in \mathcal{B}^N$, $\varphi (M, x, y) = \varphi^{Av} (M, x, y)$.*

Proof. See Appendix 6.

5. Conclusions

Next, we clarify the relations between *bifocal distributions games* and other well-known classes of TU-games, which are gathered in the Figure 5.1 below.

It is straightforward to verify that *bifocal distributions games* are minimum games of two additive games with equal worth for the grand coalition[3]. They are also a subclass of exact games (see Schmeidler (1972)). To our knowledge, no subclass of these games has been identified for the coincidence of the two prominent single-valued TU-games solutions, the Shapley value and the Nucleolus. Until the recent paper by Kar et al. (2009), we have not found results in this sense apart from the work on the so-called

[2] It can be verified that, in our context, a player who contributes nothing to every coalition, called dummy player, is an agent for which both proposals provide him with nothing. Hence, *Boundedness* (see Definition 2) implies that he receives nothing.

[3] A game $(N, V) \in \mathcal{G}^N$, with $N \in \mathcal{N}$, is *additive* if there exists $a \in \mathbb{R}^n_+$ such that for each coalition $S \subseteq N$, $V(S) = \sum_{i \in S} a_i$. The minimum game generated by a collection of games in $\mathcal{G}^N$, $\{(N, V)_t\}_{t \in T}$, denoted by $(N, \min V_T^{\min})$, is defined by $V_T^{\min}(S) = \min_{t \in T} \{V_t(S)\}$ for each coalition $S \subseteq N$.

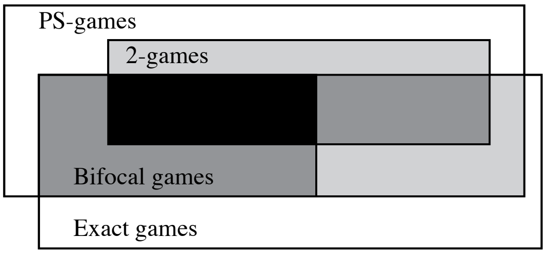

Figure 5.1: Relations among some classes of TU-games.

2-games, a special class of the k-games defined by Deng and Papadimitriou (1994). Regarding 2-games, an inclusion relation between them and *bifocal distributions games* cannot be established, although the intersection of these two classes of games is non-empty. Moreover, it is also remarkable that both 2-games and *bifocal distribution games* are PS-games, but there are PS-games that are neither 2-games nor *bifocal distribution games.*

In conclusion, this paper uses, in a simple way, cooperative game theory to support the very commonly observed collective decision of meeting people half-way. Therefore, the analysis therein combines two noteworthy characteristics, the simplicity and the match of theory with real life.

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APPENDIX 1.

Proof of Proposition 1.
Throughout this appendix, for each $x \in \mathbb{R}^n$ and each $S \subset N \in \mathcal{N}$, let $X = \sum_{i \in N} x_i$, and $X_S = \sum_{i \in S} x_i$. Given $N \in \mathcal{N}$ and $B = (M, x, y) \in \mathcal{B}^N$, let $z \in \mathbb{C}\left(V^B\right)$. Then,

$$X = Y = Z = V^B(N) = M, \tag{5.1}$$

and $z_i \geq V^B(\{i\}) = \min\{x_i, y_i\}$ for all $i \in N$. Now, we only have to prove that $z_i \leq \max\{x_i, y_i\}$ for all $i \in N$. Let us suppose that there exists $i \in N$ such that $z_i > \max\{x_i, y_i\}$ and, without loss of generality, let us assume that $x_i \leq y_i$. Then,

$$z_i > y_i. \tag{5.2}$$

Let $S = N \backslash i$. On the one hand, by Conditions 5.1 and 5.2,

$$Z_S < Y_S. \tag{5.3}$$

On the other hand, since $x_i \leq y_i$, Condition 5.1 implies

$$X_S \geq Y_S. \tag{5.4}$$

Therefore, by Conditions 5.4 and 5.3, $V^B(S) = \min\{X_S, Y_S\} = Y_S > Z_S$, contradicting that $z \in \mathbb{C}\left(V^B\right)$. Thus, $z_i \leq \max\{x_i, y_i\}$ for all $i \in N$. ■

APPENDIX 2.

