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Generalized three-sided assignment markets: consistency and the core*

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Abstract

A class of three-sided markets (and games) is considered, where value is generated by pairs or triplets of agents belonging to different sectors, as well as by individuals. For these markets we analyze the situation that arises when some agents leave the market with some payoff. To this end, we introduce the derived market (and game) and relate it to the Davis and Maschler (1965) reduced game. Consistency with respect to the derived market, together with singleness best and individual anti-monotonicity axiomatically characterize the core for these generalized three-sided assignment markets. These markets may have an empty core, but we define a balanced subclass, where the worth of each triplet is defined as the addition of the worths of the pairs it contains.

Keywords Multi-sided assignment market · Consistency · Core · Nucleolus

JEL Classification C71 · C78

1 Introduction

We consider a market with three-sectors where value is obtained by means of coalitions formed by agents of different sectors, that is, either triplets, pairs or individuals. Once the valuations of all these essential coalitions are known, a coalitional game is defined, the worth of an arbitrary coalition being the maximum worth that can be obtained by a partition of this coalition into essential ones.

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Think, for instance, of one sector formed by firms providing landline telephone and internet service, on the second sector firms providing cable TV and on the third sector firms providing mobile telephone service. A triplet formed by one firm of each sector can achieve a profit by pooling their costumers and offering them more services, but also a firm alone or a pair of firms of different sectors can attain some value. Another example, somehow inspired by Billera (1981), would consist of firms manufacturing coffee on one sector, and others producing milk and sugar, respectively, on the two other sectors.

These markets have already been considered in Tejada (2013) to see that agents of different sectors do not need to be complements and agents of the same sector do not need to be substitutes. Clearly, this class of coalitional games includes the classical three-sided assignment games of Quint (1991) where value is only generated by triplets of agents belonging to different sectors. Another possible generalization of three-sided assignment games would be just assigning a reservation value to each individual and assuming that whenever an agent does not form part of any triplet then this agent can attain his/her reservation value, in the way Owen (1992) generalizes the classical two-sided assignment game of Shapley and Shubik (1972).

The difference between the generalized three-sided markets that we consider and the three-sided assignment markets with individual reservation values is that when an agent does not form part of a triplet in the optimal partition (that we will name optimal matching), apart from being alone in an individual coalition, he/she may form part of a two-player coalition with some agent belonging to a different sector and, in that case, the value of this two-player coalition may be larger than the addition of the individual reservation value of the two agents. As a consequence, ours is a wider class since it includes games that are not strategically equivalent to a Quint (1991) three-sided assignment game. Nevertheless, as in the classical three-sided assignment games, these games may not be balanced (the core may be empty).

We restrict to the three-sided case to keep notation simpler, but all the arguments and results on the present paper can be straightforwardly extended to the multi-sided case.

In this generalized class of three-sided assignment markets, we introduce a reduced market, the derived market at a given coalition and payoff vector, which represents the situation in which members outside the coalition leave the game with a predetermined payoff and the agents that remain in the market reevaluate their coalitional worth taking into account the possibility of cooperating with the agents outside. In the case of only two sectors, this reduced market coincides with the derived market defined by Owen (1992) for two-sided assignment markets with agents' reservation values.

Our first result is that if we take a core allocation, the derived assignment game at any coalition is the superadditive cover of the Davis and Maschler (1965) reduced game at this payoff vector and coalition. This result extends the result of Owen for the two-sided case and allows us to prove that, in the class of balanced generalized three-sided assignment markets, both the core and the nucleolus are consistent with respect to the derived assignment game. Moreover, making use of derived consistency and two additional axioms, singleness best and individual anti-monotonicity, we provide an axiomatic characterization of the core on the domain of generalized three-sided assignment markets and show the independence of the three axioms. Axiomatic characterizations of the core on the domain of two-sided assignment markets are given in Sasaki (1995) and Toda (2003, 2005). Most of them make use of some monotonicity property that is not satisfied by the core in the three-sided case. The reason is that when we raise the value of a triplet, a pair or an individual in a three-sided market, the new market may fail to have core elements. This is why the previous characterizations cannot be straightforwardly extended to the three-sided case.

The last part of the paper is devoted to the study of a subclass of balanced generalized three-sided assignment markets. Besides non-negativeness, two additional properties define this subclass: a) the worth of a triplet is the addition of the worths of the three pairs that can be formed with its members and b) there is an optimal partition such that, when restricted to each pair of sectors, is also optimal for the related two-sided market. Given any market in this subclass, we construct a core element from any three selected core elements, one from each of the three associated two-sided markets.

This subclass of generalized three-sided assignment markets is inspired by the balanced subclass introduced by Quint (1991) and the supplier-firm-buyer market of Stuart (1997), where also the value of a triplet is obtained by the addition of the value of some of the pairs that can be formed with its elements. However, in their classes, such a pair cannot attain its value if not matched with an agent of the remaining sector.

The paper is organized as follows. The model is described in Section 2. The definition of the derived market and game and their main properties are given in Section 3. The derived consistency of the core and the nucleolus is proved in Section 4, and an axiomatic characterization of the core is presented in Section 5. Section 6 introduces the aforementioned subclass of balanced generalized three-sided assignment markets.

2 The model

In this section, we introduce a generalized three-sided assignment market and its corresponding assignment game.

Let $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$ be three countable disjoint sets. A *generalized three-sided assignment market* $\gamma = (M_1, M_2, M_3; v)$ consists of three different sectors, $M_1 \subseteq \mathcal{U}_1$, $M_2 \subseteq \mathcal{U}_2$, and $M_3 \subseteq \mathcal{U}_3$ with a finite number of agents each, such that $N = M_1 \cup M_2 \cup M_3 \neq \emptyset$, and a valuation fuction v. The *essential coalitions* in this market are the ones formed by exactly one agent of each sector and all their possible subcoalitions. Let us denote by $\mathcal{B}$ this set of essential coalitions,

$$\mathcal{B} = \{\{i,j,k\} \mid i \in M_1, j \in M_2, k \in M_3\} \cup \{\{i,j\} \mid i \in M_r, j \in M_s,\ r,s \in \{1,2,3\}, r \neq s\} \\ \cup \{\{i\} \mid i \in M_1 \cup M_2 \cup M_3\}.$$

The valuation function v, from the set $\mathcal{B}$ to the real numbers, $\mathbb{R}$, associates to each essential coalition a real number $v(S)$. The corresponding value $v(S)$ for each subset $S \in \mathcal{B}$ is called the *worth of the essential coalition*.

