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Folk solution for *simple* minimum cost spanning tree problems

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Abstract

A *minimum cost spanning tree* problem analyzes how to efficiently connect a group of individuals to a source. Once the efficient tree is obtained, the addressed question is how to allocate the total cost among the involved agents. One prominent solution in allocating this minimum cost is the so-called *Folk* solution. Unfortunately, in general, the *Folk* solution is not easy to compute. We identify a class of *mcst* problems in which the *Folk* solution is obtained in an easy way.

Keywords: Minimum cost spanning tree problem; *Folk* solution; Elementary cost matrix; Simple *mcst* problem;
JEL classification: C71, D63, D71.

1. Introduction

We consider a situation in which some individuals, located at different places, want to be connected to a source in order to obtain a good or service. Each link joining two individuals, or any individual to the source, has a specific fixed cost. Moreover, individuals do not mind being connected directly to the source, or indirectly through other individuals. There are several methods to obtain a way of connecting agents to the source so that *the total cost of the selected network is minimum* (Prim's algorithm (Prim, 1957), for instance). This situation is known as the *minimum cost spanning* *tree problem* (hereafter *mcst* problem) and it is used to analyze different real-life issues, from telephone and cable TV to water supply networks.

An important question is how this minimum cost should be allocated among the individuals. One prominent solution to solve the allocation of this cost is the so-called *Folk* solution. To compute this solution, first we need to calculate the irreducible costs and, in order to do that, we have to compare all paths from any two nodes (individuals) and solve a $\min - \max$ problem. Then, we have to compute the Shapley value of the cooperative game defined throughout the irreducible costs, or to apply the closed-form obtained in Bogomolnaia and Moulin (2010).

We define a class of *mcst* problems (that we call *simple mcst* problems) in which the *Folk* solution only depends on the cost of each individual to the source and the cost to the *nearest partner*, that is, the minimum connection cost of this individual. We obtain a closed-form (easy to obtain) of the *Folk* solution that does not need to compute the irreducible costs. Finally, we extend the class of *mcst* problems where this procedure can be applied.

2. Definitions

2.1. Minimum cost spanning tree

A *minimum cost spanning tree* problem involves a finite set of *agents*, $N = \{1, 2, \ldots, n\}$, who need to be connected to a *source* ω. We denote by N_ω the set of agents and the source, i.e. $N_\omega = N \cup \{\omega\}$. The agents are connected by edges and for $i \neq j$, $c_{ij} \in \mathbb{R}_+$ represents the cost of the edge $e_{ij} = (i, j)$ connecting agents $i, j \in N$. We denote by c_{ii} the cost of connecting directly agent i to the source, for all $i \in N$. Let $\mathbf{C} = [c_{ij}]_{n\times n}$ be the $n \times n$ symmetric cost matrix. The *mcst* problem is represented by the pair $(N_\omega, \mathbf{C})$.

A *spanning tree* over $(N_\omega, \mathbf{C})$ is an undirected graph p with no cycles that connects all elements of N_ω. We can identify a spanning tree with a map $p : N \to N_\omega$ so that $j = p(i)$ is the agent (or the source) whom i connects. This map p defines the edges $e^p_{ij} = (i, p(i))$ in the tree. In a spanning tree each agent is (directly or indirectly) connected to the source ω; that is, for all $i \in N$ there is some $t \in \mathbb{N}$ such that $p^t(i) = \omega$. Moreover, given a spanning tree p, there is a unique path from any i to the source for all $i \in N$, given by the edges $(i, p(i)), (p(i), p^2(i)), \ldots, (p^{t-1}(i), p^t(i) = \omega)$. The cost of building the spanning tree p is the total cost of the edges in this tree; that is,

$$C_p = \sum_{i=1}^{n} c_{ip(i)}$$

Prim (1957) provides an algorithm which solves the problem of connecting all agents to the source such that *the total cost of the network is minimum.* The achieved solution, the *minimum cost spanning tree*, may not be unique. Denote by m a tree with minimum cost and by C_m its cost. That is, for all spanning trees p

$$C_m = \sum_{i=1}^{n} c_{im(i)} \leqslant C_p = \sum_{i=1}^{n} c_{ip(i)}$$

Given a subset $S \subseteq N$, we will denote by $C_m(S)$ the minimum cost of the *mcst* sub-problem $(S_\omega, \mathbf{C}|_S)$. Let us denote by C_ω the cost of the tree in which every individual joins directly the source, $C_\omega = \sum_{i\in N} c_{ii}$. And, for any individual $i \in N$, c_{i*} represents the minimum connection cost of such an individual (interpreted as the cost to the *nearest partner*), $c_{i*} = \min_{j\in N} c_{ij}$.

