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On reduced games and the lexmax solution

Francesc Llerena and Llúcia Mauri *

Abstract

For a family of reduced games satisfying a monotonicity property, we introduced the reduced equal split-off set, an extension of the equal split-off set (Branzei et. al, 2006), and study its relation with the core. Regardless of the reduction operation we consider, the intersection between both sets is either empty or a singleton containing the lexmax solution (Arin et al., 2008). We also provide a procedure for computing the lexmax solution for a class of games that includes games with large core (Sharkey, 1982).

1 Introduction

In the context of transferable utility coalitional games (TU-games, for short), several solution concepts have been defined with the aim of accommodate together egalitarianism and particular interests. That is, to allocate the total worth of a coalition as equally as possible among its agents, while satisfying some individual requirements. One of the best known is the *weak constrained egalitarian solution* (Dutta and Ray, 1989). For convex games, Dutta and Ray (1989) devise an algorithm for finding their egalitarian allocation and show that it belongs to the core and Lorenz dominates every other core element. Unfortunately, the class of convex games is the only standard class of TU-games for which existence is guaranteed. In order to widen the domain of games for which egalitarian solutions exist, Dutta and Ray (1991) introduced the *strong constrained egalitarian solution*, a parallel concept that selects the Lorenz-maximal imputations in the *equal division core* (Selten, 1972). Related studies are Arin and Iñarra (2001), Hougaard et al. (2001) and Arin et al. (2003, 2008), who introduced other egalitarian solutions based on the notion of the *core*. Inspired by the Dutta and Ray (1989) algorithm, Branzei et al. (2006) introduce the *equal split-off set*, a non-empty set valued solution for all TU-game. In this paper, we generalize this solution concept by considering a family of reduced games, and study its relation with the core and existing egalitarian solutions.

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The paper is organized as follows. Section 2 contains notation and terminology. In Section 3 we introduce the concept of *admissible subgroup correspondence* α and the associated α-*max reduced game*. For a given α, we define the α-*reduced equal split-off set*. This set and the core have different qualitative properties. For instance, the α-reduced equal split-off set is always non-empty and finite, while the core is convex and its existence is not granted, except for balanced games. However, the intersection between them provides surprising results. For any admissible subgroup correspondence satisfying a monotonicity property, weaker than transitivity of the reduction operation, we find out that when the intersection between both sets is non-empty, then it becomes a singleton containing the *lexmax* solution of Arin et al. (2008) (Theorem 1). In Section 4, for a class of games that includes games with a large core (Sharkey, 1982), we show that the reduced equal split-off set associated with the Davis and Maschler (1965) reduced game turns out to be a singleton and it coincides with the lexmax solution (Theorem 2 and Theorem 3). Finally, we provide a procedure for computing the lexmax solution on this domain (Theorem 4).

2 Notation and terminology

The set of natural numbers $\mathbb{N}$ denotes the universe of potential players. A **coalition** is a non-empty finite subset of $\mathbb{N}$ and let $\mathcal{N} := \{N \,|\, \emptyset \neq N \subseteq \mathbb{N},\ |N| < \infty\}$ denote the set of all coalitions of $\mathbb{N}$. A **transferable utility coalitional game (a game)** is a pair (N, v) where $N \in \mathcal{N}$ is the set of players and $v : 2^N \longrightarrow \mathbb{R}$ is the characteristic function that assigns to each coalition $S \subseteq N$ a real number $v(S)$, with the convention that $v(\emptyset) = 0$. Given $S, T \in \mathcal{N}$, we use $S \subset T$ to indicate strict inclusion, that is, $S \subseteq T$ but $S \neq T$. By $|S|$ we denote the cardinality of the coalition $S \in \mathcal{N}$. By Γ we denote the class of all games.

Given $N \in \mathcal{N}$, let $\mathbb{R}^N$ stands for the space of real-valued vectors indexed by N, $x = (x_i)_{i \in N}$, and for all $S \subseteq N$, $x(S) = \sum_{i \in S} x_i$, with the convention $x(\emptyset) = 0$. For each $x \in \mathbb{R}^N$ and $T \subseteq N$, $x_{|T}$ denotes the restriction of x to T: $x_{|T} = (x_i)_{i \in T} \in \mathbb{R}^T$. Given two vectors $x, y \in \mathbb{R}^N$, $x \geq y$ if $x_i \geq y_i$, for all $i \in N$. We say that $x > y$ if $x \geq y$ and for some $j \in N$, $x_j > y_j$. Given N, a set $\pi = (P_1, \ldots, P_m)$, where $P_i \subseteq N$ for all $i \in \{1, \ldots, m\}$, with $m \leq |N|$, is a **partition** of N if the following conditions hold: (i) $P_i \neq \emptyset$ for all $i \in \{1, \ldots, m\}$, (ii) $\cup_{i=1}^m P_i = N$ and (iii) $P_i \cap P_j = \emptyset$, for all $i, j \in \{1, \ldots, m\}$, $i \neq j$.

The set of **feasible payoff vectors** of a game (N, v) is defined by $X^*(N, v) := \{x \in \mathbb{R}^N \,|\, x(N) \leq v(N)\}$. A **solution** on a class of games $\Gamma' \subseteq \Gamma$ is a mapping σ which associates with each game $(N, v) \in \Gamma'$ a subset $\sigma(N, v)$ of $X^*(N, v)$. Notice that σ is allowed to be empty. The **pre-imputation set** of (N, v) is defined by $X(N, v) := \{x \in \mathbb{R}^N \,|\, x(N) = v(N)\}$, and the set of **imputations** by $I(N, v) := \{x \in X(N, v) \,|\, x_i \geq v(\{i\}),\ \text{for all } i \in N\}$. The core of (N, v) is the set of those imputations where each coalition gets at least its worth, that is $C(N,v) = \{x \in X(N,v) \mid x(S) \geq v(S)$ for all $S \subseteq N\}$. A game (N,v) is **balanced** if it has a non-empty core. A game (N,v) is **convex** (Shapley, 1971) if, for every $S,T \subseteq N$, $v(S)+v(T) \leq v(S \cup T) + v(S \cap T)$.

Given $N \in \mathcal{N}$, for any $x \in \mathbb{R}^N$, denote by $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ the vector obtained from x by rearranging its coordinates in a non-increasing order, that is, $\hat{x}_1 \geq \hat{x}_2 \geq \ldots \geq \hat{x}_n$. In a similar way, for $\emptyset \neq S \subseteq N$, $\widehat{x_{|T}}$ denotes the vector obtained from the restriction of x to T by ordering its coordinates in a non-increasing way: $\widehat{x_{|T}}_1 \geq \widehat{x_{|T}}_2 \geq \ldots \geq \widehat{x_{|T}}_t$, where $t = |T|$. For any two vectors $y, x \in \mathbb{R}^N$ with $y(N) = x(N)$, we say that y **weakly Lorenz dominates** x, denoted by $y \succeq_{\mathcal{L}} x$, if $\sum_{j=1}^{k} \hat{y}_j \leq \sum_{j=1}^{k} \hat{x}_j$, for all $k \in \{1, \ldots, |N|\}$. We say that y **Lorenz dominates** x, denoted by $y \succ_{\mathcal{L}} x$, if at least one of the above inequalities is strict. For any two vectors $x, y \in \mathbb{R}^N$, we say that $x \preceq_{lex} y$ if $x = y$ or $x_1 < y_1$ or there exists $k \in \{2, \ldots, |N|\}$ such that $x_i = y_i$ for $1 \leq i \leq k-1$ and $x_k < y_k$. For a balanced game (N,v), the **lexmax** solution (Arin et al. 2008) is defined as $Lmax(N,v) = \{x \in C(N,v) \mid \hat{x} \preceq_{lex} \hat{y} \, \text{for all} \, y \in C(N,v)\}$. For any balanced game (N,v), the lexmax solution is a singleton and it is Lorenz undominated within the core, and then sometimes we write $x = Lmax(N,v)$.