Given a generalized three-sided assignment market $\gamma = (M_1, M_2, M_3; v)$, for each non-empty coalition $S \subseteq N = M_1 \cup M_2 \cup M_3$ we can define a *submarket* $\gamma_{|S} = (M_1 \cap S, M_2 \cap S, M_3 \cap S; v_{|S})$ where $(v_{|S})(T) = v(T)$ for all $T \in \mathcal{B}^S = \{R \in \mathcal{B} \mid R \subseteq S\}$. Notice that if one of the sectors is empty, then this generalized three-sided assignment market is a two-sided assignment market with reservation values as introduced in Owen (1992).

Given $\emptyset \neq S \subseteq N$, a *matching* μ on $S = S_1 \cup S_2 \cup S_3$, where $S_1 \subseteq M_1$, $S_2 \subseteq M_2$, and $S_3 \subseteq M_3$, is a partition of S in coalitions of $\mathcal{B}^S$. Let $\mathcal{M}(S_1, S_2, S_3)$ be the set of all possible matchings for coalition S. We define the value of coalition S at this matching as $w(S_1, S_2, S_3; \mu) = \sum_{T \in \mu} v(T)$. A matching $\mu \in \mathcal{M}(S_1, S_2, S_3)$ is *optimal* for the submarket $\gamma_{|S}$ if $w(S_1, S_2, S_3; \mu) = \max_{\mu' \in \mathcal{M}(S_1,S_2,S_3)} \{w(S_1, S_2, S_3; \mu')\}$. We denote by $\mathcal{M}_\gamma(S_1, S_2, S_3)$ the set of optimal matchings for the market $\gamma_{|S}$.

Given a generalized three-sided assignment market $\gamma = (M_1, M_2, M_3; v)$, its corresponding generalized three-sided assignment game[1] is a pair (N, w_γ) where $N = M_1 \cup M_2 \cup M_3$ is the player set and the characteristic function w_γ satisfies $w_\gamma(\emptyset) = 0$ and for all $S \subseteq N$,

$$w_\gamma(S) = \max_{\mu \in \mathcal{M}(S_1,S_2,S_3)} \{w(S_1, S_2, S_3; \mu)\} = \max_{\mu \in \mathcal{M}(S_1,S_2,S_3)} \left\{ \sum_{T \in \mu} v(T) \right\},$$

where $S_1 = S \cap M_1$, $S_2 = S \cap M_2$ and $S_3 = S \cap M_3$.

Notice that the valuation function v of a generalized three-sided assignment market can be represented by a 3-dimensional matrix A collecting the value of coalitions $\{i, j, k\}$ with $i \in M_1$, $j \in M_2$ and $k \in M_3$, a vector $Q = (q^1, q^2, q^3) \in \mathbb{R}^N$, $q^i \in \mathbb{R}^{M_i}$ for all $i \in \{1, 2, 3\}$, representing the reservation value of each agent when being alone, and one 2-dimensional matrix for each two different sectors, that represents the join reservation value of each mixed-pair of agents in these two sectors (when not matched with another agent of the third sector).

Example 1. Consider a generalized three-sided assignment market $\gamma = (M_1, M_2, M_3; v)$ where $M_1 = \{1, 2, 3\}$, $M_2 = \{1', 2', 3'\}$, $M_3 = \{1'', 2'', 3''\}$, and v is defined by one three-dimensional matrix A for the valuation of triplets,

$$A = \left[\underset{1''}{\begin{array}{c c} & \begin{array}{ccc} 1' & 2' & 3' \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} \end{array}}, \quad \underset{2''}{\begin{array}{c c} & \begin{array}{ccc} 1' & 2' & 3' \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} \end{array}}, \quad \underset{3''}{\begin{array}{c c} & \begin{array}{ccc} 1' & 2' & 3' \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} \end{array}} \right]$$

one valuation matrix for each pair of sectors

$$B^{12} = \begin{array}{c c} & \begin{array}{ccc} 1' & 2' & 3' \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}, \qquad B^{13} = \begin{array}{c c} & \begin{array}{ccc} 1'' & 2'' & 3'' \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}, \qquad B^{23} = \begin{array}{c c} & \begin{array}{ccc} 1'' & 2'' & 3'' \end{array} \\ \begin{array}{c} 1' \\ 2' \\ 3' \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}$$

and a vector of individual valuations $Q = (q^1, q^2, q^3) = (1, 0, 0; 1, 1, 0; 0, 1, 1)$. For instance, $v(\{1, 2', 1''\}) = a_{121} = 3$ while $v(\{2, 1''\}) = b^{13}_{21} = 1$ and $v(\{2'\}) = q^2_2 = 1$.

[1] A game is a pair formed by a finite set of players N and a characteristic function r that assigns a real number $r(S)$ to each coalition $S \subseteq N$, with $r(\emptyset) = 0$. The core of a coalitional game (N, r) is $C(r) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = r(N), \sum_{i \in S} x_i \geq r(S) \text{ for all } S \subseteq N\}$.

Let us remark at this point that this class of generalized assignment games is "larger" than the classical three-sided assignment games as considered for instance in Quint (1991). We mean that, different from the case of the generalized two-sided assignment games of Owen (1992), which are always strategically equivalent to a classical Shapley and Shubik (1972) assignment game, a generalized three-sided assignment game is in general not strategically equivalent to any three-sided assignment game as defined in Quint (1991), unless $v(\{i,j\}) = v(\{i\}) + v(\{j\})$ for all $\{i,j\} \in \mathcal{B}$. From now on, we denote by Γ_{3-SAG} indistinctly the set of generalized three-sided assignment markets or games.

An outcome for a generalized three-sided assignment market will be a matching and a distribution of the profits of this matching among the agents that take part.

Given $\gamma = (M_1, M_2, M_3; v)$, a *payoff vector* is $x \in \mathbb{R}^N$, where x_i stands for the payoff of player $i \in N$. We write $x_{|S}$ to denote the projection of a payoff vector x to agents in coalition $S \subseteq N$. Moreover, $x(S) = \sum_{i \in S} x_i$ with $x(\emptyset) = 0$. A payoff vector $x \in \mathbb{R}^N$ is *individually rational* for γ if $x_i \geq w_\gamma(\{i\})$ for all $i \in N$, and efficient if $x(N) = w_\gamma(N)$.

The core of a generalized three-sided assignment market $\gamma = (M_1, M_2, M_3; v)$ is the core of the asssociated assignment game (N, w_γ), being $N = M_1 \cup M_2 \cup M_3$. It is straightforward to see that this core is formed by those efficient payoff vectors that satisfy coalitional rationality for all coalitions in $\mathcal{B}$,[2]

$$C(\gamma) = \{x \in \mathbb{R}^{M_1} \times \mathbb{R}^{M_2} \times \mathbb{R}^{M_3} \mid x(N) = w_\gamma(N), x(S) \geq w_\gamma(S) \text{ for all } S \in \mathcal{B}\}.$$

As a consequence, given any optimal matching μ, if $x \in C(\gamma)$, then $x(S) = w_\gamma(S)$ for all $S \in \mu$. Since this class is a generalization of the well-known three-sided assignment games, the core may be empty. However, the reader may check that the core of the game in Example 1 above is non-empty since it contains, for instance, the allocation $(1, 1, 1; 1, 1, 1; 1, 1, 1)$.