Proof of Proposition 2.
Throughout this appendix, for each $x \in \mathbb{R}^n$ and each $S \subset N \in \mathcal{N}$, let $X = \sum_{i \in N} x_i$, and $X_S = \sum_{i \in S} x_i$. Given $N \in \mathcal{N}$ and $B = (M, x, y) \in \mathcal{B}^N$, by Definition 4:

$$\begin{aligned} V^B(\varnothing) &= 0 \text{ and} \\ V^B(S) &= \min\{X_S, Y_S\} \text{ for each } S, \varnothing \neq S \subseteq N. \end{aligned}$$

Let us consider any pair of coalitions T, T^* such that $T \cup T^* = N \backslash \{i\}$ and $T \cap T^* = \varnothing$. Note that, since $X = Y = M$, for any $i \in N$,

$$\begin{aligned} \Delta_i V^B(T) + \Delta_i V^B(T^*) &= \min\{x_i + X_T, y_i + Y_T\} - \min\{X_T, Y_T\} + \\ &+ \min\{M - X_T, M - Y_T\} - \min\{M - x_i - X_T, M - y_i - Y_T\}. \end{aligned} \tag{5.5}$$

Next, we calculate the sum of the marginal contributions of any agent $i \in N$ to T and T^*. The following four cases exhaust all the possibilities.

Case 1 : $x_i + X_T \leq y_i + Y_T$ and $X_T \leq Y_T$.

These inequalities imply that, $M - x_i - X_T \geq M - y_i - Y_T$ and $M - X_T \geq M - Y_T$. Then, by Equation 5.5, $\Delta_i V^B(T) + \Delta_i V^B(T^*) = x_i + X_T - X_T + M - Y_T - (M - y_i - Y_T) = x_i + y_i$.

A similar reasoning can be applied to **Case 2:** $x_i + X_T \leq y_i + Y_T$ and $X_T \geq Y_T$, **Case 3**: $x_i + X_T \geq y_i + Y_T$ and $X_T \leq Y_T$, and **Case 4**: $x_i + X_T \geq y_i + Y_T$ and $X_T \geq Y_T$ to conclude that, in all of them $\Delta_i V^B(T) + \Delta_i V^B(T^*) = x_i + y_i$. ■

APPENDIX 3.

Proof of Theorem 3.1.
Let $N \in \mathcal{N}$ and $B = (M, x, y) \in \mathcal{B}^N$. Considering Proposition 2 and applying to V^B the main result in Kar et al. (**?**), gathered below, we obtain that for all $i \in N$, $\gamma_i^{Sh}(V^B) = \gamma_i^{PNu}(V^B) = (x_i + y_i)/2$, where γ^{PNu} denotes the Prenucleolus. Now, given that, by Definition 4, $\gamma^{PNu}(V^B)$ satisfies individual rationality, that is, $\gamma_i^{PNu}(V^B) \geq V^B(\{i\})$ for all $i \in N$, we have that $\gamma^{Nu}(V^B) = \gamma^{PNu}(V^B)$.

Main Result in Kar, **Mitra and Wutuswami (2009)**: For each $N \in \mathcal{N}$, if a TU-game (N, V) is a PS-game, then for all $i \in N$, $\gamma_i^{Sh}(N, V) = \gamma_i^{PNu}(N, V) = k_i/2$, where

γ^{PNu} denotes the Prenucleolus and k_i is the player i's specific constant corresponding to the sum of his marginal contribution to any pair of coalitions T, T^* such that $T \cup T^* = N \backslash \{i\}$ and $T \cap T^* = \varnothing$. ■

APPENDIX 4. *General Claims*

We present two claims which are used in the proofs of appendices 5 and 6.

Claim 1: If a *bifocal distribution solution*, φ, satisfies *Anonymity* and *A-Impartiality*, for each $N \in \mathcal{N}$ and each $(M, x, y) \in \mathcal{B}_*^N$,

$$\varphi_h(M, x, y) = (x_h + y_h)/2 \text{ for each } h \in N,$$

where for each $N \in \mathcal{N}$, $\mathcal{B}_*^N$ denotes the following subclass of *bifocal distribution problems*,

$$\mathcal{B}_*^N = \{(M, x, y) \in \mathcal{B}^N, N = N^1 \cup N^2 \cup N^3 \mid x_k = y_k \text{ for each } k \in N^1,$$

$$\text{and } x_i = y_j \neq x_j = y_i \text{ for each } i \in N^2 \text{ and each } j \in N^3\}.$$

Proof. Given $N \in \mathcal{N}$ and $(M, x, y) \in \mathcal{B}_*^N$, let p denote the cardinality of N^2, that is, $p = |N^2|$. Note that $p = |N^3|$.