Once a minimum cost spanning tree is constructed, the important issue is how to allocate the associated cost C_m among the agents.

A *cost sharing rule* for *mcst* problems is a function that proposes for any *mcst* problem $(N_\omega, \mathbf{C}) \in \mathcal{N}_n$ an allocation $\alpha(N_\omega, \mathbf{C}) = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$, such that

$$\sum_{i=1}^{n} \alpha_i = C_m.$$

2.2. The ***Folk*** *solution*

Many solutions have been defined in the *mcst* literature (see, for instance, Bergantiños and Vidal-Puga (2008) for definitions and a comparative analysis). We will focus on the so-called *Folk* solution proposed independently by Feltkamp et al. (1994) and Bergantiños and Vidal-Puga (2007). We will denote this solution by $F(N_\omega, \mathbf{C})$. It can be obtained as the Shapley value of the stand-alone game associated with the irreducible cost matrix defined by:

$$c^*_{ij} = \min_{P_{ij}} \max_{e\in P_{ij}} \{c(e)\}$$

where P_{ij} are paths from i to j, $e \in P_{ij}$ is an edge in this path, and $c(e)$ is the cost of this edge. Bogomolnaia and Moulin (2010) provide a closed-form expression of the *Folk* solution: for individual i order increasingly the irreducible costs of connecting this individual to other $n-1$ agents, so that $c_i^{*1} \leqslant c_i^{*2} \leqslant \ldots \leqslant c_i^{*(n-1)}$. Then, the *Folk* solution is

$$F_i(N_\omega, \mathbf{C}) = \frac{c_{ii}^*}{n} + \sum_{k=1}^{n-1} \frac{1}{k(k+1)} \min\{c_i^{*k}, c_{ii}^*\} \tag{1}$$

2.3. ***Simple*** *mcst* ***problems***

Definition 1. ***Elementary cost matrix***
A mcst problem $(N_\omega, \mathbf{C})$ *is said to be an* ***elementary cost*** *mcst* ***problem*** *if for all* $i, j \in N$, $c_{ij} \in \{c_1, c_2\}$. *We will denote an elementary cost mcst problem by* $(N_\omega, \mathbf{C}^e)$.

Remark 1. *Usually, elementary cost matrices are defined such that* $c_1 = 0$ *and* $c_2 = 1$. *The general case* $c_1 \leqslant c_2$*, low and high cost, is also known as* $2-$*mcst problems (Estévez-Fernández and Reijnierse, 2014).*

Definition 2. ***Autonomous component***
Given a mcst problem $(N_\omega, \mathbf{C})$*, with minimum connecting cost* C_m*, a subset* $S \subseteq N$ *is said to be:*

- ***autonomous*** *if* $C_m = C_m(S) + C_m(N \backslash S)$*;*
- ***an autonomous component*** *if it is autonomous and has no autonomous subset; if* $T \subseteq S$, $T \neq S$, *then* T *is not autonomous.*

Remark 2. *Note that if* S *is autonomous, so is* $N \backslash S$. *The Folk solution provides the same allocation to individual* i *in the whole problem or if applied to an autonomous component* S*, with* $i \in S$

$$F_i\left(S_\omega, \mathbf{C}|_S\right) = F_i(N, \mathbf{C}), \quad \text{for all } i \in S,\ S \text{ autonomous component}$$

Therefore we can solve separately the (smaller) problems $(S_\omega, \mathbf{C}|_S)$ *for autonomous components. Note that mcst problems with elementary cost matrices may have several autonomous components (see Example 1).*

Definition 3. ***Simple*** *mcst* ***problem***
Given a mcst problem $(N_\omega, \mathbf{C})$*, it is said to be* ***simple*** *if the cost matrix* $\mathbf{C}$ *is elementary and the set of all individuals* N *is an autonomous component. We will denote a simple mcst problem by* $(N_\omega, \mathbf{C}^s)$.