3 The derived market

In this section, we introduce the corresponding derived market (and game) for the generalized three-sided assignment market.

Given any coalitional game, and given a particular distribution of the worth of the grand coalition, we may ask what happens when some agents leave the market after being paid according to that given distribution. The agents that remain must reevaluate the worth of all the coalitions they can form. The different ways in which this reevaluation is done correspond to the different notions of reduced game that exist in the literature.

Maybe the best known notion of reduced game is that of Davis and Maschler (1965), where the remaining coalitions take into account what they could obtain by joining some agents that have left with the condition of preserving for them the amount they have already been paid.

[2]Notice that our essential coalitions are precisely the "essential" coalitions of the game (N, w_γ) in the sense of Huberman (1980).

Definition 2 (Davis and Maschler, 1965)**.** Given a generalized three-sided assignment game (N, w_γ), a non-empty coalition S and a payoff vector $x \in \mathbb{R}^{N\setminus S}$, the *Davis and Maschler reduced game* for the coalition S at x is the game $(S, w_\gamma^{S,x})$ that is defined by

$$w_\gamma^{S,x}(T) = \begin{cases} 0 & \text{if } T = \emptyset, \\ w_\gamma(N) - x(N \setminus S) & \text{if } T = S, \\ \max\limits_{Q \subseteq N\setminus S} \{w_\gamma(T \cup Q) - x(Q)\} & \text{otherwise.} \end{cases}$$

In general, the reduced game of a generalized three-sided assignment game is not superadditive as the following example illustrates.

Example 3. Take again the assignment market of Example 1 and consider the coalition $S = \{2, 3, 1', 2', 2'', 3''\}$ and the core allocation $x = (1, 1, 1; 1, 1, 1; 1, 1, 1)$. Consider also subcoalitions $T_1 = \{2\}$, $T_2 = \{3\}$ and $T_3 = \{2, 3\}$. When computing the worth of these subcoalitions in the reduced game for the coalition S at x, we find

$$w_\gamma^{S,x}(\{2, 3\}) = 1 < 2 = w_\gamma^{S,x}(\{2\}) + w_\gamma^{S,x}(\{3\}).$$

Thus, this game is not superadditive. This implies that the Davis and Maschler reduced game of a generalized three-sided assignment game is in general not a generalized three-sided assignment game.

To solve this, we introduce a new reduced generalized three-sided assignment market (and game) that extends the derived game introduced by Owen (1992) for the two-sided case. We will see that this notion of reduced game is closely related to the Davis and Machler reduced game.

Definition 4. Given a generalized three-sided assignment market $\gamma = (M_1, M_2, M_3; v)$, $\emptyset \neq S = S_1 \cup S_2 \cup S_3$, $S \neq N$, where $S_1 \subseteq M_1$, $S_2 \subseteq M_2$, $S_3 \subseteq M_3$ and $x \in \mathbb{R}^{N\setminus S}$, the *derived market* at S and x is $\hat{\gamma}^{S,x} = (S_1, S_2, S_3; \hat{v}^{S,x})$ where

$$\hat{v}^{S,x}(T) = \max_{\substack{Q \subseteq N\setminus S \\ T \cup Q \in \mathcal{B}}} \{v(T \cup Q) - x(Q)\} \text{ for all } T \in \mathcal{B}^S. \tag{1}$$

Then, the corresponding *derived game* at S and x is $(S, w_{\hat{\gamma}^{S,x}})$ where for all $R \subseteq S$,

$$w_{\hat{\gamma}^{S,x}}(R) = \max_{\mu \in \mathcal{M}(M_1 \cap R, M_2 \cap R, M_3 \cap R)} \Big\{ \sum_{T \in \mu} \hat{v}^{S,x}(T) \Big\}.$$

Notice that to obtain the derived game, we first consider the valuation in the reduced situation of the essential coalitions of the submarket. The valuation of these essential coalitions of the submarket is obtained by allowing them to cooperate only with agents that have left but with whom they can form an essential coalition of the initial market. In particular, when $T = \{i, j, k\}$ with $i \in S_1$, $j \in S_2$ and $k \in S_3$, then $\hat{v}^{S,x}(\{i, j, k\}) = v(\{i, j, k\})$. Thus, the worth $w_{\hat{\gamma}^{S,x}}(R)$ in the derived game for any coalition $R \subseteq S$ is obtained from the valuations $\hat{v}^{S,x}$ of the essential coalitions in $\mathcal{B}^S$ by imposing superadditivity. Hence, the derived assignment game is always a superadditive game.

Notice that if one of the sectors is empty, then the market is a two-sided market (with individual reservation values) and the definition of derived game coincides with the one given by Owen (1992) for these markets.

Given a game (N, w), its *superadditive cover* is the minimal superadditive game $(N, \tilde{w})$ such that $\tilde{w} \geq w$. Next theorem shows that for any generalized three-sided assignment game (N, w_γ), its derived game $(S, w_{\hat{\gamma}^{S,x}})$ at any coalition S and core allocation x is the superadditive cover of the corresponding Davis and Maschler reduced game $(S, w_\gamma^{S,x})$.

Theorem 5. *Let $\gamma = (M_1, M_2, M_3; v)$ be a generalized three-sided assignment market, $N = M_1 \cup M_2 \cup M_3$, (N, w_γ) the associated generalized three-sided game and $x \in C(\gamma)$. Then for any $\emptyset \neq S \subsetneq N$, the derived game $(S, w_{\hat{\gamma}^{S,x}})$, where $\hat{\gamma}^{S,x} = (M_1 \cap S, M_2 \cap S, M_3 \cap S; \hat{v}^{S,x})$, is the superadditive cover of the Davis and Maschler reduced game $(S, w_\gamma^{S,x})$.*

Proof. Let us write $\hat{w} = w_{\hat{\gamma}^{S,x}}$. We have to show that $\hat{w}$ is superadditive, $\hat{w} \geq w_\gamma^{S,x}$ and $\hat{w}$ is minimal with these two properties.

