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On the coincidence of the Mas-Colell bargaining set and the core

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Abstract

In this paper we prove that the Mas-Colell bargaining set coincides with the core for three-player balanced and superadditive cooperative games. This is no longer true without the superadditivity condition or for games with more than three-players. Furthermore, under the same assumptions, the coincidence between the Mas-Collel and the individual rational bargaining set (Vohra (1991)) is revealed.

Keywords: Cooperative game, Mas-Colell bargaining set, balancedness, individual rational bargaining set *JEL classification:* C71, D63, D71.

1. Introduction

The bargaining set is a solution of transferable utility cooperative games that selects those payoff vectors guaranteeing the formation of stable coalition structures. Specifically, a payoff vector is in the classical bargaining set (Davis and Maschler (1963, 1967)) if each of its objections has a counterobjection. This bargaining set is non-empty for transferable utility cooperative games with a non-empty set of imputations (Davis and Maschler (1963, 1967) and Peleg (1967)). Furthermore, Solymosi (1999) gives a general necessary and sufficient condition for its coincidence with the core.

From this initial bargaining set, several variants appeared in the literature. One of the furthest studied is the so-called Mas-Colell bargaining set (Mas-Colell (1989)). The main difference between these two solutions is that, in the classical bargaining set, objections and counterobjections are defined of one explicit player against another. Meanwhile, in the Mas-Colell bargaining set, the objections and counterobjections are defined for coalitions and the explicitly condition of one player against another is not required. Since, in transferable utility cooperative games, the Mas-Colell bargaining set includes the classical bargaining set (Holzman (2001)), non-emptiness of the Mas-Colell bargaining set follows directly.

Coincidence between the Mas-Colell bargaining set and the core is proved by Dutta et al. (1989) for the class of convex games and by Mas-Colell Mas-Colell (1989), under standard assumptions, in exchange economies with a continuum of agents. It was also shown by Mas-Colell (1989) that for balanced games with at least four players, the Mas-Colell bargaining set may be larger than the core. Finally, Holzman (2001) gives the necessary and sufficient conditions for the coincidence of the Mas-Colell bargaining set and the core.

In the current approach, we prove that the Mas-Colell bargaining set coincides with the core of every balanced and superadditive transferable utility cooperative game with three-players. In addition, we show that without superadditivity this is no longer true. Note that for the classical bargaining set this coincidence is well-known (Izquierdo and Rafels (2012)). Nonetheless, for the Mas-Colell bargaining set this result in the case of balanced and superadditive three-player transferable utility cooperative games is not so straightforward.

The paper is organized as follows: Section 2 presents the definitions and preliminaries, Section 3 provides the results, and Section 4 contains some final remarks.

2. Preliminaries

Let $N = \{1, 2, \ldots, n\}$, be a set of players. For any coalition $S \subseteq N$, a transferable utility cooperative game, a **game**, with player set N is a pair (N, v) where N is the set of players and v is the **characteristic function** $v : 2^N \to \mathbb{R}$ assigning to every coalition a real number $v(S)$, the **worth** of the coalition S, with $v(\emptyset) := 0$. The worth of the coalition, $v(S)$, is interpreted as what the coalition S can obtain on its own.

A **payoff allocation** is a vector $x = (x_i)_{i \in N} \in \mathbb{R}^N$, where x_i is the payoff to player $i \in N$. We write $x(S) = \sum_{i \in S} x_i$ for all non-empty coalitions $S \subseteq N$ and $x(\emptyset) = 0$. We use $S \subset T$ to indicate strict inclusion, that is $S \subseteq T$ but

$S \neq T$. By $|S|$ we denote the cardinality of the coalition $S \subseteq N$. The set of all games is denoted by Γ.

The **pre-imputation set** of a game (N, v) is defined by $I^*(N, v) := \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$.

For a game (N, v), the set of **imputations** is $I(N, v) := \{x \in I^*(N, v) \mid x(i) \geq v(i), \text{ for all } i \in N\}$. The **core** of a game (N, v) is the set of those imputations where each coalition gets at least its worth, that is $\mathcal{C}(N, v) := \{x \in I(N, v) \mid x(S) \geq v(S) \text{ for all } S \subseteq N\}$. A game with a non-empty core is called **balanced**. A game is **superadditive** if, for every $S, T \subseteq N,\ S \cap T = \emptyset$ we have $v(S) + v(T) \leq v(S \cup T)$.

