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Brownian Signals: Information Quality, Quantity and Timing in Repeated Games.*

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Abstract

This paper examines different Brownian information structures for varying time intervals. We focus on the non-limit case and on the trade-offs between information quantity and quality to efficiently establish incentives. These two dimensions of information tend to complement each other when signals quality is sufficiently high. Otherwise, information quantity tends to replace information quality. Any conclusion depends crucially on the rate at which information quality improves or decays with respect to the discounting incentives.

JEL: C73, D82, D86.

KEYWORDS: Repeated Games, Frequent Monitoring, Information Quantity, Information Quality.

1 Introduction

Economic relations depend crucially on the frequency of interaction between the involved parties. This fact affects the agents' behavior and the value of these relations. Frequent actions favor coordination in the same way as discounting. However, there are additional considerations regarding the effect of time and the information structure. In other words, how the frequency of access to information and the individuals' actions feedback into signals?

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For instance, agents' actions may affect the fundamental value of a given variable (e.g., price or output) or they may simply create instability with no effect on the fundamental value.

In the present paper we examine different information structures and how the associated signals distribution depends upon period length. It is not our goal to present a general theory, but to illustrate some crucial aspects through a simple game. Contrary to the existing literature (Abreu et al., 1991; Fundenberg and Levine, 2007; Sannikov and Skrzypacz, 2007),[1] we focus on the non-limit case. We want to understand the trade-offs between information quantity and quality and how they are used to establish incentives (Mirrlees, 1974).

In particular, we consider that information quantity and quality are **complements** if they move in the same direction; otherwise they are **substitutes**.

While the limit case is interesting for theoretical and tractable reasons, the non-limit case studied in the present paper is more realistic but the results are mainly numerical. However, the relevance of the existing information trade-offs justify a careful analysis.[2]

Our findings depend on how actions affect the distribution of the Brownian public signals. We consider three different information structures:

In section 3.1 **actions impact on the drift** and information quality improves with time.[3] For small time intervals there is a predominant use of information quantity that is substituted by quality as the time interval increases. However, for sufficiently large time intervals the deviation incentives are not compensated by information gains. Consequently, the monitoring tightens, complementing information quality with a larger set of signals.

In section 3.2 **a deviation increases the noise** of the process and information quality worsens with time.[4] As the time interval increases the loss in information quality is complemented with a selective reduction in the number of signals used to sustain cooperation. For sufficiently large time intervals the deviation incentives become sufficiently strong, monitoring employs an increasing number of signals in substitution of the decreasing quality.

[1] Other relevant contributions are Faingold and Sannikov (2011), Fundenberg and Levine (2009), Osório (2012), Sannikov (2007), and Sannikov and Skrzypacz (2010). A notable variation of the original model is Fudenberg and Olszewski (2011) that study repeated games with stochastic asynchronous monitoring. Outside the limit case, Fudenberg et al. (2014) show that if players wait long enough, then it is likely that everybody has observed the same signal and a folk theorem may be possible. Osório (2015) focus on Poisson signals *à la* Abreu et al. (1991).

[2] Kamada and Kominers (2010) consider a time varying information structure. However, their argument and information notions are different. See also Kandori (1992).

[3] Under this information structure Sannikov and Skrzypacz (2007) show that in the limit (but also for small time intervals) the equilibrium degenerates. A different approach and implications are shown in Osório (2012).

[4] Under this information structure Fudenberg and Levine (2007) show that full efficiency is possible in the limit.

	$e_{2t}=1$	$e_{2t}=0$
$e_{1t}=1$	g, g	$-u, g+u$
$e_{1t}=0$	$g+u, -u$	$0, 0$

Table 1: The Prisoners' Dilemma Stage Game Payoffs.

In section 3.3 **a deviation decreases the noise** of the process and information quality improves with the time.[5] In this case, information quality alone cannot compensate the increasing deviation incentives. Information quantity complements quality in the provision of incentives.

Section 4 concludes.

2 The model

We explore frequent monitoring in a simple prisoners' dilemma game with two long-run players $i = 1, 2$. At moments in time $t = 0, \tau, 2\tau, ...$, players simultaneously choose from two different effort levels $e_{it} = 1$, or $e_{it} = 0$. In the former, player i provides effort; in the latter, player i shirks. We consider the stage game payoffs in Table 1, with $g > u > 0$. Shirking is a dominate strategy for both players.

In the subsequent period $t+\tau$, an imperfect public signal $s_{t+\tau}$, generated by an *arithmetic Brownian motion* (ABM) process is observed:

$$s_{t+\tau} - s_t = \mu_t \tau + \sigma_t \int_t^{t+\tau} dW_x, \; with \; W_t = 0 \; and \; t = 0, \tau, 2\tau, ..., \tag{1}$$

where $\{W_x; x \geq 0\}$ is a standard Brownian motion and $s_t = 0$ is the initial value. The parameters μ_t and σ_t are the drift and the noise of the process at time t, respectively. The drift and the noise depend on the profile of efforts $e_t = (e_{1t}, e_{2t})$ and the information structure.