By definition, $\hat{w}$ is superadditive. Now, we show that $\hat{w}(T) \geq w_\gamma^{S,x}(T)$ for all $T \subseteq S$. Notice that, for all $T \subseteq S$ there exists $Q \subseteq N \setminus S$ such that

$$\begin{aligned} w_\gamma^{S,x}(T) &= w_\gamma(T \cup Q) - \sum_{l \in Q} x_l \\ &= w(T \cup Q; \mu) - \sum_{l \in Q} x_l \end{aligned} \tag{2}$$

for some matching μ on $T \cup Q$. We introduce the following partition of the set of coalitions in μ:
$I_1 = \{\{i,j,k\} \in \mu \mid i \in T, j \in T, k \in T\}$
$I_2 = \{\{i,j,k\} \in \mu \mid i \notin T, j \notin T, k \notin T\}$
$I_3 = \{\{i,j,k\} \in \mu \mid i \in T, j \in T, k \notin T\}$
$I_4 = \{\{i,j,k\} \in \mu \mid i \in T, j \notin T, k \notin T\}$
$I_5 = \{\{i,j\} \in \mu \mid i \in T, j \in T\}$
$I_6 = \{\{i,j\} \in \mu \mid i \notin T, j \notin T\}$
$I_7 = \{\{i,j\} \in \mu \mid i \in T, j \notin T\}$
$I_8 = \{\{i\} \in \mu \mid i \in T\}$.
$I_9 = \{\{i\} \in \mu \mid i \notin T\}$.

We write $w(T \cup Q; \mu)$ that appears in equation (2) in terms of the above partition.

$$\begin{aligned} w(T \cup Q; \mu) = &\sum_{\{i,j,k\} \in I_1} v(\{i,j,k\}) + \sum_{\{i,j,k\} \in I_2} v(\{i,j,k\}) + \sum_{\{i,j,k\} \in I_3} v(\{i,j,k\}) \\ &+ \sum_{\{i,j,k\} \in I_4} v(\{i,j,k\}) + \sum_{\{i,j\} \in I_5} v(\{i,j\}) + \sum_{\{i,j\} \in I_6} v(\{i,j\}) \\ &+ \sum_{\{i,j\} \in I_7} v(\{i,j\}) + \sum_{\{i\} \in I_8} v(\{i\}) + \sum_{\{i\} \in I_9} v(\{i\}). \end{aligned} \tag{3}$$

Then, substitute (3) in equation (2) and distribute $\sum_{l\in Q} x_l$ among the sets of the partition.

$$\begin{aligned}
w_\gamma^{S,x}(T) &= v(T\cup Q;\mu) - \sum_{i\in Q} x_i \\
&= \sum_{\{i,j,k\}\in I_1} v(\{i,j,k\}) + \sum_{\{i,j,k\}\in I_2} v(\{i,j,k\}) - x_i - x_j - x_k \\
&+ \sum_{\{i,j,k\}\in I_3} v(\{i,j,k\}) - x_k + \sum_{\{i,j,k\}\in I_4} v(\{i,j,k\}) - x_j - x_k \\
&+ \sum_{\{i,j\}\in I_5} v(\{i,j\}) + \sum_{\{i,j\}\in I_6} v(\{i,j\}) - x_i - x_j + \sum_{\{i,j\}\in I_7} v(\{i,j\}) - x_j \\
&+ \sum_{\{i\}\in I_8} v(\{i\}) + \sum_{\{i\}\in I_9} v(\{i\}) - x_i.
\end{aligned}$$

Since $x \in C(\gamma)$, the second, the sixth and the last term are non-positive.

Let it be $\hat{v} = \hat{v}^{S,x}$ as defined in (1). For all $t, r, s \in \{1,2,3\}$ such that $r \neq s$, $r \neq t$, $s \neq t$ and all $i \in M_r \cap T$, $j \in M_s \cap T$,

$$\hat{v}(\{i,j\}) = \max_{k\in Q\cap M_t} \{v(\{i,j,k\}) - x_k, v(\{i,j\})\}.$$

As a consequence, for all $\{i,j,k\} \in I_3$, $v(\{i,j,k\}) - x_k \leq \hat{v}(\{i,j\})$ and for all $\{i,j\} \in I_5$, $v(\{i,j\}) \leq \hat{v}(\{i,j\})$.

Also, for all $t \in \{1,2,3\}$ and $l \in M_t \cap T$, if r, s are such that $r \neq s$, $s \neq t$ and $s \neq r$, then,

$$\hat{v}(\{l\}) = \max_{\substack{i\in M_r\cap Q\\ j\in M_s\cap Q}} \{v(\{i,j,l\}) - x_i - x_j, v(\{i,l\}) - x_i, v(\{j,l\}) - x_j, v(\{l\})\}.$$

As a consequence, for all $\{i,j,k\} \in I_4$, $v(\{i,j,k\}) - x_j - x_k \leq \hat{v}(\{i\})$; for all $\{i,j\} \in I_7$, $v(\{i,j\}) - x_j \leq \hat{v}(\{i\})$ and trivially $v(\{i\}) \leq \hat{v}(\{i\})$ for all $\{i\} \in I_8$.

To sum up, taking into account that $\hat{w}$ is superadditive by definition,

$$w_\gamma^{S,x}(T) \leq \sum_{\{i,j,k\}\in I_1} \hat{v}(\{i,j,k\}) + \sum_{\substack{\{i,j,k\}\in I_3\\ \{i,j\}\in I_5}} \hat{v}(\{i,j\}) + \sum_{\substack{\{i,j,k\}\in I_4\\ \{i,j\}\in I_7\\ \{i\}\in I_8}} \hat{v}(\{i\}) \leq \hat{w}(T).$$

Now, we only need to show that $\hat{w}$ is the minimal superadditive game satisfying the above inequality. First, consider $\{k\} \in \mathcal{B}^S$. Then,

$$\begin{aligned}
w_\gamma^{S,x}(\{k\}) &= \max_{Q\subseteq N\setminus S} \{w_\gamma(\{k\}\cup Q) - x(Q)\} \\
&\geq \max_{\substack{Q\subseteq N\setminus S\\ \{k\}\cup Q\in\mathcal{B}}} \{w_\gamma(\{k\}\cup Q) - x(Q)\} \\
&\geq \max_{\substack{Q\subseteq N\setminus S\\ \{k\}\cup Q\in\mathcal{B}}} \{v(\{k\}\cup Q) - x(Q)\} \qquad (4)\\
&= \hat{v}(\{k\}).
\end{aligned}$$

Secondly, for all $\{i,j\} \in \mathcal{B}^S$,

$$\begin{aligned} w_\gamma^{S,x}\{i,j\} &= \max_{Q\subseteq N\setminus S} \{w_\gamma(\{i,j\}\cup Q) - x(Q)\} \\ &\geq \max_{\substack{Q\subseteq N\setminus S \\ \{i,j\}\cup Q\in\mathcal{B}}} \{w_\gamma(\{i,j\}\cup Q) - x(Q)\} \\ &\geq \max_{\substack{Q\subseteq N\setminus S \\ \{i,j\}\cup Q\in\mathcal{B}}} \{v(\{i,j\}\cup Q) - x(Q)\} \qquad (5) \\ &= \hat{v}(\{i,j\}). \end{aligned}$$