We define the classical bargaining set (Davis and Maschler (1963, 1967)). Let (N, v) be a game, $x \in I(N, v)$ and i and j be distinct players. An **objection** of a player i against j at x is a pair (S, y) where $i \in S$ but $j \notin S$ and $y(S) = v(S)$ and satisfies $y_k > x_k$ for all $k \in S$.

A **counterobjection** to this objection (S, y), is a pair (T, z), where T is a coalition containing j but not i, and $z(T) = v(T)$ and satisfies $z_k \geq x_k$, for all $k \in T \setminus S$ and $z_k \geq y_k$ for all $k \in T \cap S$.

An objection is **justified** if there is no counter-objection to it.

Definition 1. *Let (N, v) be a game. The classical* **bargaining set** *(Davis and Maschler (1963, 1967)) of (N, v), $\mathcal{M}(N, v)$* [1]*, is the set of all imputations $x \in I(N, v)$ such that there are no justified objections at x.*

Next we define the Mas-Colell bargaining set (Mas-Colell (1989)). Let (N, v) be a game and $x \in I^*(N, v)$. An **objection** at x is a pair (S, y), where S is a non-empty coalition, $y(S) = v(S)$ and satisfies $y_k \geq x_k$ for all $k \in S$ and at least one of the inequalities is strict.

A **counterobjection** to this objection is a pair (T, z), where T is a non-empty coalition, $z(T) = v(T)$ and satisfies $z_k \geq x_k$, for all $k \in T \setminus S$ and $z_k \geq y_k$ for all $k \in T \cap S$ and at least one of the inequalities is strict. An objection is **justified** if there is no counterobjection to it.

Definition 2. *Let (N, v) be a game. The* **Mas-Colell bargaining set** *of (N, v), $\mathcal{MB}(N, v)$, is the set of all preimputations, $x \in I^*(N, v)$, such that there are no justified objections at x.*

The **individual rational bargaining set** (Vohra,1991), $\mathcal{IRMB}(N, v)$, is the set of imputations contained in the Mas-Colell bargaining set.

[1] We write simply $\mathcal{M}(N, v)$ instead of $\mathcal{M}_1^i(N, v)$.

3. The Mas-Colell bargaining set and the core

In this section we state and prove our result: for three-player balanced and superadditive games, the Mas-Colell bargaining set coincides with the core of the game.

Theorem 1. *Let (N,v) be a three-player game. If the game is balanced and superadditive, then $\mathcal{MB}(N,v) = \mathcal{C}(N,v)$.*

Proof. It is proved by Holzman (2001) that for superadditive games $\mathcal{M}(N,v) \subseteq \mathcal{MB}(N,v)$. Since the game is balanced it is easy to see that in this case $C(N,v) = \mathcal{M}(N,v)$ and we have $\mathcal{C}(N,v) = \mathcal{M}(N,v) \subseteq \mathcal{MB}(N,v)$. Hence, it is enough to prove that if we take a pre-imputation which is not in the core of the game, this will not belong to the Mas-Colell bargaining set.

Let $x \in I^*(N,v) \setminus \mathcal{C}(N,v)$ and $N = \{i,j,k\}$. We will distinguish between $x \in I(N,v)$ and $x \notin I(N,v)$.[2]

Case 1: $x \in I(N,v)$. Since $x \notin \mathcal{C}(N,v)$, there exists a coalition S such that $x(S) < v(S)$. Without loss of generality we have

$$S = \{i,j\} \text{ such that } x_i + x_j < v(ij).^3 \tag{1}$$

By superadditivity and (1),

$$x_k > v(k); \tag{2}$$

and, due to balanced condition and (1) there exists a coalition T with two players $T \neq \{i,j\}$ such that

$$v(T) < x(T). \tag{3}$$

Without loss of generality, suppose $T = \{i,k\}$, thus $v(ik) < x_i + x_k$ (similarly with $T = \{j,k\}$). Hence, there are two possibilities:

(a) $v(jk) \leq x_j + x_k$. We have: by (1) $x(ij) < v(ij)$ and $v(jk) \leq x(jk)$, $v(ik) < x(ik)$. Therefore, $S = \{i,j\}$ is the unique coalition who can object at x via (S,y) with $y(S) = v(S)$ and there is no any coalition who can counter-object to it.

[2] For simplicity, we consider the game (N,v) zero-normalized.