By *Boundedness*,

$$\varphi_k(M, x, y) = x_k = y_k = (x_k + y_k)/2 \text{ for each } k \in N^1.$$

Now, consider the *bifocal distribution problem* (M, x^*, y^*) in which $x^* = y$ and $y^* = x$. Note that $(M, x^*, y^*) = (M, \pi(x), \pi(y))$ where π is a permutation such that for each $k \in N^1$, $\pi(k) = k$ and, for each $i \in N^2$ and each $j \in N^3$, $\pi(i) = j$ and $\pi(j) = i$.

By *A-Impartiality* and *Anonymity*, for each $i, i' \in N^2$ and each $j, j' \in N^3$,

$$\varphi_{i'}(M, x, y) = \varphi_i(M, x, y) = \varphi_i(M, x^*, y^*) = \varphi_j(M, x, y) = \varphi_{j'}(M, x, y).$$

By *Efficiency*,

$$\sum_{i \in N^2} \varphi_i(M, x, y) + \sum_{j \in N^3} \varphi_j(M, x, y) = p(x_i + y_i) = p(x_j + y_j).$$

Therefore, for each $i \in N^2$ and each $j \in N^3$,

$$\varphi_i(M, x, y) = p(x_i + y_i)/2p = (x_i + y_i)/2 = (x_j + y_j)/2 = p(x_j + y_j)/2p = \varphi_j(M, x, y).$$

So we conclude that

$$\varphi_h(M,x,y) = (x_h + y_h)/2 \text{ for each } h \in N.$$

■

Next Claim can be proved by straightforwardly adapting to our context the proof of Proposition 3 in de Frutos (1999), which is a similar result but for the context of bankruptcy problems.

Claim 2: If a *bifocal distribution solution*, φ, satisfies *No Advantageous Merging or Splitting*, then it also fulfills *Anonymity.*

APPENDIX 5.

Proof of Theorem 4.1

It is straightforward to verify that the *Average value* satisfies *Anonymity*, *A-Impartiality* and *Additivity.* Let φ be a *bifocal distribution rule* satisfying these axioms. Given $N \in \mathcal{N}$ and $B = (M,x,y) \in \mathcal{B}^N$, let us consider three cases.

Case 1: $(M,x,y) \in \mathcal{B}_U^N$, where for each $N \in \mathcal{N}$, $\mathcal{B}_U^N$ denotes the subclass of *unanimity bifocal distribution problems*, that is,

$$\mathcal{B}_U^N = \{(M,x,y) \in \mathcal{B}^N \mid x_i = y_i \text{ for each } i \in N\}.$$

Then, by *Boundedness*,

$$\varphi_i(M,x,y) = x_i = y_i = (x_i + y_i)/2 \text{ for each } i \in N.$$

Case 2: $(M,x,y) \in \mathcal{B}_*^N$, where for each $N \in \mathcal{N}$, $\mathcal{B}_*^N$ denotes the following subclass of *bifocal distribution problems*,

$$\mathcal{B}_*^N = \{(M,x,y) \in \mathcal{B}^N, N = N^1 \cup N^2 \cup N^3 \mid x_k = y_k \text{ for each } k \in N^1,$$

$$\text{and } x_i = y_j \neq x_j = y_i \text{ for each } i \in N^2 \text{ and each } j \in N^3\}.$$

Taking into account that φ satisfies *A-Impartiality* and *Anonymity,* by Claim 1,

$$\varphi_i(M,x,y) = (x_i + y_i)/2 \text{ for each } i \in N.$$

Case 3: $(M, x, y) \in \mathcal{B}^N \backslash \{\mathcal{B}_U^N \cup \mathcal{B}_*^N\}$.