Finally, for all $\{i,j,k\} \in \mathcal{B}^S$,

$$\begin{aligned} w_\gamma^{S,x}(\{i,j,k\}) &= \max_{Q\subseteq N\setminus S} \{w_\gamma(\{i,j,k\}\cup Q) - x(Q)\} \\ &\geq \max_{\substack{Q\subseteq N\setminus S \\ \{i,j,k\}\cup Q\in\mathcal{B}}} \{w_\gamma(\{i,j,k\}\cup Q) - x(Q)\} \\ &\geq \max_{\substack{Q\subseteq N\setminus S \\ \{i,j,k\}\cup Q\in\mathcal{B}}} \{v(\{i,j,k\}\cup Q) - x(Q)\} \qquad (6) \\ &= \hat{v}(\{i,j,k\}). \end{aligned}$$

Assume now (N,w) is superadditive and $w \geq w_\gamma^{S,x}$ and for all $T \subseteq S$, let μ be an optimal matching for $\hat{\gamma}_{|T}^{S,x}$, $\mu \in \mathcal{M}_\gamma(M_1\cap T, M_2\cap T, M_3\cap T)$. Then,

$$\begin{aligned} w(T) &\geq \sum_{\{i,j,k\}\in\mu} w(\{i,j,k\}) + \sum_{\{i,j\}\in\mu} w(\{i,j\}) + \sum_{\{k\}\in\mu} w(\{k\}) \\ &\geq \sum_{\{i,j,k\}\in\mu} w_\gamma^{S,x}(\{i,j,k\}) + \sum_{\{i,j\}\in\mu} w_\gamma^{S,x}(\{i,j\}) + \sum_{\{k\}\in\mu} w_\gamma^{S,x}(\{i,j\}) \\ &\geq \sum_{\{i,j,k\}\in\mu} \hat{v}(\{i,j,k\}) + \sum_{\{i,j\}\in\mu} \hat{v}(\{i,j\}) + \sum_{\{k\}\in\mu} \hat{v}(\{k\}) \\ &= \hat{w}(T), \end{aligned}$$

where the last inequality follows from (4), (5) and (6).

This shows that $\hat{w}$ is the minimal superadditive game such that $\hat{w} \geq w_\gamma^{S,x}$, which implies that $\hat{w}$ is the superadditive cover of $w_\gamma^{S,x}$. □

4 Derived consistency of the core and the nucleolus

In this section, for the class of generalized three-sided assignment markets, we introduce a consistency property with respect to the derived market. We name this property *derived consistency.*

Before doing that, we need to introduce the notion of solution in the class Γ_{3-SAG} of generalized three-sided assignment markets or games. Next definition extends to our setting the notion of feasibility that is usual in two-sided assignment markets. See, for instance, Toda (2005).

Definition 6. Given a generalized three-sided assignment market $\gamma = (M_1, M_2, M_3; v)$, an allocation $x \in \mathbb{R}^{M_1} \times \mathbb{R}^{M_2} \times \mathbb{R}^{M_3}$ is *feasible-by-matching* if there exists a matching $\mu \in \mathcal{M}(M_1, M_2, M_3)$ such that for all $S \in \mu$, $x(S) = v(S)$.

In that case, we say that x and μ are compatible. Notice that a matching μ compatible with x may not be optimal. Moreover, the set of feasible-by-matching allocations is always non-empty since we can take the matching $\mu = \{\{i\}\}_{i\in N}$ and then $x = (v(\{i\}))_{i\in N}$ is feasible with respect to μ.

Definition 7. A *solution* on a class $\Gamma \subseteq \Gamma_{3-SAG}$ is a correspondence σ that assigns a subset of feasible-by-matching payoff vectors to each $\gamma \in \Gamma$.

Given $\gamma \in \Gamma$, we write $\sigma(\gamma)$ to denote the subset of feasible-by-matching payoff vectors assigned by solution σ to the assignment market γ. Notice that a solution σ is allowed to be empty. The core correspondence and the mapping that gives to each agent his/her individual value (compatible with the empty matching) are examples of solutions on the class of generalized three-sided assignment markets. Similarly, the nucleolus, which will be defined below, is a solution on the subclass of balanced generalized three-sided assignment markets.

Definition 8. A solution σ on the class of generalized assignment markets satisfies *derived consistency* if for all $\gamma = (M_1, M_2, M_3; v)$, all $\emptyset \neq S \subsetneq N$ and all $x \in \sigma(\gamma)$, then $x_{|S} \in \sigma(\hat{\gamma}^{S,x})$.

Next theorem shows that the core satisfies derived consistency on the domain of generalized three-sided assignment markets.

Theorem 9. *On the domain of generalized three-sided assignment markets, the core satisfies derived consistency.*

Proof. Let $\gamma = (M_1, M_2, M_3; v)$ be a generalized three-sided assignment market, let x be a core allocation and $\emptyset \neq S \subsetneq M_1 \cup M_2 \cup M_3$. To simplify notation, let us write $\hat{v} = \hat{v}^{S,x}$ and $\hat{w} = w_{\hat{\gamma}^{S,x}}$.

Consider all possible coalitions in $\mathcal{B}^S$. First, for all $\{i,j,k\} \in M_1 \cap S \times M_2 \cap S \times M_3 \cap S$, $x_i + x_j + x_k \geq v(\{i,j,k\}) = \hat{v}(\{i,j,k\})$. Secondly, for all $\{i,j\} \in (M_1 \cap S) \times (M_2 \cap S)$, $x_i + x_j \geq v(\{i,j\})$ and $x_i + x_j \geq v(\{i,j,k\}) - x_k$ for all $k \in M_3 \setminus S$. Hence, $x_i + x_j \geq \hat{v}(\{i,j\})$. Finally, for all $i \in M_1 \cap S$, $x_i \geq v(\{i\})$, and $x_i \geq v(\{i,j\}) - x_j$ for all $j \in M_2 \setminus S$, and $x_i \geq v(\{i,k\}) - x_k$ for all $k \in M_3 \setminus S$, and $x_i \geq v(\{i,j,k\}) - x_j - x_k$ for all $j \in M_2 \setminus S$ and for all $k \in M_3 \setminus S$. Hence, $x_i \geq \hat{v}(\{i\})$. Proceeding similarly for the remaining $T \in \mathcal{B}^S$, we obtain

$$x(T) \geq \hat{v}(T) \text{ for all } T \in \mathcal{B}^S. \tag{7}$$

Finally, it remains to show that $x(S) = \hat{w}(S)$. Expression (7) implies $x(R) \geq \hat{w}(R)$ for all $R \subseteq S$. Let us denote by $(S, \widetilde{w_\gamma^{S,x}})$ the superadditive cover of the Davis and Maschler reduced game $(S, w_\gamma^{S,x})$. Now, appyling Theorem 5 we obtain

$$x(S) \geq \hat{w}(S) = \widetilde{w_\gamma^{S,x}}(S) \geq w_\gamma^{S,x}(S) = x(S),$$

where the last equality follows from the Davis and Maschler reduced game property of the core (see Peleg, 1986). Thus, $x(S) = \hat{w}(S)$ and this completes the proof of $x_{|S} \in C(\hat{\gamma}^{S,x})$. $\square$

The *nucleolus* is a well-known single-valued solution for coalitional games introduced by Schmeidler (1969). When the game is balanced, the nucleolus is the unique core allocation that lexicographically minimizes the vector of decreasingly-ordered excesses of coalitions.[3]

The nucleolus of a generalized three-sided assignment market $\gamma = (M_1, M_2, M_3; v)$ is the nucleolus of the associated assignment game (N, w_γ), and it will be denoted by $\eta(\gamma)$. Next, we show that when a generalized three-sided assignment market is balanced the nucleolus also satisfies derived consistency.