Remark 3. *Obviously* N *is always autonomous. If it is an autonomous component, it is the unique autonomous component in the mcst problem.*

3. The result

The following result shows that in *simple mcst problems* it is possible to obtain the *Folk* solution only taking into account, for each individual $i \in N$ the cost of connecting this individual to the source, c_{ii}, and the cost to connect with the nearest partner c_{i_*}.

Theorem 1. *Given a **simple** mcst problem* $(N_\omega, \mathbf{C}^s)$,

a) *If* $c_{ii} = c_1 \quad \Rightarrow \quad F_i(N_\omega, \mathbf{C}^s) = c_1$

b) *If* $c_{i_*} = c_2 \quad \Rightarrow \quad F_i(N_\omega, \mathbf{C}^s) = c_2$

c) *If* $c_{ii} = c_2$ *and* $c_{i_*} = c_1 \quad \Rightarrow \quad F_i(N_\omega, \mathbf{C}^s) = c_2 - \dfrac{C_\omega - C_m}{n_3}$,

where $n_3 = |\{i \in N : c_{ii} = c_2 \text{ and } c_{i_*} = c_1\}|$

Proof. Let us consider a simple *mcst* problem $(N_\omega, \mathbf{C}^s)$, and let m a minimum cost spanning tree with cost C_m.

a) If an individual i is such that $c_{ii} = c_1$. In this case, $c^*_{ii} = c_1$ and, for all k, $\min\{c_i^{*k}, c^*_{ii}\} = c_1$. Then, if we apply equation (1) to obtain the *Folk* solution we get

$$F_i(N_\omega, \mathbf{C}^s) = \frac{c^*_{ii}}{n} + \sum_{k=1}^{n-1} \frac{1}{k(k+1)} \min\{c_i^{*k}, c^*_{ii}\} = \frac{c_1}{n} + \sum_{k=1}^{n-1} \frac{1}{k(k+1)} c_1 =$$

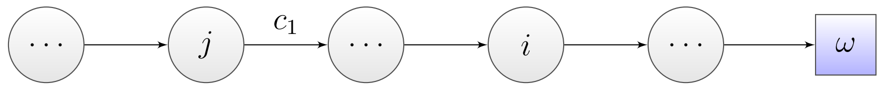
$$= \frac{c_1}{n} + c_1 \sum_{k=1}^{n-1} \left[\frac{1}{k} - \frac{1}{k+1}\right] = \frac{c_1}{n} + c_1 \left(1 - \frac{1}{n}\right) = c_1$$

b) If $c_{i_*} = c_2$, then $c^*_{ij} = c_2$, for all $j = 1, 2, \ldots, n$ and by reasoning as in the previous case we obtain

$$F_i(N_\omega, \mathbf{C}^s) = c_2$$

c) Let us suppose the existence of two individuals in this case such that for some k, $c^*_{ik} < c^*_{jk}$. This implies $c^*_{ik} = c_{ik} = c_1$, $c^*_{jk} = c_{jk} = c_2$, and $c^*_{ij} = c_{ij} = c_2$. If $c_{jm(j)} = c_2$, we may define the spanning tree p such that it coincides with m except in that $p(j) = \omega$. Then $C_m = C_p$ and N is not an autonomous component, a contradiction. In other case, if $c_{jm(j)} = c_1$, we have two possibilities for the minimum cost spanning tree m:

c1) Individual i is *closer* to the source than individual j, that is



Then, there is some j' *between* j and i such that $c_{j'm(j')} = c_2$ and we may define the spanning tree p such that it coincides with m except in that $p(j') = \omega$. Then $C_m = C_p$ and N is not an autonomous component, a contradiction.

c2) Individual j is *closer* to the source than individual i. By reasoning in a similar way as in the previous case $c1$), we obtain a contradiction.

So, for all k, $c^*_{ik} = c^*_{jk}$ and applying equation (1) the *Folk* solution allocates the same amount to both individuals.