Firstly, let us decompose B as a sum of two *bifocal distribution problems*, $B = B^1 + B^{'}$, in the following way.

Starting from $B = (M, x, y)$, let $r \in N$ such that

$$\min\{x_r, y_r\} = \min_{j \in N}\{\min\{x_j, y_j\}_{x_j \neq 0, y_j \neq 0}; \{x_j\}_{y_j=0}; \{y_j\}_{x_j=0}\},$$

and let $k \in N$ such that

$$y_k = \max_{j \in N}\{y_j\}, \text{ if } \min\{x_r, y_r\} = x_r$$

and

$$x_k = \max_{j \in N}\{x_j\}, \text{ if } \min\{x_r, y_r\} = y_r.$$

We define $B^1 = \left(M^1, x^1, y^1\right)$, with $M^1 = \min\{x_r, y_r\}$ and x^1, y^1 such that:
if $\min\{x_r, y_r\} = x_r$,

$$\begin{aligned} x_r^1 &= x_r, \; x_i^1 = 0 \text{ for each } i \neq r, \text{ and} \\ y_k^1 &= x_r, \; y_i^1 = 0 \text{ for each } i \neq k; \end{aligned}$$

and if $\min\{x_r, y_r\} = y_r$,

$$\begin{aligned} y_r^1 &= y_r, \; y_i^1 = 0 \text{ for each } i \neq r, \text{ and} \\ x_k^1 &= y_r, \; x_i^1 = 0 \text{ for each } i \neq k. \end{aligned}$$

Then, $B = B^1 + B'$ where $B' = (M', x', y')$ with $M' = M - M^1, x' = x - x^1$ and $y' = y - y^1$. On the one hand, by construction, $B^1 = \left(M^1, x^1, y^1\right) \in \left\{\mathcal{B}_U^N \cup \mathcal{B}_*^N\right\}$. On the other hand, if $B' = (M', x', y') \in \left\{\mathcal{B}_U^N \cup \mathcal{B}_*^N\right\}$, some of the previous cases can be applied to both B^1 and B', and we have that, for each $i \in N$, $\varphi_i\left(M^1, x^1, y^1\right) = (x_i^1 + y_i^1)/2$ and $\varphi_i\left(M', x', y'\right) = (x_i' + y_i')/2$. Therefore, by *Additivity*, for each $i \in N$,

$$\varphi_i\left(M, x, y\right) = \left[(x_i^1 + y_i^1)/2\right] + \left[(x_i' + y_i')/2\right] = (x_i + y_i)/2.$$

If $B' = (M', x', y') \in \mathcal{B}^N \backslash \left\{\mathcal{B}_U^N \cup \mathcal{B}_*\right\}$, we decompose B' as a sum of two *bifocal distribution problems*, $B^{'} = B^2 + B''$, in the same way as we did it for B, but now starting from $B' = (M', x', y')$. Again, by construction, $B^2 = \left(M^2, x^2, y^2\right) \in \left\{\mathcal{B}_U^N \cup \mathcal{B}_*^N\right\}$ and $B'' = (M'', x'', y'')$ is such that $M'' = M' - M^2, x'' = x' - x^2$ and $y'' = y' - y^2$. Now, if $B'' \in \left\{\mathcal{B}_U^N \cup \mathcal{B}_*^N\right\}$, some of the previous cases can be applied to both B^2 and B'', and we have that, for each $i \in N$, $\varphi_i\left(M^2, x^2, y^2\right) = (x_i^2 + y_i^2)/2$ and $\varphi_i\left(M'', x'', y''\right) = (x_i'' + y_i'')/2$. Therefore, by *Additivity*, for each $i \in N$,

$$\varphi_i\left(M', x', y'\right) = \left[(x_i^2 + y_i^2)/2\right] + \left[(x_i'' + y_i'')/2\right] = (x_i' + y_i')/2$$

and for each $i \in N$,

$$\varphi_i\left(M, x, y\right) = \left[(x_i^1 + y_i^1)/2\right] + \left[(x_i' + y_i')/2\right] = (x_i + y_i)/2.$$