3 Reduced equal split-off set and the core

Branzei et al. (2006) propose a set valued solution concept for arbitrary coalitional games, called the **equal split-off set**, inspired by the Dutta-Ray (1989) algorithm for finding their egalitarian solution for convex games. Each equal split-off allocation is the output of a sequential procedure where the game is reduced each time the payoffs to players in a coalition maximizing average worth are assigned. Then, a reduced game is defined by only taking into account the whole group of players outside the game. Following this idea, but considering reduced games allowing more coalitional options, we define a family of solutions that generalize the equal split-off set and study its relation with the core.

Next we introduce the concept of **admissible subgroup correspondence** inspired by the work of Thomson (1990) and also used by Izquierdo et al. (2005).

Definition 1. *An admissible subgroup correspondence $\alpha : \mathcal{N} \to \mathcal{N}$ is a correspondence associating with each $N \in \mathcal{N}$ a non-empty list $\alpha(N)$ of coalitions of N.*

We denote by $\mathcal{A}$ the set of all admissible subgroup correspondences. Given $\alpha, \alpha' \in \mathcal{A}$, we write $\alpha \leq \alpha'$ if for all $N \in \mathcal{N}$, $\alpha(N) \subseteq \alpha'(N)$. For each $\alpha \in \mathcal{A}$, we define the associated $\alpha-$**max reduced game**.

Definition 2. *Let (N,v) be a game, $\alpha \in \mathcal{A}$, $\emptyset \neq N' \subset N$ and $x \in \mathbb{R}^K$ where $N \setminus N' \subseteq K \subseteq N$. The $\alpha-$max reduced game relative to N' at x is the game*

$\left(N', r^{N'}_{\alpha,x}(v)\right)$ *defined by*

$$r^{N'}_{\alpha,x}(v)(S) = \begin{cases} 0 & \text{if } S = \emptyset, \\ \max\limits_{Q \in \alpha(N \setminus N')} \{v(S \cup Q) - x(Q)\} & \text{if } \emptyset \neq S \subset N', \\ v(N) - x(N \setminus N') & \text{if } S = N'. \end{cases} \tag{1}$$

The interpretation of the α-max reduced game is as in Davis and Maschler (1965) but here the options of members in N' to cooperate with members in $N \setminus N'$ are restricted by the admissible subgroup correspondence α. The **Davis and Maschler reduced game** is a particular case when $\alpha(N) = 2^N$ for all $N \in \mathcal{N}$. Other well-known reduced games can also be obtained by taking a suitable admissible subgroup correspondence. For instance, the **complement reduced game** proposed by Moulin (1985) is defined by $\alpha(N) = \{N\}$ for all $N \in \mathcal{N}$, or the **projected reduced game** (Funaki, 1998) by $\alpha(N) = \{\emptyset\}$ for all $N \in \mathcal{N}$. Another example is $\alpha(N) = \{\emptyset, N\}$, for all $N \in \mathcal{N}$. This correspondence formalizes a dichotomous situation where any coalition may stand alone or join the whole group of players. Other examples of admissible subgroup correspondences can be given by taking into account several aspects of coordination between players: communication, hierarchies, geographical areas, or the size of the subgroups. The above reduction operations will be denoted by α_{DM}, α_M, α_P and α_D, respectively.

A well-known property related with the notion of reduced game is **consistency**.

Definition 3. *Let σ be a solution on $\Gamma' \subseteq \Gamma$. Given $\alpha \in \mathcal{A}$, we say that σ satisfies α-consistency on Γ' if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$, all $N' \subset N$, $N' \neq \emptyset$, and all $x \in \sigma(N, v)$, then $\left(N', r^{N'}_{\alpha,x}(v)\right) \in \Gamma'$ and $x_{|N'} \in \sigma\left(N', r^{N'}_{\alpha,x}(v)\right)$.*

On the domain of convex games, the weak constrained egalitarian solution of Dutta and Ray (1989) satisfies α_{DM}-consistency (Dutta, 1990). On the domain of balanced games, the core also satisfies α_{DM}-consistency (Peleg, 1986). Using the same proof as Peleg (1986), it can be easily shown that the core satisfies α-consistency for all $\alpha \in \mathcal{A}$.

Proposition 1. *On the domain of balanced games, the core satisfies $\alpha-$consistency, for all $\alpha \in \mathcal{A}$.*

It is quite straightforward to see that the lexmax solution is α_{DM}-consistent on the domain of balanced games. Let (N, v) be a balanced game and $x = Lmax(N, v)$. Take $\emptyset \neq N' \subset N$ and suppose $x_{|N'} \neq Lmax\left(N', r^{N'}_{\alpha_{DM},x}(v)\right)$. Since $x_{|N'} \in C\left(N', r^{N'}_{\alpha_{DM},x}(v)\right)$, it holds $\hat{y} \preceq_{lex} \widehat{x_{|N'}}$, where $y = Lmax\left(N', r^{N'}_{\alpha_{DM},x}(v)\right)$. Notice that $z = \left(y, x_{|N \setminus N'}\right) \in C(N, v)$. But $\hat{z} \preceq_{lex} \hat{x}$, which leads a contradiction.[1]

[1]The following property is well known (see, for instance, Potters and Tijs, 1992). For any $n \in \mathbb{N}$ we define the map $\theta : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which arranges the coordinates of a point in $\mathbb{R}^n$ in non-increasing order. Take $x, y \in \mathbb{R}^n$ such that $\theta(x)$ is lexicographically not greater than $\theta(y)$. Take now any $z \in \mathbb{R}^p$ and consider the vectors $(x, z), (y, z) \in \mathbb{R}^{n+p}$. Then, $\theta(x, z)$ is lexicographically not greater than $\theta(y, z)$.

This complete the proof of the following proposition.

Proposition 2. *On the domain of balanced games, the lexmax solution satisfies* $\alpha_{DM}-$*consistency.*

Associated with $\alpha \in \mathcal{A}$ we introduce the α-reduced equal split-off set.

Definition 4. *Let* (N, v) *be a game,* $\alpha \in \mathcal{A}$ *and* $\pi = (T_1, \ldots, T_t)$ *be a partition of* N*. We say that* π *is an* α**-ordered partition** *of* N *if*

$$T_1 \in \arg \max_{\emptyset \neq S \subseteq N} \left\{ \frac{v(S)}{|S|} \right\} \text{ and } T_k \in \arg \max_{\emptyset \neq S \subseteq N \setminus T_1 \cup \ldots \cup T_{k-1}} \left\{ \frac{r^{N \setminus T_1 \cup \ldots \cup T_{k-1}}_{\alpha, x_{k-1}}(v)(S)}{|S|} \right\}$$

for each $k = 2, \ldots, t$*, where*

- $x_1 = \left(\frac{v(T_1)}{|T_1|}, \ldots, \frac{v(T_1)}{|T_1|} \right) \in \mathbb{R}^{T_1}$ *and*
- $x_k \in \mathbb{R}^{T_1 \cup \ldots \cup T_k}$ *is recursively defined as follows:*

$$x_{k,i} = \begin{cases} x_{k-1,i} & \text{if } i \in T_1 \cup \ldots \cup T_{k-1}, \\ \\ \dfrac{r^{N \setminus T_1 \cup \ldots \cup T_{k-1}}_{\alpha, x_{k-1}}(v)(T_k)}{|T_k|} & \text{if } i \in T_k. \end{cases} \tag{2}$$

We call the payoff vector $x_t \in \mathbb{R}^N$ *as the* α**-reduced equal split-off allocation** *generated by* π*.*

Definition 5. *Let* (N, v) *be a game and* $\alpha \in \mathcal{A}$*. The* α**-reduced equal split-off set** *of a game* (N, v)*, denoted by* $RESO(N, v, \alpha)$*, is the set of all* α*-reduced equal split-off allocations.*

For $\alpha = \alpha_M$ we recover the equal split-off set of Branzei et al. (2006). Example 1 illustrates the above procedure.