To obtain the allocation of these individuals, note that if we call n_1 the number of individuals that are in case a) and n_2 the number of individuals in case b), then individuals in case c) should pay

$$R = C_m - n_1 c_1 - n_2 c_2 = C_m - C_\omega + n_3 c_2$$

As the *Folk* solution allocates the same amount to any individual in this group, then

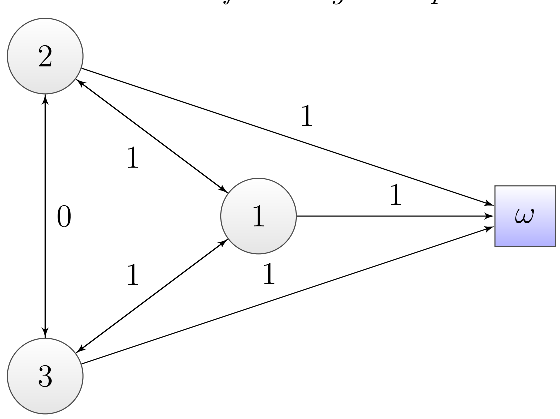
$$F_i(N_\omega, \mathbf{C}^\mathbf{s}) = \frac{R}{n_3} = c_2 - \frac{C_\omega - C_m}{n_3}$$

the required result.

■

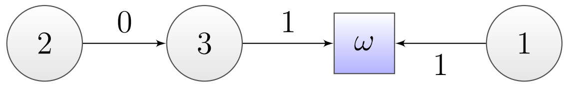
The following example shows that the result in Theorem 1 cannot be applied to elementary cost *mcst* problems, since they can lead to different autonomous components. Nevertheless, every elementary cost *mcst* problem can be obtained as union of simple problems.

Example 1. *Let us consider the following mcst problem*



A minimum cost spanning tree ($C_m = 2$) is given by function m defined as:

$$m(1) = \omega \qquad m(2) = 3 \qquad m(3) = \omega; \qquad C_m = c_{11} + c_{23} + c_{33} = 2.$$



The Folk solution provides the allocation $F = (1, 1/2, 1/2)$. Observe that the problem is not simple since it has two autonomous components, $N_1 = \{1\}$, $N_2 = \{2, 3\}$. However, the proposal provided in Theorem 1 coincides with the Folk solution.

Now, we extend the class of *mcst* problems in which the result in Theorem 1 can be obtained by allowing problems with several autonomous components. It is important to note that, as every elementary cost *mcst* problem, or $2-$*mcst* problem, can be decomposed in autonomous components, Corollary 1 provides a way of finding the *Folk* solution in this class of problems.

Definition 4. *A mcst problem $(N_\omega, \mathbf{C})$ is* ***simple-decomposable*** *if it is possible to split N*

$N = N_1 \cup N_2 \cup \ldots \cup N_r, \quad N_i \cap N_j = \emptyset$, *for $i \neq j$*

such that

$$C_m\left(N_\omega, \mathbf{C}\right) = \sum_{t=1}^{r} C_m\left((N_t)_\omega, \mathbf{C}|_{N_t}\right)$$

and every mcst sub-problem $\left((N_t)_\omega, \mathbf{C}|_{N_t}\right)$ is simple. We will denote a simple-decomposable mcst problem by $(N_\omega, \mathbf{C}^{\textit{s-dec}})$. Each mcst simple sub-problem $\left((N_t)_\omega, \mathbf{C}|_{N_t}\right)$ is called a ***simple component*** *of $(N_\omega, \mathbf{C})$.*

We will denote the high cost in every simple component $\left((N_t)_\omega, \mathbf{C}|_{N_t}\right)$ by $c_2(t)$ and the lower cost by $c_1(t)$. Then, as a direct consequence of Theorem 1 and Remark 2, we obtain the following result.

Corollary 1. *Given a* ***simple-decomposable*** *mcst problem $(N_\omega, \mathbf{C}^{\textit{s-dec}})$, for any individual $i \in N$, let N_t the simple component such that $i \in N_t$. Then*

1) *If $c_{ii} = c_1(t) \quad \Rightarrow \quad F_i(N_\omega, \mathbf{C}^s) = c_1(t)$*
2) *If $c_{i*} = c_2(t) \quad \Rightarrow \quad F_i(N_\omega, \mathbf{C}^s) = c_2(t)$*
3) *If $c_{ii} = c_2(t)$ and $c_{i*} = c_1(t) \quad \Rightarrow \quad F_i(N_\omega, \mathbf{C}^s) = c_2(t) - \dfrac{E(t)}{n_3(t)}$,*

   where

   $$E(t) = C_\omega(N_t) - C_m(N_t),\ n_3(t) = |\{i \in N_t : c_{ii} = c_2(t) \ \text{and} \ c_{i*} = c_1(t)\}|$$

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