Theorem 10. *On the class of balanced generalized three-sided assignment markets, the nucleolus satisfies derived consistency.*

Proof. Let $\gamma = (M_1, M_2, M_3; v)$ be a balanced generalized three-sided assignment market, $\eta(\gamma) = \eta$ be the nucleolus and $\emptyset \neq S \subsetneq M_1 \cup M_2 \cup M_3$. Since the nucleolus satisfies the Davis and Maschler reduced game property (Potters, 1991), $\eta_{|S} = \eta(w_\gamma^{S,\eta})$ which implies $\eta(S) = w_\gamma^{S,\eta}(S)$. On the other hand, since $\eta \in C(\gamma)$, by Theorem 9 we know that $\eta_{|S} \in C(w_{\hat{\gamma}^{S,\eta}})$ which implies $\eta(S) = w_{\hat{\gamma}^{S,\eta}}(S)$. Hence, taking into account Theorem 5, we have $\widetilde{w_\gamma^{S,\eta}}(S) = w_{\hat{\gamma}^{S,\eta}}(S) = \eta(S) = w_\gamma^{S,\eta}(S)$, being $(S, \widetilde{w_\gamma^{S,\eta}})$ the superadditive cover of the Davis and Maschler reduced game $(S, w_\gamma^{S,\eta})$. Miquel and Núñez (2011) show that when a balanced game and its superadditive cover have the same efficiency level, then the nucleolus of both games coincide. Therefore, $\eta_{|S} = \eta(w_{\hat{\gamma}^{S,\eta}})$. $\square$

Next proposition shows that any solution σ on the domain Γ_{3-SAG} that satisfies derived consistency always selects a subset of the core, that is, $\sigma(\gamma) \subseteq C(\gamma)$ for all $\gamma \in \Gamma_{3-SAG}$.

Proposition 11. *On the domain of generalized three-sided assignment markets, derived consistency implies core selection.*

Proof. We want to show that any non-empty solution σ on the domain of Γ_{3-SAG}, that satisfies derived consistency, consists of core elements.

Let σ be a solution on Γ_{3-SAG} satisfying derived consistency and take $x \in \sigma(\gamma)$, being $\gamma = (M_1, M_2, M_3; v)$. We need to show that x satisfies coalitional rationality and efficiency. Notice that if two sides of the market are empty, then the game is modular[4] and since σ is feasible-by-matching, $\sigma(\gamma) = \{(v(\{i\}))_{i\in N}\} = C(\gamma)$. When only one side is empty, the game is a two-sided assignment market and the statement follows from Llerena et al (2014). So, we can assume without loss of generality that $M_l \neq \emptyset$ for all $l \in \{1, 2, 3\}$. Then, for all $i \in M_1 \cup M_2 \cup M_3$ consider the derived market relative to $S = \{i\}$ at x. By derived consistency of σ, $x_i \in \sigma(\hat{\gamma}^{\{i\},x})$. Moreover, feasibility-by-matching of σ implies that $x_i = \hat{v}^{\{i\},x}(\{i\})$. Now, let $E \in \mathcal{B}$ be any

[3] Given a game (N, r), the excess of a coalition $S \subseteq N$ at a payoff vector $x \in \mathbb{R}^N$ is $r(S) - \sum_{i\in S} x_i$.

[4] A game (N, v) is modular if, for all $S \subseteq N$, $v(S) = \sum_{i\in S} v(\{i\})$.

essential coalition such that $i \in E$. By definition of derived market at $\{i\}$ and x we have $x_i = \hat{v}^{\{i\},x}(\{i\}) \geq v(E) - \sum_{k\in E\setminus\{i\}} x_k$. Hence, $\sum_{k\in E} x_k \geq v(E)$ which states that x satisfies coalitional rationality.

In order to prove efficiency, let μ be an optimal matching and μ' be a matching compatible with x. Then, $w_\gamma(N) = \sum_{S\in\mu} v(S) \leq \sum_{S\in\mu}(\sum_{i\in S} x_i) = \sum_{S\in\mu'}(\sum_{i\in S} x_i) = \sum_{S\in\mu'} v(S)$, where the last equality follows from the fact that μ' is compatible with x. Since μ is optimal and $w_\gamma(N) \leq \sum_{S\in\mu'} v(S)$, we get that μ' is also optimal and x is efficient, which concludes the proof. □

5 An axiomatic characterization of the core

In this section, we give an axiomatic characterization of the core on the class of generalized three-sided assignment markets, Γ_{3-SAG}. As mentioned in the introduction, other known characterizations of the core of two-sided assignment markets rely on monotonicity properties that are not satisfied by the core on the domain of three-sided assignment markets. In the present characterization we make use of derived consistency and two additional properties, *singleness best* and *individual anti-monotonicity*, that are introduced in the sequel.

Definition 12. A solution σ on $\Gamma \subseteq \Gamma_{3-SAG}$ satisfies *singleness best* if whenever the partition in singletons is optimal, $(v(\{i\}))_{i\in N} \in \sigma(\gamma)$ holds.

Given two payoff vectors $x = (x_i)_{i\in N}$, $x' = (x'_i)_{i\in N}$ in $\mathbb{R}^N$ and $\mu \in \mathcal{M}(M_1, M_2, M_3)$, we write $x' \geq_\mu x$ when $x_i = x'_i$ for all $\{i\} \in \mu$ and $x'_i \geq x_i$ if $\{i\} \notin \mu$. That is, x' is greater than x with respect to μ when agents that are matched with some other partner receive at least as much in x' than in x, while agents that are alone receive the same payoff in both allocations.