Example 1. *Let* (N, v) *be a balanced game with set of players* $N = \{1, 2, 3, 4\}$ *and characteristic function:*

S	$v(S)$	S	$v(S)$	S	$v(S)$	S	$v(S)$
{1}	0	{12}	10	{123}	13	{1234}	15
{2}	5	{13}	8	{124}	11		
{3}	3	{14}	6	{134}	10		
{4}	2	{23}	8	{234}	10		
		{24}	6				
		{34}	5				

It is not difficult to verify that $RESO(N,v,\alpha_P) = \{x = (5,5,3,2), y = (4,5,4,2)\}$, *where* x *is generated by* $\pi^x = (\{1,2\},\{3\},\{4\})$ *and* y *by* $\pi^y = (\{2\},\{1,3\},\{4\})$. *Moreover,* $RESO(N,v,\alpha_M) = RESO(N,v,\alpha_D) = RESO(N,v,\alpha_{DM}) = \{(5,5,3,2)\}$.

In order to analyze the relation between the α-reduced equal split-off set and the core of a game, we consider a family of admissible subgroup correspondences that satisfies a monotonicity property .

Definition 6. *Let* $\alpha \in \mathcal{A}$. *We say that* α *satisfies* **monotonicity in payments** *if for all* $N \in \mathcal{N}$, *all* $(N,v) \in \Gamma$, *and all* $x \in RESO(N,v,\alpha)$ *generated by* $\pi = (T_1,\ldots,T_t)$, *it holds* $x_i \geq x_j$ *for all* $i \in T_k$, *all* $j \in T_h$ *and all* $k < h \leq t$.

We denote by $\mathcal{A}_{mon}$ the set of admissible subgroup correspondences satisfying monotonicity in payments.

A natural requirement on $\alpha \in \mathcal{A}$ is that the associated α-max reduced game should be transitive, in the sense that the repeated use of the reduced game does not depend on the order that players leave the game.

Definition 7. *Let* $\alpha \in \mathcal{A}$. *The* α*-max reduced game is said to be transitive if* $r^{N''}_{\alpha,x_{|N'}}\left(r^{N'}_{\alpha,x}(v)\right) = r^{N''}_{\alpha,x}(v)$, *for all* $N \in \mathcal{N}$, *all* $(N,v) \in \Gamma$, *all coalitions* $\emptyset \neq N'' \subset N' \subset N$ *and all payoff vector* $x \in \mathbb{R}^K$ *with* $N \setminus N'' \subseteq K \subseteq N$.

We denote by $\mathcal{A}_t$ the set of admissible subgroup correspondences such that the associated α-max reduced games is transitive. It can be easily checked that $\alpha_P, \alpha_M \in \mathcal{A}_t$. To show that $\alpha_{DM} \in \mathcal{A}_t$ see, for instance, Chang and Hu (2007). The next two propositions state that transitivity is a sufficient but not necessary condition to satisfy monotonicity in payments. The proofs are given in the Appendix.

Proposition 3. $\mathcal{A}_t \subset \mathcal{A}_{mon}$.

Proposition 4. $\alpha_D \in \mathcal{A}_{mon}$ *but* $\alpha_D \notin \mathcal{A}_t$

Our main result in this section (Theorem 1) states that for any $\alpha \in \mathcal{A}_{mon}$, the intersection $RESO(N,v,\alpha) \cap C(N,v)$ is either the empty set or the lexmax solution. Before to do this, we need some preliminary results. The first one states that if the grand coalition N is a coalition maximizing average worth, then any α-reduced equal split-off set, $\alpha \in \mathcal{A}_{mon}$, is a singleton containing the equal split-off allocation.

Proposition 5. *Let* (N,v) *be a game and* $\alpha \in \mathcal{A}_{mon}$. *If* $N \in \arg\max_{\emptyset \neq S \subseteq N} \left\{ \frac{v(S)}{|S|} \right\}$, *then* $RESO(N,v,\alpha) = \left\{ \left(\frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|} \right) \right\}$.

Proof. Let (N, v) be a game. If $N \in \arg\max_{\emptyset \neq S \subseteq N} \left\{\frac{v(S)}{|S|}\right\}$, then $x = \left(\frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|}\right) \in RESO(N, v, \alpha)$. Suppose there is $y \in RESO(N, v, \alpha)$, $y \neq x$, generated by $\pi_y = (S_1, \ldots, S_s)$. For all $i \in S_1$, $y_i = x_i = \frac{v(N)}{|N|}$. By efficiency, $y(N) = x(N)$, and thus $y(N \setminus S_1) = x(N \setminus S_1)$. Moreover, since $\alpha \in \mathcal{A}_{mon}$, $y_i \leq \frac{v(S_1)}{|S_1|} = \frac{v(N)}{|N|} = x_i$, for all $i \in N \setminus S_1$. This inequality, together with $y(N \setminus S_1) = x(N \setminus S_1)$, imply $x_i = y_i$, for all $i \in N$. □

Combining monotonicity in payments with Proposition 5, we obtain an order relation between admissible subgroup correspondences $\alpha \in \mathcal{A}_{mon}$ and the intersection of the α-reduced equal split-off set with the core.

Proposition 6. *Let $\alpha, \alpha' \in \mathcal{A}_{mon}$ such that $\alpha \leq \alpha'$. Let (N, v) be a balanced game and $x \in RESO(N, v, \alpha) \cap C(N, v)$. Then, $x \in RESO(N, v, \alpha') \cap C(N, v)$.*

Proof. Let (N, v) be a balanced game and $\alpha, \alpha' \in \mathcal{A}_{mon}$ with $\alpha \leq \alpha'$. Notice first that for all $\emptyset \neq N' \subset N$ and all $y \in \mathbb{R}^N$, it holds

$$r^{N'}_{\alpha',y}(v)(R) \geq r^{N'}_{\alpha,y}(v)(R), \tag{3}$$

for all $R \subseteq N'$.

Let $x \in RESO(N, v, \alpha) \cap C(N, v)$ generated by $\pi_x = (T_1, T_2, T_3, \ldots, T_t)$ and $z^1 \in RESO(N, v, \alpha')$ generated by $\pi_{z^1} = (T_1, S_2, \ldots, S_s)$. If $t = 1$ then, by Proposition 5, $RESO(N, v, \alpha) = RESO(N, v, \alpha') = \left\{\left(\frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|}\right)\right\}$. Assume $t > 1$. For all $i \in T_1$, $x_i = z^1_i$, and by $\alpha'-$consistency of the core

$$x_{|N \setminus T_1} \in C\left(N \setminus T_1, r^{N \setminus T_1}_{\alpha', z^1}(v)\right). \tag{4}$$

By monotonicity in payments, for all $i \in T_2$ and all $j \in S_2$, $x_i \geq x_j$. Since $x_i = x_k$ for all $i, k \in T_2$, we have $x_i = \frac{x(T_2)}{|T_2|} \geq \max_{j \in S_2}\{x_j\} \geq \frac{x(S_2)}{|S_2|}$. Thus, taking all of this into account together with (3) and (4), we obtain the chain of inequalities

$$\begin{aligned} \frac{r^{N \setminus T_1}_{\alpha,x}(v)(T_2)}{|T_2|} = \frac{x(T_2)}{|T_2|} \geq \frac{x(S_2)}{|S_2|} \quad &\geq \quad \frac{r^{N \setminus T_1}_{\alpha',z^1}(v)(S_2)}{|S_2|} \geq \frac{r^{N \setminus T_1}_{\alpha',z^1}(v)(T_2)}{|T_2|} \\ &\geq \quad \frac{r^{N \setminus T_1}_{\alpha,z^1}(v)(T_2)}{|T_2|} = \frac{r^{N \setminus T_1}_{\alpha,x}(v)(T_2)}{|T_2|}, \end{aligned}$$

which implies

$$\frac{r^{N \setminus T_1}_{\alpha,x}(v)(T_2)}{|T_2|} = \frac{r^{N \setminus T_1}_{\alpha',z^1}(v)(S_2)}{|S_2|} = \frac{r^{N \setminus T_1}_{\alpha',z^1}(v)(T_2)}{|T_2|}.$$