[3] If no confusion arises, from now on, we will write $v(ij)$ instead of $v(\{i,j\})$.

(b) $v(jk) > x_j + x_k$. By (1) $v(ij) > x_i + x_j$, therefore in this case, only coalitions $\{i,j\}$ and $\{j,k\}$ can make objections at x:

• If $S = \{i,j\}$ and $y_S = (x_i, v(ij) - x_i)$, by (1) $y_S \geq x_S$ and $y_j > x_j$. The only possible counterobjection to (S,y) is by $(T = \{j,k\}, z)$ where $z_j \geq v(ij) - x_i > x_i$ and $z_k \geq x_k$. Then,

$$z_j + z_k = v(jk) > v(ij) - x_i + x_k. \tag{4}$$

• If $S = \{j,k\}$ and $y_S = (v(jk) - x_k, x_k)$, since $v(jk) > x_j + x_k$ we have, $y_S \geq x_S$ and $y_j > x_j$. The only possible counterobjection to (S,y) is by $(T = \{i,j\}, z)$ where $z_j \geq v(jk) - x_k > x_k$ and $z_i \geq x_i$. Then,

$$z_j + z_k = v(jk) < v(ij) - x_i + x_k. \tag{5}$$

Note that (4) and (5) cannot be satisfied at the same time. If (4) fails then, $(S = \{i,j\}, y)$ is an objection without a counterobjection, and if (5) fails the same apply for $(S = \{j,k\}, y)$. Thus, in any case we have an objection to x that cannot be countered.

Case 2: If $x \notin I(N,v)$. Since $x \in I^*(N,v) \setminus I(N,v)$, $x(N) = v(N)$, and by normalization of the game, w.l.o.g., we have

$$x_i < v(i) = 0. \tag{6}$$

Since the game is balanced, there exists a coalition $T \subset N$, with $|T| = 2$, such that $v(T) \leq x(T)$. Next we analyze the different possibilities for this coalition T.

(a) $T = \{i,k\}$ (similarly with $T = \{i,j\}$, since $i \in T$). Then, $v(ik) \leq x_i + x_k$. Suppose that $x_j + x_k < v(jk)$, by balancedness $x_j + x_k + v(i) < v(jk) + v(i) \leq v(N)$. Then, $v(i) < x_i$, therefore $x_i > o$ which contradicts (6). Therefore, $v(jk) \leq x_j + x_k$. Hence, there are two possibilities depending on $v(ij)$:

(a.1) $v(ij) \geq x_i + x_j$. Balancedness and $v(ij) \geq x_i + x_j$ implies $x_k \geq 0$ and by (6) we have $x_i < 0$. We must distinguish three cases:

• $x_j < 0$. Let $S = \{i,j\}$ and $y_S = (\frac{v(ij)}{2}, \frac{v(ij)}{2}) \geq x_S$ be an objection to x. Since $x_i < 0$, $x_j < 0$ and $v(ij) \geq 0$ this implies $y_i + y_j = v(ij) > x_i + x_j$ and at least one inequality is strict. Note that this an objection to x without a counterobjection.

• $x_j \geq 0$ and $v(ij) > x_i + x_j$. If $v(ij) < x_j$, $(S = \{i\}, y)$, with $y_i = 0 > x_i$ is an objection to x without a counterobjection. If $v(ij) \leq x_j$ $(S = \{i, j\}, y)$, with $y_i = 0 > x_i$ and $j_j = v(ij) \geq x_j$ is an objection to x without a counterobjection.

• $x_j \geq 0$ and $v(ij) = x_i + x_j$. Only coalition $\{i\}$ can object at x and the pair $(S = \{i\}, y)$, with $y_i = 0 > x_i$ is an objection to x without a counterobjection.

(a.2) $v(ij) < x_i + x_j$. Since $v(ik) \leq x_i + x_k$ and by superadditivity $v(ik) \geq 0$, we have $0 \leq x_i + x_k$ and by (6) $x_i < 0$. Therefore, $x_k > 0$. Equivalently, since $v(ij) < x_i + x_j$ we have $x_j > 0$. Therefore coalition $\{i\}$ is the unique who can object at x and the pair $(S = \{i\}, y)$, with $y_i = 0 > x_i$ is an objection to x without a counterobjection.

(b) $T = \{j, k\}$. Then, $v(jk) \leq x_j + x_k$. We have four possibilities:

(b.1) $v(ik) \geq x_i + x_k$ and $v(ij) > x_i + x_j$.