Definition 13. A solution σ on $\Gamma \subseteq \Gamma_{3-SAG}$ satisfies *individual anti-monotonicity* if for all $\gamma' = (M_1, M_2, M_3; v') \in \Gamma$, all $\gamma = (M_1, M_2, M_3; v) \in \Gamma$, all $u \in \sigma(\gamma')$ and μ compatible with u, if $v(E) = v'(E)$ for all $E \in \mathcal{B}$ with $|E| > 1$ and $(v'(\{i\}))_{i\in N} \geq_\mu (v(\{i\}))_{i\in N}$, then it holds $u \in \sigma(\gamma)$.

Singleness best simply says that if remaining unmatched is optimal for every player, then the vector of individual values should be an outcome of the solution. This axiom has some resemblance with the *zero inessential game property* of Hwang and Sudhölter (2001) in the sense that it is a non-emptiness axiom for generalized three-sided assignment games that are trivial or inessential. Individual anti-monotonicity says that if the individual values decrease (in the sense defined above) any payoff vector in the solution of the original market should remain in the solution of the new market. Notice that the value of pairs and triplets coincide in both markets. Individual anti-monotinicity is a weaker version of *anti-monotonicity* introduced by Keiding (1986) and also used by Toda (2003).

Now, we characterize the core on the class of generalized three-sided assignment games, Γ_{3-SAG}, by means of derived consistency, singleness best and individual anti-monotonicity.

Theorem 14. *On the domain Γ_{3-SAG}, the core is the unique solution that satisfies derived consistency, singleness best and individual anti-monotonicity.*

Proof. It is straightforward that the core satisfies all three axioms. Assume now that σ is a solution on Γ_{3-SAG} also satisfying all three axioms. Take any $\gamma = (M_1, M_2, M_3; v) \in \Gamma_{3-SAG}$. By Proposition 11, we know that $\sigma(\gamma) \subseteq C(\gamma)$. We have to show that $C(\gamma) \subseteq \sigma(\gamma)$. Take $u \in C(\gamma)$ and $\mu \in \mathcal{M}(M_1, M_2, M_3)$ compatible with u. Then, μ is optimal for γ. Now, define a market $\gamma' = (M_1, M_2, M_3; v')$ where $v'(E) = v(E)$ for all $E \in \mathcal{B}$ such that $|E| > 1$ and $v'(E) = u_i$ for all $E = \{i\}$. Notice that $v'(\{i\}) = u_i = v(\{i\})$ for all $\{i\} \in \mu$ and $v'(\{i\}) = u_i \geq v(\{i\})$ for all $\{i\} \notin \mu$. Hence, $(v'(\{i\}))_{i\in N} \geq_\mu (v(\{i\}))_{i\in N}$. Let us see that $\mu' = \{\{i\} \mid i \in N\}$ is optimal for γ'. To this end, take any matching $\mu'' \in \mathcal{M}(M_1, M_2, M_3)$. Then,

$$\begin{aligned}\sum_{E\in\mu'} v'(E) = \sum_{i\in N} v'(\{i\}) = \sum_{i\in N} u_i &= \sum_{\substack{E\in\mu''\\|E|>1}} \sum_{i\in E} u_i + \sum_{\substack{E\in\mu''\\|E|=1}} \sum_{i\in E} u_i \\ &\geq \sum_{\substack{E\in\mu''\\|E|>1}} v'(E) + \sum_{\substack{E\in\mu''\\|E|=1}} v'(E) = \sum_{E\in\mu''} v'(E).\end{aligned}$$

The inequality follows from the fact that $u \in C(\gamma)$ and the relationship between v and v'. Thus, μ' is optimal for γ'. By singleness best, $u = (u_i)_{i\in N} = (v'(\{i\}))_{i\in N} \in \sigma(\gamma')$ and then, by individual anti-monotonicity, $u \in \sigma(\gamma)$. Hence, $C(\gamma) \subseteq \sigma(\gamma)$. Together with the reverse inclusion, $\sigma(\gamma) \subseteq C(\gamma)$, we conclude that $C(\gamma) = \sigma(\gamma)$. $\square$

We now show that no axiom in the above characterization is implied by the others. To this end, we introduce different solutions satisfying all axioms but one.

Example 15. For all $\gamma = (M_1, M_2, M_3; v) \in \Gamma_{3-SAG}$, let us consider

$$\sigma_1(\gamma) = \emptyset.$$

Clearly, σ_1 satisfies derived consistency and individual anti-monotonicity but not singleness best.

Example 16. For all $\gamma = (M_1, M_2, M_3; v) \in \Gamma_{3-SAG}$, let us consider

$$\sigma_2(\gamma) = \left\{ u \in \mathbb{R}^N \;\middle|\; \begin{array}{l} u \text{ is feasible-by-matching for } \gamma, \\ u_i \geq w_\gamma(\{i\}), \text{ for all } i \in N, \\ u(N) = w_\gamma(N) \end{array} \right\}.$$

Notice that if $u \in \sigma_2(\gamma)$, every matching μ that is compatible with u is optimal. It can be easily checked that σ_2 satisfies singleness best and individual anti-monotonicity but, as a consequence of Theorem 14, it does not satisfy derived consistency.

Example 17. For all $\gamma = (M_1, M_2, M_3; v) \in \Gamma_{3-SAG}$, let $\eta(\gamma)$ be the nucleolus of γ and consider

$$\sigma_3(\gamma) = \begin{cases} \emptyset & \text{if } C(\gamma) = \emptyset, \\ \{\eta(\gamma)\} & \text{if } C(\gamma) \neq \emptyset. \end{cases}$$

The solution σ_3 satisfies singleness best and derived consistency (see Theorem 10), but, as a consequence of Theorem 14, it does not satisfy individual anti-monotonicity.

These three examples prove that none of the axioms is redundant in the above characterization of the core.

6 2-additive generalized three-sided assignment markets

In this section, we introduce a subclass of generalized three-sided assignment markets that we will name *2-additive generalized three-sided assignment markets* and denote by Γ^{add}_{3-SAG}. This subclass is basically defined by three conditions. The first one requires non-negativeness of the valuation function. Secondly, the valuation of each triplet $(i,j,k) \in M_1 \times M_2 \times M_3$ is the addition of the valuations of all pairs of agents in the triplet. Finally, we require the existence of an optimal matching that remains optimal for the projection to each two-sided market.

The reader will notice that the spirit of this class of 2-additive generalized three-sided assignment markets is similar to that of the balanced classes of multi-sided assignment games in Quint (1991) and Stuart (1997). In both cases, the authors impose that the worth of a triplet is the addition of some numbers attached to its pairs. The difference is that in their models a pair cannot attain its worth if not matched with a third agent of the remaining sector, while in our case there is an underlying two-sided market for each pair of sectors.

As in Quint (1991), we will assume from now on that the market is square, that is $|M_1| = |M_2| = |M_3|$.