Thus, there is $z^2 \in RESO(N, v, \alpha')$ generated by $\pi_{z^2} = (T_1, T_2, R_3, \ldots, R_r)$ and such that $z^2_i = x_i$ for all $i \in T_1 \cup T_2$. Again by α'-consistency of the core we have $x_{|N \setminus T_1 \cup T_2} \in C\left(N \setminus T_1 \cup T_2, r^{N \setminus T_1 \cup T_2}_{\alpha', z^2}(v)\right)$, and by monotonicity in payments

$x_i \geq x_j$, for all $i \in T_3$, $j \in R_3$. Since $x_i = x_k$ for all $i, k \in T_3$, we have $x_i = \frac{x(T_3)}{|T_3|} \geq \max_{j \in R_3}\{x_j\} \geq \frac{x(R_3)}{|R_3|}$. Thus, as before, we have that

$$\frac{r_{\alpha,x}^{N \setminus T_1 \cup T_2}(v)(T_3)}{|T_3|} = \frac{r_{\alpha',z^2}^{N \setminus T_1 \cup T_2}(v)(R_3)}{|R_3|} = \frac{r_{\alpha',z^2}^{N \setminus T_1 \cup T_2}(v)(T_3)}{|T_3|}.$$

Hence, there is $z^3 \in RESO(N, v, \alpha')$ generated by $\pi_{z^3} = (T_1, T_2, T_3, P_4, \dots, P_p)$ and such that $z_i^3 = x_i$ for all $i \in T_1 \cup T_2 \cup T_3$.

Following this process step by step we find that $x \in RESO(N, v, \alpha') \cap C(N, v)$. $\square$

Remark 1. *Observe that $RESO(N, v, \alpha) \cap C(N, v) \subseteq RESO(N, v, \alpha') \cap C(N, v)$, whenever $\alpha \leq \alpha'$. However, in general, $RESO(N, v, \alpha) \nsubseteq RESO(N, v, \alpha')$, as shown Example 1.*

Now we have all the tools to state the main result of this section.

Theorem 1. *Let (N, v) be a balanced game, $\alpha \in \mathcal{A}_{mon}$ and $x \in RESO(N, v, \alpha) \cap C(N, v)$. Then, $Lmax(N, v) = \{x\}$.*

Proof. Let (N, v) be a balanced game, $\alpha \in \mathcal{A}_{mon}$ and $x \in RESO(N, v, \alpha) \cap C(N, v)$. Since $\alpha \leq \alpha_{DM}$, from Proposition 6 we know that $x \in RESO(N, v, \alpha_{DM})$. Let $\pi = (S_1, S_2, \dots, S_s)$ be an $\alpha_{DM}-$ordered partition of N generating x. If $s = 1$, then, by Proposition 5, $x = \left(\frac{v(N)}{|N|}, \dots, \frac{v(N)}{|N|}\right) = Lmax(N, v)$. If $s > 1$ suppose, on the contrary, $x \neq Lmax(N, v)$. Let $y = Lmax(N, v)$. As $\alpha_{DM} \in \mathcal{A}_{mon}$, we know that for all $i \in S_1$, $x_i = \frac{v(S_1)}{|S_1|} \geq x_j$ for all $j \in N$. Since $y \in C(N, v)$, there is $i_1 \in S_1$ such that $y_{i_1} \geq \frac{v(S_1)}{|S_1|}$, and thus $\hat{y}_1 \geq y_{i_1} \geq \frac{v(S_1)}{|S_1|}$. This inequality together with the fact that $\hat{y} \preceq_{lex} \hat{x}$ imply $y_{i_1} = x_{i_1}$. If $S_1 \setminus \{i_1\} \neq \emptyset$, then $y(S_1 \setminus \{i_1\}) = y(S_1) - \frac{v(S_1)}{|S_1|} \geq v(S_1) - \frac{v(S_1)}{|S_1|} = |S_1 \setminus \{i_1\}| \frac{v(S_1)}{|S_1|}$. Hence, there exists at least some player $i_2 \in S_1 \setminus \{i_1\}$ such that $y_{i_2} \geq \frac{v(S_1)}{|S_1|} = x_{i_2}$. Since $\widehat{y_{|N \setminus \{i_1\}}} \preceq_{lex} \widehat{x_{|N \setminus \{i_1\}}}$, we conclude that $y_{i_2} = x_{i_2}$. Following this process we can check that $y_k = x_k$ for all $k \in S_1$, and so $\widehat{y_{|N'}} \preceq_{lex} \widehat{x_{|N'}}$ where $N' = N \setminus S_1$. Now consider the reduced game $\left(N', r_{\alpha_{DM},y}^{N'}(v)\right)$. Since $y_{|S_1} = x_{|S_1}$, by $\alpha_{DM}-$consistency of the core, $x_{|N'}, y_{|N'} \in C\left(N', r_{\alpha_{DM},y}^{N'}(v)\right)$. Moreover, as $\alpha_{DM} \in \mathcal{A}_t$, $x_{|N'} \in RESO\left(N', r_{\alpha_{DM},y}^{N'}(v), \alpha_{DM}\right)$ being $\pi_{|N'} = (S_2, \dots, S_s)$ a α_{DM}-ordered partition of N' generating $x_{|N'}$. On the other hand, by α_{DM}-consistency of the lexmax solution $y_{|N'} = Lmax\left(N', r_{\alpha_{DM},y}^{N'}(v)\right)$. Now from the reasoning above we can see that $y_k = x_k$ for all $k \in S_2$. Following this line of argument we conclude that $x = y$. $\square$

From Theorem 1 a natural question arises: given a balanced game (N, v), is there some $\alpha \in \mathcal{A}_{mon}$ such that $Lmax(N, v) \in RESO(N, v, \alpha)$? Although in general this fact is not true (see Example 2 below), in Section 4 we will see that for some classes of games the lexmax solution can be interpreted as an α_{DM}-reduced equal split-off allocation.

Example 2. *Let (N, v) be a balanced game with set of players $N = \{1, 2, 3\}$ and characteristic function:*

S	*v(S)*	*S*	*v(S)*	*S*	*v(S)*
$\{1\}$	0	$\{12\}$	1	$\{123\}$	1
$\{2\}$	0	$\{13\}$	1		
$\{3\}$	0	$\{23\}$	0		

For all $\alpha \in \mathcal{A}$, $RESO(N, v, \alpha) = \{(0.5, 0.5, 0), (0.5, 0, 0.5)\}$ and $Lmax(N, v) = (1, 0, 0)$.

4 Davis and Maschler reduced equal split-off set and the lexmax solution

In this section, we show that on a class of games that includes games with large core (Sharkey, 1982) the lexmax solution turns out to be the unique α_{DM}-reduced equal split-off allocation. This result provides an alternative procedure for computing the lexmax solution for games with large core.[2] Before proving it, we need a technical lemma.

Lemma 1. *Let (N, v) be a game, $M_1 = \arg\max_{\emptyset \neq T \subseteq N} \left\{\frac{(T)}{|T|}\right\}$, $N_1 = \{i \in S \mid S \in M_1\}$ and $x \in RESO(N, v, \alpha_{DM})$ generated by the α_{DM}-ordered partition $\pi_x = (T_1, \ldots, T_t)$. Let $T_1 \cup \ldots \cup T_{q^*} = \{i \in N \mid x_i \geq x_j \text{ for all } j \in N\}$. If $N_1 \neq N$, then $N_1 = T_1 \cup \ldots \cup T_{q^*}$.*[3]

The proof of Lemma 1 is given in the Appendix.