• $v(ik) > x_i + x_k$. Coalitions $\{i\}$, $\{i, k\}$,$\{i, j\}$ can object at x. The first possible objection is $(S = \{i\}, y)$, with $y_i = 0 > x_i$. Coalitions that can counter-object are $T = \{i, k\}$ and $T = \{i, j\}$. If $T = \{i, k\}$ wishes to counter-object needs $v(ik) > x_k$, and if $T = \{i, j\}$ needs $v(ij) > x_i$. The second one is $(S = \{i, k\}, y)$ with $y_i = v(ik) - x_k > x_i$, $y_k = x_k \geq x_k$. The coalitions that can counter-object are $T = \{i\}$ and $T = \{i, j\}$. If $T = \{i\}$ we need $v(ik) < x_k$, and if $T = \{i, j\}$ we need $v(ij) - x_j > v(ik) - x_k$. The third one is $(S = \{i, j\}, y)$ with $y_i = v(ij) - x_j > x_i$, $y_j = x_j \geq x_j$. The coalitions that can counter-object are $T = \{i\}$ and $T = \{i, k\}$. If $T = \{i\}$ we need $v(ij) < x_j$, and if $T = \{i, k\}$ we need $v(ij) - x_j < v(ik) - x_k$. Thus, there are three possibilities:

The objection $(S = \{i\}, y)$ has no counterobjection if $v(ik) \leq x_k$ and $v(ij) \leq x_j$.

The objection $(S = \{i, k\}, y)$ has no counterobjection if $v(ik) \geq x_k$ and $v(ij) - x_j \leq v(ik) - x_k$.

The objection $(S = \{i, j\}, y)$ has no counterobjection if $v(ij) \geq x_j$ and $v(ij) - x_j \geq v(ik) - x_k$.

Since one of these possibilities always occurs, there will always be an objection without a counterobjection.

- $v(ik) = x_i + x_k$. There are two possible coalitions who can object at x. The first one is $(S = \{i\}, y)$, with $y_i = 0 > x_i$. The only possible coalition who can counter-object is $T = \{i, j\}$ and needs $v(ij) > x_j$. The second one is $(S = \{i, j\}, y)$ with $y_i = v(ij) - x_j > x_i$, $y_j = x_j \geq x_j$. The only coalition who can counter-object is $T = \{i\}$ and must satisfy $v(ij) < x_j$. So that, in any case there is an objection without a counterobjection.

(b.2) $v(ik) \geq x_i + x_k$ and $v(ij) \leq x_i + x_j$. This is equivalent to (a.1).

(b.3) $v(ik) < x_i + x_k$ and $v(ij) \leq x_i + x_j$. This is equivalent to (a.2).

(b.4) $v(ik) < x_i + x_k$ and $v(ij) > x_i + x_j$. This is equivalent to (a.1).

Thus, we have seen that in all the cases there is an objection without a counterobjection. Therefore, $x \notin \mathcal{MB}(N, v)$, and the coincidence result holds. ■

Next, we give an example to show that superadditivity condition is needed in Theorem 1

Example 1. *Consider the following three-player game with player set* $N = \{1, 2, 3\}$ *where* $v(\{i\}) = 0$ *for all* $i \in N$, $v(S) = -1$ *for all* $S \subset N$ *with* $|S| = 2$, *and* $v(N) = 1$. *It is easy to see that the game is balanced, not superadditive and the core is the entire imputation set. The pre-imputation* $x = (-0.5, -0.5, 2)$ *does not belong to the core since it can be improved upon by several coalitions. Nonetheless, every objection against it admits a counterobjection and therefore it belongs to the Mas-Colell bargaining set.*

As a consequence of the above theorem, under the same conditions, the Mas-Colell bargaining set is included in the imputations set, that is, it coincides with the individual rational bargaining set.

Corollary 1. *Let* (N, v) *be a three-player game. If the game is balanced and superadditive, then* $\mathcal{MB}(N, v) = \mathcal{IRMB}(N, v) = \mathcal{C}(N, v)$.

Finally, as shown by the last example in Mas-Colell (1989), for balanced and superadditive games with at least four players the bargaining set may be larger than the core.

4. Final remarks

Theorem 1 determines the coincidence between the core and the Mas-Collel bargaining set for every balanced and superadditive three-player cooperative game. A natural extension of the current approach consists on its general study to the case of NTU games.

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