If $B'' = (M'', x'', y'') \in \mathcal{B}^N \backslash \left\{\mathcal{B}_U^N \cup \mathcal{B}_*\right\}$, we continue the process until we have decomposed B as a sum of p bifocal distribution problems, $B = B^1 + B^2 + ... + B^p$, with $B^j = \left(M^j, x^j, y^j\right) \in \left\{\mathcal{B}_U^N \cup \mathcal{B}_*\right\}$ for each $j \in \{1, ...p\}$. Note that, since this process ends in at most $2(n-1)$ steps, $p \leq 2n-1$. Then, by using a similar reasoning to the previous one, we have that for each $i \in N$,

$$\varphi_i\left(M, x, y\right) = \left[(x_i^1 + y_i^1)/2\right] + \left[(x_i^2 + y_i^2)/2\right] + \cdots + \left[(x_i^p + y_i^p)/2\right] = (x_i + y_i)/2.$$

Therefore, we conclude that

$$\varphi\left(M, x, y\right) = (x + y)/2 = \varphi^{Av}\left(M, x, y\right).$$

Next, we show the independence of the axioms. Let us consider the function φ_ω^{Av} that associates to each $N \in \mathcal{N}$ and each $B = (M, x, y) \in \mathcal{B}^N$, $\varphi_\omega^{Av}(M, x, y) = \gamma_\omega^{Sh}(V^B)$ where γ_ω^{Sh} is the weighted Shapley value with weight system ω (see Shapley Shapley (1953a)). It can be easily checked that φ_ω^{Av} is a *bifocal distribution rule* that satisfies *A-Impartiality* and *Additivity*, but fails to satisfy *Anonymity*. Obviously, the *bifocal distribution rule* φ^x, defined by associating to each $N \in \mathcal{N}$ and each $B = (M, x, y) \in \mathcal{B}^N$ $\varphi^x(M, x, y) = x$, satisfies *Anonymity* and *Additivity*, but fails to satisfy *A-Impartiality*. Finally, let φ^{CEL} denote the function that associate to each $N \in \mathcal{N}$, each $B = (M, x, y) \in \mathcal{B}^N$, and each $i \in N$,

$$\varphi_i^{CEL}(M, x, y) = \max\{\min\{x_i, y_i\}, \max\{x_i, y_i\} - \beta\},$$

where β is such that $\sum\limits_{i \in N} \max\{\min\{x_i, y_i\}, \max\{x_i, y_i\}\text{-}\beta\} = M$. It is straightforward to verify that φ^{CEL} is a *bifocal distribution rule* that satisfies *Anonymity* and *A-Impartiality*, but fails to satisfy *Additivity.* ■

APPENDIX 6.

Proof of Theorem 4.2

It is straightforward to verify that the *Average value* satisfies *A-Impartiality* and *No Advantageous Merging or Splitting.* Let φ be a *bifocal distribution rule* satisfying these axioms. Given $N \in \mathcal{N}$ and $B = (M, x, y) \in \mathcal{B}^N$, let us consider $N = N^1 \cup N^2$, where $N^1 = \{i \in N : x_i = y_i\}$ and $N^2 = \{i \in N : x_i \neq y_i\}$. For each $i \in N^2$, let us define $\alpha_i = min\{x_i, y_i\}$.

Let us assume that each agent $i \in N^2$ splits what proposals x and y provide him, under the names of i_1 and i_2, as follows: $x_i = x'_{i_1} + x'_{i_2}$ and $y_i = y'_{i_1} + y'_{i_2}$, where $x'_{i_1} = y'_{i_1} = \alpha_i$, $x'_{i_2} = x_i - \alpha_i$ and $y'_{i_2} = y_i - \alpha_i$. Let N' denote the new set of agents, $N' = N^1 \cup P \cup Q$, where $P = \{j = i_1 : i \in N^2\}$ and $Q = \{k = i_2 : i \in N^2\}$. For each $i \in N^1$, let $x'_i = y'_i = x_i = y_i$. Then, we have that $N' \in \mathcal{N}$ and $(M, x', y') \in \mathcal{B}^{N'}$. Note that for each $k \in Q$ either $x'_k = 0$ and $y'_k > 0$ or vice versa.