The next result states that the Davis and Maschler reduced equal split-off set becomes a singleton when intersects with the core.[4]

Theorem 2. *Let (N, v) be a balanced game. If $x \in RESO(N, v, \alpha_{DM}) \cap C(N, v)$, then $RESO(N, v, \alpha_{DM}) = Lmax(N, v) = \{x\}$.*

Proof. Let $x \in RESO(N, v, \alpha_{DM}) \cap C(N, v)$. From Theorem 1 we know that $Lmax(N, v) = \{x\}$. Suppose there is $y \in RESO(N, v, \alpha_{DM}) \setminus C(N, v)$. Let $\pi_x =$

[2]On the class of games with large core, Arin et al. (2003) design a procedure for finding the lexmax solution. Klijn et al. (2003) provide an algorithm for calculating the lexmax solution of neighbor games.

[3]As shown Example 2, Lemma 1 does not hold if $N_1 = N$.

[4]For arbitrary $\alpha \in \mathcal{A}_{mon}$ this statement is not true. Indeed, in Example 1, $RESO(N, v, \alpha_P) = \{(5, 5, 3, 2), (4, 5, 4, 2)\}$ and $(5, 5, 3, 2) \in C(N, v)$. Moreover, in general the α-reduced equal split-off set is a discrete set containing more than one element (see Example 2).

$(T_1, \ldots, T_t)$ and $\pi_y = (S_1, \ldots, S_s)$ be two α_{DM}-ordered partitions of N generating x and y, respectively. Let

$$\begin{aligned} T_1 \cup \ldots \cup T_{q^*} &= \{i \in N \mid x_i \geq x_j \text{ for all } j \in N\} \\ S_1 \cup \ldots \cup S_{p^*} &= \{i \in N \mid y_i \geq y_j \text{ for all } j \in N\}. \end{aligned} \tag{5}$$

Let $M_1 = \arg \max_{\emptyset \neq T \subseteq N} \left\{ \frac{v(T)}{|T|} \right\}$ and $N_1 = \{i \in S \,|\, S \in M_1\}$. We distinguish two cases.

- **Case 1**: $N_1 = N$
If $T_1 = N$ then, by Proposition 5, $RESO(N, v, \alpha_{DM}) = \{x\}$. If $T_1 \neq N$, for all $i \in T_1$, $x_i = \frac{v(T_1)}{|T_1|}$. Let $k \in \{2, \ldots, t\}$ and $i \in T_k$. Since $N = N_1$, there is $R \in M_1$ such that $i \in R$. As $x \in C(N, v)$, $x(R) = x(R \backslash T_1) + x(R \cap T_1) \geq v(R)$ or, equivalently, $x(R \setminus T_1) \geq v(R) - x(R \cap T_1) = v(R) - |R \cap T_1| \frac{v(T_1)}{|T_1|} = v(R)\left(1 - \frac{|R \cap T_1|}{|R|}\right) = \frac{v(R)}{|R|}|R \setminus T_1| = \frac{v(T_1)}{|T_1|}|R \setminus T_1|$. Since $\alpha_{DM} \in \mathcal{A}_{mon}$, for all $i \in R \backslash T_1$, $x_i \leq \frac{v(T_1)}{|T_1|}$. Combining both inequalities we obtain, for all $i \in R \backslash T_1$, $x_i = \frac{v(T_1)}{|T_1|}$. Therefore, for all $i, j \in N$, $x_i = x_j$. Finally, by efficiency, $x = \left(\frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|}\right)$ and, by Proposition 5, we conclude $RESO(N, v, \alpha_{DM}) = \{x\}$.

- **Case 2**: $N_1 \neq N$
Let q^* and p^* as defined in (5). Notice that $q^* < t$ and $p^* < s$ since, otherwise, $N \in M_1$ contradicting $N_1 \neq N$. From Lemma 1, $N_1 = T_1 \cup \ldots \cup T_{q^*} = S_1 \cup \ldots \cup S_{p^*}$, which implies $x_i = y_i$ for all $i \in N_1$. Thus, the reduced games $\left(N \setminus N_1, r^{N \setminus N_1}_{\alpha_{DM}, x}(v)\right)$ and $\left(N \setminus N_1, r^{N \setminus N_1}_{\alpha_{DM}, y}(v)\right)$ coincide. By α_{DM}-consistency of the core, $x_{|N \setminus N_1} \in C\left(N \setminus N_1, r^{N \setminus N_1}_{\alpha_{DM}, x}(v)\right)$. Since $\alpha_{DM} \in \mathcal{A}_t$,
$$x_{|N \setminus N_1}, y_{|N \setminus N_1} \in RESO\left(N \setminus N_1, r^{N \setminus N_1}_{\alpha_{DM}, x}(v), \alpha_{DM}\right).$$
Now define
$$\begin{aligned} T_{q^*+1} \cup \ldots \cup T_k &= \{i \in N \setminus N_1 \mid x_i \geq x_j \text{ for all } j \in N \setminus N_1\} \\ S_{p^*+1} \cup \ldots \cup S_h &= \{i \in N \setminus N_1 \mid y_i \geq y_j \text{ for all } j \in N \setminus N_1\}. \end{aligned}$$
Let $M_2 = \arg \max_{\emptyset \neq T \subseteq N \setminus N_1} \left\{ \frac{r^{N \setminus N_1}_{\alpha_{DM}, x}(v)(T)}{|T|} \right\}$ and $N_2 = \{i \in S \,|\, S \in M_2\}$.
If $N_2 = N \backslash N_1$ then, as in Case 1, $x_i = y_i$ for all $i \in N \backslash N_1$, and thus $x = y$. If not, again from Lemma 1, we have that $N_2 = T_{q^*+1} \cup \ldots \cup T_k = S_{p^*+1} \cup \ldots \cup S_h$ and $x_i = y_i$ for all $i \in N_2$. Repeating this line of reasoning we conclude that $x = y$. □

Now, we show that for games with large core the α_{DM}-reduced equal split-off set coincides with the lexmax solution.

The concept of large core is based on the notion of aspiration. An **aspiration** of the game (N, v) is a vector $x \in \mathbb{R}^N$ such that $x(S) \geq v(S)$ for all $S \subseteq N$. We denote by $A(N, v)$ the set of aspirations of the game (N, v).

Definition 8. *The core of a game (N, v) is* **large** *if for all $y \in A(N, v)$, there exists $x \in C(N, v)$ such that $x \leq y$.*

Theorem 3. *Let (N, v) be a game with large core. Then, $RESO(N, v, \alpha_{DM}) = Lmax(N, v)$.*

Proof. Let (N, v) be a game with large core and $x \in RESO(N, v, \alpha_{DM})$ generated by $\pi = (T_1, \ldots, T_t)$.

If $t = 1$, by Proposition 5, $RESO(N, v, \alpha_{DM}) = Lmax(N, v) = \left\{\left(\frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|}\right)\right\}$.

If $t > 1$, let $M_1 = \arg \max_{\emptyset \neq S \subseteq N} \left\{\frac{v(S)}{|S|}\right\}$ and $N_1 = \{i \in S \mid S \in M_1\}$.

Take $S \in M_1$ and define $y^1 \in \mathbb{R}^{N_1}$ as follows:

$$y_i^1 := \frac{v(S)}{|S|}, \text{ for all } i \in N_1. \tag{6}$$

We distinguish two cases.

- **Case 1**: $N_1 = N$

  Notice first that $y^1 \in A(N, v)$. Since (N, v) has a large core, there exists $z \in C(N, v)$ such that $z \leq y^1$. Take $i \in N$. Since, by assumption, $N_1 = N$, there exists $R^i \in M_1$ such that $i \in R^i$ and $y^1(R^i) = v(R^i)$. Thus, $z(R^i) = v(R^i)$ and $y_i^1 = y^1(R^i) - y^1(R^i \setminus \{i\}) \leq z(R^i) - z(R^i \setminus \{i\}) = z_i$, which implies $y_i^1 = z_i$, for all $i \in N$. Hence, $y^1 \in C(N, v)$. Now we claim that $N \in M_1$. Indeed, suppose $N \notin M_1$. Let $S \in M_1$. For all $i \in N$, $y_i^1 = \frac{v(S)}{|S|}$. By efficiency, $v(N) = y^1(N) = |N|\frac{v(S)}{|S|} > v(N)$, getting a contradiction. Hence, $N \in M_1$ and, by Proposition 5, we have $RESO(N, v, \alpha_{DM}) = Lmax(N, v) = \left\{\left(\frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|}\right)\right\}$.