Let us introduce some notation: given a generalized three-sided assignment market $\gamma = (M_1, M_2, M_3; v)$, for all $r, s \in \{1,2,3\}$, we consider the two-sided market $\gamma^{rs} = (M_r, M_s; v_{|\mathcal{B}^{M_r \cup M_s}})$. Then, we denote by $\mathcal{M}_{\gamma^{rs}}(M_r, M_s)$ the set of optimal matchings for the two-sided market γ^{rs}, that is, partitions of $M_r \cup M_s$ in mixed pairs and singletons that maximize the sum of the valuations of the coalitions in the partition. And $C(\gamma^{rs})$ stands for the core of the underlying two-sided assignment game $(M_r \cup M_s, w_{\gamma^{rs}})$.

Definition 18. A generalized three-sided assignment market $\gamma = (M_1, M_2, M_3; v)$, with $|M_1| = |M_2| = |M_3|$, belongs to the class Γ^{add}_{3-SAG} if and only if

1. $v \geq 0$,
2. $v(\{i,j,k\}) = v(\{i,j\}) + v(\{i,k\}) + v(\{j,k\})$ for all $(i,j,k) \in M_1 \times M_2 \times M_3$, and $v(\{k\}) = 0$ for all $k \in M_1 \cup M_2 \cup M_3$,
3. there exists $\mu \in \mathcal{M}_\gamma(M_1, M_2, M_3)$ such that $\mu_{|M_r \times M_s} \in \mathcal{M}_{\gamma^{rs}}(M_r, M_s)$.

Conditions (1) and (2) imply that v is superadditive and this guarantees that there exists an optimal matching only containing triplets. It is easy to find examples that show that conditions (1) and (2) are not sufficient to guarantee the non-emptiness of the core. However, next proposition shows that the three conditions together guarantee that the core of any generalized three-sided assignment market in the class Γ^{add}_{3-SAG} is non-empty.

Proposition 19. *The class of 2-additive generalized three-sided assignment markets is balanced.*

Proof. Let $\gamma = (M_1, M_2, M_3; v) \in \Gamma^{add}_{3-SAG}$ and let μ be an optimal matching only containing triplets, $\mu \in \mathcal{M}_\gamma(M_1, M_2, M_3)$. The existence of such a matching μ follows from conditions (1) and (2). From Shapley and Shubik (1972) it is known that each two-sided assignment market is balanced. So, take core allocations, $(u^1, v^1) \in C(\gamma^{12})$, $(u^2, w^2) \in C(\gamma^{13})$ and $(v^3, w^3) \in C(\gamma^{23})$. We will see that $(u^1 + u^2, v^1 + v^3, w^2 + w^3) \in C(\gamma)$. Indeed, take any $\{i, j, k\} \in \mathcal{B}$ and notice that

$$\begin{aligned} u^1_i + u^2_i + v^1_j + v^3_j + w^2_k + w^3_k &= (u^1_i + v^1_j) + (u^2_i + w^2_k) + (v^3_j + w^3_k) \\ &\geq v(\{i, j\}) + v(\{i, k\}) + v(\{j, k\}) = v(\{i, j, k\}) \end{aligned}$$

where the inequality follows from the core constraints of (u^1, v^1), (u^2, w^2) and (v^3, w^3) in each two-sided market and, because of condition (3), it becomes an equality when $\{i, j, k\} \in \mu$.

Similarly, if $\{i, j\} \in \mathcal{B}$, we may assume without loss of generality that $i \in M_1$ and $j \in M_2$, and hence, taking into account $u^2_i \geq v(\{i\}) = 0$ and $v^3_j \geq v(\{j\}) = 0$, we get

$$u^1_i + u^2_i + v^1_j + v^3_j = (u^1_i + v^1_j) + u^2_i + v^3_j \geq v(\{i, j\}).$$

Finally, if $\{i\} \in \mathcal{B}$, let us assume without loss of generality that $i \in M_1$. Then $u^1_i + u^2_i \geq 0 = v(\{i\})$ follows also from the individual rationality of (u^1, v^1) and (u^2, v^2). $\square$

In the above proposition we have deduced the existence of core elements for $\gamma \in \Gamma^{add}_{3-SAG}$ by operating with three core elements of the related two-sided markets. However not all elements of $C(\gamma)$ can be obtained in this way as the next example illustrates.

Example 20. Let $\gamma = (M_1, M_2, M_3; v)$ be a 2-additive three-sided assignment market where the values of pairs of agents in different sectors are given by the three following matrices and the optimal matchings are in boldface:

$$B^{12} = \begin{array}{c} \\ 1 \\ 2 \end{array} \!\!\begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} \mathbf{7} & 6 \\ 0 & \mathbf{3} \end{pmatrix} \end{array}, \qquad B^{13} = \begin{array}{c} \\ 1 \\ 2 \end{array} \!\!\begin{array}{c} \begin{array}{cc} 1'' & 2'' \end{array} \\ \begin{pmatrix} \mathbf{3} & 0 \\ 6 & \mathbf{7} \end{pmatrix} \end{array}, \qquad B^{23} = \begin{array}{c} \\ 1' \\ 2' \end{array} \!\!\begin{array}{c} \begin{array}{cc} 1'' & 2'' \end{array} \\ \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \end{array}.$$

Then, the three-dimensional matrix that gathers the values of the triplets of agents of different sectors by imposing 2-additivity is:

$$A = \begin{array}{c} \\ 1 \\ 2 \\ \\ \end{array} \!\!\begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} \mathbf{11} & 9 \\ 7 & 9 \end{pmatrix} \\ 1'' \end{array} \qquad \begin{array}{c} \\ 1 \\ 2 \\ \\ \end{array} \!\!\begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} 7 & 7 \\ 7 & \mathbf{11} \end{pmatrix} \\ 2'' \end{array}.$$

Notice that this game belongs to Γ^{add}_{3-SAG} since also condition (3) of Definition 18 is satisfied.

Consider $z = (3,1;2,2;6,8) \in C(\gamma)$ and assume there exist $(u^1, v^1) \in C(\gamma^{12})$, $(u^2, w^2) \in C(\gamma^{13})$ and $(v^3, w^3) \in C(\gamma^{23})$ such that

$$z = (u_1^1 + u_1^2, u_2^1 + u_2^2; v_1^1 + v_1^3, v_2^1 + v_2^3; w_1^2 + w_1^3, w_2^2 + w_2^3).$$

Then, $u_1^1 + u_1^2 = 3$ and $v_2^1 + v_2^3 = 2$ would add up to $u_1^1 + v_2^1 = 5 - u_1^2 - v_2^3$. But on the other hand, $(u^1, v^1) \in C(\gamma^{12})$ implies $u_1^1 + v_2^1 \geq 6$. Finally, $5 - u_1^2 - v_2^3 \geq 6$ leads to the contradiction $u_1^2 + v_2^3 \leq -1$.

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