For each $k \in Q$, let $z_k = \max\{x'_k, y'_k\}$. Then, we express z_k as the ratio of the two smallest natural numbers, that is, $z_k = n_k/m_k$ such that there are no $n'_k, m'_k \in \mathbb{N}$ verifying $z_k = n'_k/m'_k$ and $n'_k < n_k$. Now, let us consider the least common multiple of the set $\{m_k : k \in Q\}$, denoted by h. That is, $h = LCM(\{m_k : k \in Q\})$. Next, let us define $h_k = (hn_k)/m_k$ for each $k \in Q$.

Let us assume that each agent $k \in Q$ is split in h_k agents, each of one receiving an identical part of both x'_k and y'_k. Let N'' denote the new set of agents,

$$N'' = N^1 \cup P \bigcup_{k \in Q} H_k,$$

where for each $k \in Q$, $H_k = \{l^k = k_r, \, r = 1, .., h_k\}$.

Now, for each $i \in N^1$, let $x''_i = x'_i$ and $y''_i = y'_i$. For each $j \in P$, let $x''_i = x'_i$ and $y''_i = y'_i$, and for each $k \in Q$ and each $l^k \in H_k$, let $x''_{l^k} = x'_k/h_k$ and $y''_{l^k} = y'_k/h_k$. Note that, by construction, for each agent belonging to any H_k, for $k \in Q$, either $x''_{l^k} = 0$ and $y''_{l^k} = 1/h$ or vice versa. Then, we have that $N'' \in \mathcal{N}$ and $(M, x'', y'') \in \mathcal{B}_*^{N''}$.

By *Boundedness*,

$$\varphi_i \left(M, x'', y''\right) = x_i = y_i = (x_i + y_i)/2 \text{ for each } i \in N^1,$$

and

$$\varphi_j \left(M, x'', y''\right) = \alpha_j \text{ for each } j \in P.$$

Moreover, given that φ satisfies *No Advantageous Merging or Splitting*, by Claim 2, it also fulfills *Anonymity*. Then, by *Efficiency* and Claim 1,

$$\varphi_{l^k} \left(M, x'', y''\right) = M' / \sum_{k \in Q} h_k \text{ for each } k \in Q \text{ and each } l^k \in H_k,$$

where $M' = M - \sum_{i \in N^1} x_i - \sum_{j \in P} \alpha_j$.

Now, let us consider the *bifocal distribution problem* (M, x', y'), in which each group of agents in each H_k, for each $k \in Q$, merges under the name of k. Then, by *Boundedness*,

$$\varphi_i \left(M, x', y'\right) = x_i = y_i = (x_i + y_i)/2 \text{ for each } i \in N^1,$$

and

$$\varphi_j \left(M, x', y'\right) = \alpha_j \text{ for each } j \in P.$$

Furthermore, by *No Advantageous Merging or Splitting*,

$$\begin{aligned} \varphi_{l^k} \left(M, x', y'\right) &= h_k M' / \sum_{k \in Q} h_k = h z_k M' / \sum_{k \in Q} h z_k = z_k M' / \sum_{k \in Q} z_k = \\ &= z_k M' / 2M' = z_k / 2 = (x'_k + y'_k)/2 = [(x_k + y_k)/2] - \alpha_k. \end{aligned}$$

Finally, let us consider the *bifocal distribution problem* $B = (M, x, y)$, in which each pair of agents $\{j = i_1, k = i_2\}$ with $i_1 \in P$ and $i_2 \in Q$, merges under the name of i, for each $i \in N^2$. Then, by *Boundedness*,

$$\varphi_i \left(M, x, y\right) = x_i = y_i = (x_i + y_i)/2 \text{ for each } i \in N^1,$$

and by *No Advantageous Merging or Splitting*,

$$\varphi_i \left(M, x, y\right) = (x_i + y_i)/2 - \alpha_i + \alpha_i = (x_i + y_i)/2 \text{ for each } i \in N^2.$$

Therefore,

$$\varphi \left(M, x, y\right) = (x + y)/2 = \varphi^{Av} \left(M, x, y\right).$$

To show the independence of the axioms it is enough to consider the *bifocal distribution rule* φ^y defined by associating to each $N \in \mathcal{N}$ and each $B = (M, x, y) \in \mathcal{B}^N$, $\varphi^y(M, x, y) = y$. Obviously, φ^y satisfies *No Advantageous Merging or Splitting*, but fails to satisfy *A-Impartiality.* ■