- **Case 2**: $N_1 \neq N$

  Let $T_1 \cup \ldots \cup T_{q^*} = \{i \in N \mid x_i \geq x_j \text{ for all } j \in N\}$. Notice that $q^* < t$ since, otherwise, $N \in M_1$, in contradiction with $N_1 \neq N$. Since $T_1 \cup \ldots \cup T_{q^*} = N_1$ (Lemma 1), we have that $x_{|N_1} = y^1$.

  Let $(N \setminus N_1, w^1)$ be the reduced game relative to $N \setminus N_1$ at y^1 defined as follows:

$$w^1(\emptyset) = 0 \text{ and } w^1(R) = \max_{Q \subseteq N_1} \{v(R \cup Q) - y^1(Q)\}, \text{ for all } R \subseteq N \setminus N_1. \tag{7}$$

Let $M_2 = \arg\max_{\emptyset \neq S \subseteq N} \left\{ \dfrac{w^1(S)}{|S|} \right\}$ and $N_2 = \{ i \in S \,|\, S \in M_2 \}$.

Take $S \in M_2$ and define $y^2 \in \mathbb{R}^{N_1 \cup N_2}$ as follows:

$$y_i^2 := y_i^1 \;\text{ if } i \in N_1, \;\text{ and } y_i^2 := \frac{w^1(S)}{|S|}, \text{ if } i \in N_2, \tag{8}$$

where y^1 is defined in (6).

• If $N_2 = N \setminus N_1$, from (7) and (8) it is not difficult to verify that (a): $y^2 \in A(N, v)$ and (b): for a given $i \in N$, there is $R^i \subseteq N$ such that $i \in R^i$ and $y^2(R^i) = v(R^i)$. Since (N, v) has a large core, there is $z \in C(N, v)$ such that $z \leq y^2$. This inequality, together with both conditions (a) and (b), imply $y_i^2 \leq z_i$ for all $i \in N$. Hence, $y^2 = z \in C(N, v)$. From the efficiency of y^2, it follows that

$$\begin{gathered} M_2 = \arg\max_{\emptyset \neq S \subseteq N} \left\{ \tfrac{w^1(S)}{|S|} \right\} = \arg\max_{\emptyset \neq S \subseteq N} \left\{ \tfrac{r^{N \setminus N_1}_{\alpha_{DM}, x}(v)(S)}{|S|} \right\}, \\ \text{and } \; r^{N \setminus N_1}_{\alpha_{DM}, x}(v)(S) = w^1(S), \text{ for all } S \in M_2. \end{gathered} \tag{9}$$

We claim that $N_2 = N \setminus N_1 \in M_2$. Indeed, suppose that $N_2 \notin M_2$. For all $i \in N_2$, $y_i^2 = \frac{w^1(S)}{|S|}$, where $S \in M_2$. Since y^2 is efficient, $v(N) = y^2(N_1) + y^2(N_2) = y^1(N_1) + |N_2| \frac{w^1(S)}{|S|} > y^1(N_1) + w^1(N_2) \geq y^1(N_1) + v(N_1 \cup N_2) - y^1(N_1) = v(N)$, getting a contradiction. Thus, $N_2 \in M_2$. By Proposition 5, and taking into account (9), we have that $RESO\left(N \setminus N_1, r^{N \setminus N_1}_{\alpha_{DM}, x}(v), \alpha_{DM}\right) = \left\{ y^2_{|N \setminus N_1} \right\}$. By definition, and considering that $N_1 = T_1 \cup \ldots \cup T_{q^*}$ and $\alpha_{DM} \in \mathcal{A}_t$, we get $x_{|N \setminus N_1} \in RESO\left(N \setminus N_1, r^{N \setminus N_1}_{\alpha_{DM}, x}(v), \alpha_{DM}\right)$. Thus, $x_{|N \setminus N_1} = y^2_{|N \setminus N_1}$. Since $x_{|N_1} = y^1$, we have that $x = y^2$. As $y^2 \in C(N, v)$, from Theorem 2 we conclude that $RESO\,(N, v, \alpha_{DM}) = Lmax(N, v) = \{x\}$.

• If $N_2 \neq N \setminus N_1$, first observe that expression (9) holds. Let $T_{q^*+1} \cup \ldots \cup T_{p^*} = \{ i \in N \setminus N_1 \,|\, x_i \geq x_j \text{ for all } j \in N \setminus N_1 \}$. From Lemma 1 we know that $N_2 = T_{q^*+1} \cup \ldots \cup T_{p^*}$.

Let $(N \setminus N_1 \cup N_2, w^2)$ be the reduced game relative to $N \setminus N_1 \cup N_2$ at y^2 defined as follows:

$$w^2(\emptyset) = 0 \text{ and } w^2(R) = \max_{Q \subseteq N_1 \cup N_2} \{ v(R \cup Q) - y^2(Q) \}, \text{ for all } R \subseteq N \setminus N_1 \cup N_2. \tag{10}$$

Let $M_3 = \arg\max_{\emptyset \neq S \subseteq N} \left\{ \dfrac{w^2(S)}{|S|} \right\}$ and $N_3 = \{ i \in S \,|\, S \in M_3 \}$.

Take $S \in M_3$ and define $y^3 \in \mathbb{R}^{N_1 \cup N_2 \cup N_3}$ as follows:

$$y_i^3 := y_i^2 \;\text{ if }\; i \in N_1 \cup N_2, \;\text{ and }\; y_i^3 := \frac{w^2(S)}{|S|}, \;\text{ if }\; i \in N_3, \tag{11}$$

where y^2 is defined in (8).

• If $N_3 = N \setminus N_1 \cup N_2$, following the arguments above, we obtain that $x = y^3 \in C(N,v)$ and $RESO(N,v,\alpha_{DM}) = Lmax(N,v) = \{x\}$.

• If $N_3 \neq N \setminus N_1 \cup N_2$, repeating the same procedure, in a finite number of steps we will get the result. $\square$

Remark 2. *The proofs of the above theorems provides a* **procedure** *for calculating the lexmax solution for some classes of games working as follows. Let (N,v) be a balanced game:* **Step 1***: Let $M_1 = \arg\max_{\emptyset \neq S \subseteq N} \left\{ \frac{v(S)}{|S|} \right\}$ and $N_1 = \{i \in S \,|\, S \in M_1\}$. Every player in N_1 receives $\frac{v(T_1}{|T_1|}$, where $T_1 \in \arg\max_{\emptyset \neq S \subseteq N} \left\{ \frac{v(S)}{|S|} \right\}$.* **Step 2***: If $N_1 \neq N$, let us denote $w = r^{N\setminus N_1}_{\alpha_{DM},x^1}(v)$, being $x^1 = \left(\frac{v(T_1)}{|T_1|}, \ldots, \frac{v(T_1)}{|T_1|} \right) \in \mathbb{R}^{N_1}$. Let $M_2 = \arg\max_{\emptyset \neq S \subseteq N\setminus N_1} \left\{ \frac{w(S)}{|S|} \right\}$ and $N_2 = \{i \in S \,|\, S \in M_2\}$. Every player in N_2 receives $\frac{w(T_2)}{|T_2|}$, where $T_2 \in M_2$. The process stops when a partition of N of the form $(N_1, N_2, \ldots, N_t)$, for some $1 \leq t \leq |N|$, is reached.*

Let us denote $\mathbf{F}^v$ the payoff vector generated by the above procedure. Note that $\mathbf{F}^v$ is just the allocation constructed to proof Theorem 2 and Theorem 3. In general, $\mathbf{F}^v$ is not a core element. For instance, in Example 2, $N_1 = N$ and $\mathbf{F}^v = (0.5, 0.5, 0.5)$ is not efficient. We claim that when $\mathbf{F}^v$ belongs to the core it coincides with the lexmax solution.

Theorem 4. *Let (N,v) be a balanced game. If $\mathbf{F}^v \in C(N,v)$, then $\mathbf{F}^v = Lmax(N,v)$.*

Proof. Let (N,v) be a balanced game and $\pi = (N_1, \ldots, N_t)$ be the partition of N generating $\mathbf{F}^v$. Recall that $N_1 = \{i \in S \,|\, S \in M_1\}$, where $M_1 = \arg\max_{\emptyset \neq S \subseteq N} \left\{ \frac{v(S)}{|S|} \right\}$. If $N_1 = N$ then, by efficiency, $\mathbf{F}^v = \left(\frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|} \right)$, and thus $\mathbf{F}^v = Lmax(N,v)$. If $N_1 \neq N$, let $y = Lmax(N,v)$ and suppose $y \neq \mathbf{F}^v$. Let $S \in M_1$. Since $\alpha_{DM} \in \mathcal{A}_{mon}$, the same argument used in the proof of Theorem 1 leads to $\mathbf{F}^v_i = y_i$ for all $i \in S$. Consequently, $\mathbf{F}^v_i = y_i$ for all $i \in N_1$. Hence, $\widehat{y_{|N\setminus N_1}} \preceq_{lex} \widehat{\mathbf{F}^v_{|N\setminus N_1}}$. Now consider the reduced game $(N \setminus N_1, w)$, where $w = r^{N\setminus N_1}_{\alpha_{DM},\mathbf{F}^v}(v)$. Since $y_{|N_1} = \mathbf{F}^v_{|N_1}$, by $\alpha_{DM}-$consistency of the core $\mathbf{F}^v_{|N\setminus N_1}, y_{|N\setminus N_1} \in C\left(N \setminus N_1, w\right)$. Moreover, by $\alpha_{DM}-$consistency of the lexmax solution $y_{|N\setminus N_1} = Lmax\left(N \setminus N_1, w\right)$. Since $\mathbf{F}^v_{|N\setminus N_1} = \mathbf{F}^w$, as before we can check that $y_i = \mathbf{F}^v_i$ for all $i \in N_2$. Following this process step by step, and considering that $\alpha_{DM} \in \mathcal{A}_t$, we conclude that $\mathbf{F}^v = Lmax(N,v)$. $\square$

As we have seen in the Theorem 3's proof, for games with large core $\mathbf{F}^v$ is a core element. This fact, together with Theorem 4, provide a necessary condition for a game to have large core.

Corollary 1. *Let (N,v) be a balanced game. If $\mathbf{F}^v \notin C(N,v)$, then $C(N,v)$ is not large.*

We end the paper linking the α_{DM}-reduced equal split-off set with the egalitarian solution of Dutta and Ray (1989).

On the domain of convex games, Dutta and Ray (1989) show that the weak constrained egalitarian solution (WCES, for short) is the unique Lorenz maximal allocation in the core, and hence it coincides with the lexmax solution. This, together with the fact that convex games have large core, lead to the following corollary.

Corollary 2. *Let* (N, v) *be a convex game. Then,* $RESO(N, v, \alpha_{DM}) = WCES(N, v)$.

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Appendix

Proof of Proposition 3. Let (N, v) be a game, $\alpha \in \mathcal{A}_t$ and $x \in RESO(N, v, \alpha)$ generated by $\pi = (T_1, \ldots, T_t)$, with $t > 1$. For $k \in \{1, \ldots, t-1\}$, let us denote $N_k = N \setminus T_1 \cup \ldots \cup T_k$. Let $N_0 = N$ and $v = r^{N_0}_{\alpha,x}(v)$. For $k \leq t-1$, $i \in T_k$ and $j \in T_{k+1}$, we have

$$x_i = \frac{r^{N_{k-1}}_{\alpha,x}(v)(T_k)}{|T_k|} \text{ and } x_j = \frac{r^{N_k}_{\alpha,x}(v)(T_{k+1})}{|T_{k+1}|} = \frac{r^{N_k}_{\alpha,x|N_{k-1}}\left(r^{N_{k-1}}_{\alpha,x}(v)\right)(T_{k+1})}{|T_{k+1}|}. \tag{12}$$

We distinguish two cases:

- **Case 1**: $T_{k+1} = N_k$. In this situation, for $j \in T_{k+1}$,

$$x_j = \frac{r^{N_{k-1}}_{\alpha,x}(v)(N_{k-1}) - r^{N_{k-1}}_{\alpha,x}(v)(T_k)}{|N_k|}. \tag{13}$$

  Suppose $x_j > x_i$, for $i \in T_k$ and $j \in T_{k+1}$. Then, combining (12) and (13) we obtain

$$r^{N_{k-1}}_{\alpha,x}(v)(N_{k-1}) > \frac{r^{N_{k-1}}_{\alpha,x}(v)(T_k)}{|T_k|}(|N_k| + |T_k|)$$

  or, equivalently,

$$\frac{r^{N_{k-1}}_{\alpha,x}(v)(N_{k-1})}{|N_{k-1}|} > \frac{r^{N_{k-1}}_{\alpha,x}(v)(T_k)}{|T_k|},$$

  in contradiction with the fact that $T_k \in \arg\max_{\emptyset \neq T \subseteq N_{k-1}} \left\{\frac{r^{N_{k-1}}_{\alpha,x}(v)(T)}{|T|}\right\}$.

- **Case 2**: $T_{k+1} \subset N_k$. In this case, there is $Q^* \in \alpha(T_k)$ such that, for all $j \in T_{k+1}$,

$$x_j = \frac{r_{\alpha,x}^{N_{k-1}}(v)(T_{k+1} \cup Q^*) - |Q^*| \frac{r_{\alpha,x}^{N_{k-1}}(v)(T_k)}{|T_k|}}{|T_{k+1}|}. \tag{14}$$

If $x_j > x_i$ for $i \in T_k$, then combining (12) and (14) we have

$$r_{\alpha,x}^{N_{k-1}}(v)(T_{k+1} \cup Q^*) > \frac{r_{\alpha,x}^{N_{k-1}}(v)(T_k)}{|T_k|}(|T_{k+1}| + |Q^*|)$$

or, equivalently,

$$\frac{r_{\alpha,x}^{N_{k-1}}(v)(T_{k+1} \cup Q^*)}{|T_{k+1} \cup Q^*|} > \frac{r_{\alpha,x}^{N_{k-1}}(v)(T_k)}{|T_k|},$$

in contradiction with the fact that $T_k \in \arg\max_{\emptyset \neq T \subseteq N_{k-1}} \left\{ \frac{r_{\alpha,x}^{N_{k-1}}(v)(T)}{|T|} \right\}$.

Hence, $x_j \leq x_i$ for all $i \in T_k$, all $j \in T_{k+1}$ and all $k \in \{1, \ldots, t-1\}$, which concludes the proof. $\square$

Proof of Proposition 4. Let (N, v) be a game and $x \in RESO(N, v, \alpha_D)$ generated by $\pi = (T_1, \ldots, T_t)$, with $t > 1$. For $k \in \{1, \ldots, t-1\}$ let us denote $N_k = N \setminus T_1 \cup \ldots \cup T_k$. Let $N_0 = N$ and $v = r_{\alpha_D,x}^{N_0}(v)$. For $k \leq t-1$, $i \in T_k$ and $j \in T_{k+1}$ we have

$$x_i = \frac{r_{\alpha_D,x}^{N_{k-1}}(v)(T_k)}{|T_k|} \text{ and } x_j = \frac{r_{\alpha_D,x}^{N_k}(v)(T_{k+1})}{|T_{k+1}|}. \tag{15}$$

If $k < t-1$, we distinguish two cases:

- **Case 1**: $x_j = \frac{v(T_{k+1})}{|T_{k+1}|}$.

In this situation,

$$x_j = \frac{v\,(T_{k+1})}{|T_{k+1}|} \leq \frac{r_{\alpha_D,x}^{N_{k-1}}(v)(T_{k+1})}{|T_{k+1}|} \leq \frac{r_{\alpha_D,x}^{N_{k-1}}(v)(T_k)}{|T_k|} = x_i, \tag{16}$$

where the first inequality follows from the definition of α_D and the second one from the fact that $T_k \in \arg\max_{\emptyset \neq T \subseteq N_{k-1}} \left\{ \frac{r_{\alpha_D,x}^{N_{k-1}}(v)(T)}{|T|} \right\}$.

- **Case 2**: $x_j = \frac{v(T_1 \cup \ldots \cup T_k \cup T_{k+1}) - x(T_1 \cup \ldots, \cup T_k)}{|T_{k+1}|}$.

Notice first that $x(T_k) = r^{N_{k-1}}_{\alpha_D,x}(v)(T_k)$. Then,

$$\begin{aligned}
x_j &= \frac{v\left(T_1 \cup \ldots \cup T_k \cup T_{k+1}\right) - x\left(T_1 \cup \ldots, \cup T_{k-1}\right) - x(T_k)}{|T_{k+1}|} \\
&\leq \frac{r^{N_{k-1}}_{\alpha_D,x}(v)(T_k \cup T_{k+1}) - x(T_k)}{|T_{k+1}|} \\
&= \frac{r^{N_{k-1}}_{\alpha_D,x}(v)(T_k \cup T_{k+1}) - r^{N_{k-1}}_{\alpha_D,x}(v)(T_k)}{|T_{k+1}|} \\
&\leq \frac{r^{N_{k-1}}_{\alpha_D,x}(v)(T_k)}{|T_k|} = x_i,
\end{aligned}$$

where the first inequality follows from the definition of α_D and the second one from the fact that $T_k \in \arg\max_{\emptyset \neq T \subseteq N_{k-1}} \left\{ \frac{r^{N_{k-1}}_{\alpha_D,x}(v)(T)}{|T|} \right\}$.

If $i \in T_{t-1}$ and $j \in T_t$, then $x_j = \frac{v(T_1 \cup \ldots \cup T_{t-1} \cup T_t) - x(T_1 \cup \ldots, \cup T_{t-1})}{|T_t|}$. Thus, as in the above Case 2 it can be shown that $x_j \leq x_i$.

To see that $\alpha_D \notin \mathcal{A}_t$, consider the game (N, v) with set of player $N = \{12345\}$ and characteristic function as follows:

$$v(\{4\}) = 0.95, v(\{14\}) = v(\{134\}) = 1.9, v(\{23\}) = v(\{123\}) = 1.05, v(\{34\}) = 1,$$

$$v(\{234\}) = 2.8, v(\{1234\}) = 2, v(\{12345\}) = 3.8 \text{ and } v(S) = 0, \text{ otherwise.}$$

Take $\left(0.95, 0.6\hat{3}, 0.6\hat{3}, 0.95, 0.6\hat{3}\right)$. Routine verification shows that

$$r^{\{235\}}_{\alpha_D, x_{|\{1235\}}}\left(r^{\{1235\}}_{\alpha_D,x}(v)\right)(\{23\}) = 1.85 > r^{\{235\}}_{\alpha_D,x}(v)(\{23\}) = 1.05.$$

$\square$

Proof of Lemma 1. Let (N, v) be a game and $x \in RESO(N, v, \alpha_{DM})$ generated by $\pi_x = (T_1, \ldots, T_t)$. Let $T_1 \cup \ldots \cup T_{q^*} = \{i \in N \,|\, x_i \geq x_j \text{ for all } j \in N\}$. Notice first that $q^* < t$ since, otherwise, $x = \left(\frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|}\right)$ which implies $N \in \arg\max_{\emptyset \neq T \subseteq N} \left\{\frac{v(T)}{|T|}\right\}$, in contradiction with $N_1 \neq N$.

First we show that $T_1 \cup \ldots \cup T_{q^*} \subseteq N_1$. Let $i \in T_1 \cup \ldots \cup T_{q^*}$. If $i \in T_1$, clearly $i \in N_1$. If $i \in T_h$ for some $h \in \{2, \ldots, q^*\}$, then there is $Q^* \subseteq T_1 \cup \ldots \cup T_{h-1}$ such that

$$x_i = \frac{v(T_1)}{|T_1|} = \frac{r^{N \setminus T_1 \cup \ldots \cup T_{h-1}}_{\alpha_{DM},x}(v)(T_h)}{|T_h|} = \frac{v(T_h \cup Q^*) - |Q^*| \frac{v(T_1)}{|T_1|}}{|T_h|}. \tag{17}$$

Reordering terms in (17), we have that $\frac{v(T_1)}{|T_1|} = \frac{v(T_h \cup Q^*)}{|T_h \cup Q^*|}$, which implies $T_h \cup Q^* \in \arg\max_{\emptyset \neq T \subseteq N} \left\{\frac{v(T)}{|T|}\right\}$, and thus $i \in N_1$.

To show the reverse inclusion, take $i \in N_1$ and suppose $i \notin T_1 \cup \ldots \cup T_{q^*}$. Then, there is $R^* \in M_1$ such that $i \in R^*$. Next we show that $R^* \setminus T_1 \cup \ldots \cup$

$T_{q^*} \neq T_{q^*+1} \cup \ldots \cup T_t$. Indeed, if $R^* \setminus T_1 \cup \ldots \cup T_{q^*} = T_{q^*+1} \cup \ldots \cup T_t$, then $N = T_1 \cup \ldots \cup T_{q^*} \cup R^*$. As we have seen before, $T_1 \cup \ldots \cup T_{q^*} \subseteq N_1$. This inclusion, together with $R^* \in \arg\max_{\emptyset \neq T \subseteq N} \left\{ \frac{v(T)}{|T|} \right\}$, imply $N_1 = N$, a contradiction. Hence,

$$\begin{aligned}
\frac{v(T_1)}{|T_1|} \quad &> \quad \frac{r^{N \setminus T_1 \cup \ldots \cup T_{q^*}}_{\alpha_{DM},x}(v)(T_{q^*+1})}{|T_{q^*+1}|} \geq \frac{r^{N \setminus T_1 \cup \ldots \cup T_{q^*}}_{\alpha_{DM},x}(v)(R^* \setminus T_1 \cup \ldots \cup T_{q^*})}{|R^* \setminus T_1 \cup \ldots \cup T_{q^*}|} \\
&\geq \quad \frac{v(R^*) - x(R^* \cap \{T_1 \cup \ldots \cup T_{q^*}\}}{|R^* \setminus T_1 \cup \ldots \cup T_{q^*}|} = \frac{v(R^*) - |R^* \cap \{T_1 \cup \ldots \cup T_{q^*}\}| \frac{v(T_1)}{|T_1|}}{|R^* \setminus T_1 \cup \ldots \cup T_{q^*}|},
\end{aligned} \tag{18}$$

where the first inequality follows from the definition of $T_1 \cup \ldots \cup T_{q^*}$, the second one from $T_{q^*+1} \in \arg\max_{\emptyset \neq T \subseteq N \setminus T_1 \cup \ldots \cup T_{q^*}} \left\{ \frac{r^{N \setminus T_1 \cup \ldots \cup T_{q^*}}_{\alpha_{DM},x}(v)(T)}{|T|} \right\}$, and the last one from the definition of the α_{DM}-max reduced game and the fact that $R^* \setminus T_1 \cup \ldots \cup T_{q^*} \neq T_{q^*+1} \cup \ldots \cup T_t$. From (18) it follows $\frac{v(T_1)}{|T_1|} > \frac{v(R^*)}{|R^*|}$, in contradiction with $R^* \in \arg\max_{\emptyset \neq T \subseteq N} \left\{ \frac{v(T)}{|T|} \right\}$. Hence, $i \in T_1 \cup \ldots \cup T_{q^*}$ and $N_1 = T_1 \cup \ldots \cup T_{q^*}$. $\square$

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