Let S denote the set of signals $s_{t+\tau}$ that suggest defection, i.e., realizations of the process inside some region bounded by some threshold/s $\underline{s}$ and/or $\overline{s}$. The probability that the state of the public process (1) appears in the set S is Gaussian distributed and given by:

$$p(e_{1t}, e_{2t}) = \Pr(s_{t+\tau} \in S | e_t) = \int_S \exp(-(s - \mu_t \tau)^2 / (2\sigma_t^2 \tau)) / (2\pi \sigma_t^2 \tau)^{1/2} ds.$$

The probability that the state of the public process (1) appears outside S, i.e., it is interpreted

[5] Under this information structure Fudenberg and Levine (2007) show that non-trivial but not full efficient payoffs are possible in the limit.

as a signal of mutual cooperation, is given by $1 - p(e_{1t}, e_{2t})$.

The common discount factor is $\delta \equiv e^{-r\tau}$, where $r \in (0, \infty)$ denotes the discount rate.[6]

We look at profiles of strategies that form a *perfect public equilibrium.*[7]

3 Action-signal feedback and the information structure

Our objective is to understand how information quality and quantity are used to enforce cooperation. The highest non-trivial equilibrium payoff is:

$$v = g - up(1,1)/(p(1,0) - p(1,1)),$$

subject to be non-negative and to the incentives condition:

$$g/u \geq (1 - \delta(1 - p(1,1)))/\delta(p(1,0) - p(1,1)). \tag{2}$$

Otherwise, we say that the equilibrium is trivial or degenerates.[8]

Optimally requires condition (2) to bind in equilibrium. This condition determines how information quality and quantity are used to enforce cooperation while maximizing the expected payoffs.

In order to proceed with the analysis we define the meaning of information quality and quantity in our setup.

Definition 1 (information quantity) *The information quantity increases with the cardinality of the set S, and decreases otherwise.*

Information quantity corresponds to the number of signals used to infer or monitor the players' actions. Figure 1 provides an illustration of optimal threshold strategies for two particular information structures. These thresholds strategies vary with τ, therefore, so thus the cardinality of the set S.

[6] We restrict our analysis to the simplest setting. The results generalize straightforwardly for more general discount factors, payoff structures and games.

[7] A strategy is public if depends only on the public history (of signals) and not on the private history (of signals and personal effort). Given a public history, a profile of public strategies that induces a Nash equilibrium on the continuation game from that time on is called a PPE.

[8] A step by step derivation of this result can be found in Fudenberg and Levine (2007).

Definition 2 (information quality) *The information quality of the signal s increases with the likelihood ratio:*

$$l = \Pr\left(s \in S | (1,0)\right) / \Pr\left(s \in S | (1,1)\right),$$

and decreases otherwise.

Intuitively, information quality is measured by the likelihood of correct punishment with respect to mistaken punishment (Mirrlees, 1974).

Definition 3 (substitutes & complements) *Information quantity and quality are substitutes (complements, respectively) if for varying time interval they move in opposed (same, respectively) directions.*

In other words, complementarity implies either a simultaneously increase or decrease in the cardinality of the set S and the likelihood ratio l in response to a change in the period length τ. Otherwise, they are substitutes in the provision of incentives.

We now consider three time varying information structures that have been discussed in the literature. As standardization the intuition is presented assuming that the time interval increases. The reader is free to consider the inverse exercise.

3.1 Information structure: deviation impacts on the drift

In this information structure a deviation decrease the drift of the process (1). For instance, we could have $\mu_t = c\left(e_{1t} + e_{2t}\right)$ with $c > 0$ and $\sigma_t = \sigma > 0$. In this setting Sannikov and Skrzypacz (2007) show that one-side threshold strategies are optimal for detecting deviations. Players use a common threshold $\underline{s}$ to distinguish realizations suggesting cooperation, i.e., $\{s_{t+\tau} > \underline{s}\}$, from realizations suggesting defection, i.e., $S = \{s_{t+\tau} \leq \underline{s}\}$.

Sannikov and Skryzpacz (2007) and Fudenberg and Levine (2007) found an equilibrium degeneracy for small time intervals. The reason is that in spite of the low deviation incentives through discounting, the information quality is so bad that monitoring is impossible.

However, information quality improves for larger time intervals. Therefore, as we move away from the limit, non-trivial equilibria are possible if the informational quality improvements compensate the lower discount factor. In this case, as the time interval increases, incentives are sustained with less but more informative signals. Information quantity is smoothly substituted by quality. The information gains decreases the likelihood of mistaken punishment and increase payoffs.

However, at a certain point, the informational quality improvements are not enough to compensate the increasing deviation incentives. Consequently, the monitoring technology is complemented with an increasing set of signals of increasing quality. Payoffs tend to increase because the information quality has a positive value creation effect, except when monitoring becomes sufficiently tighten through a larger set of punishment signals. Finally, for large time intervals the equilibrium degenerates. Overall, there is an inverted u-shape relation between payoffs and time.

The following result resumes our findings.

Proposition 4 *If the individuals' actions impact on the drift:*

$$\begin{array}{lcl} 0 < \tau \leq \tau^1 & : & \textit{the equilibrium degenerates} \\ \tau^1 < \tau \leq \tau^2 & : & \textit{information quantity and quality are substitutes} \\ \tau^2 < \tau \leq \tau^3 & : & \textit{information quantity and quality are complements} \\ \tau^3 < \tau < \infty & : & \textit{the equilibrium degenerates} \end{array}$$

for some cutoffs $0 < \tau^1 \leq \tau^2 \leq \tau^3 < \infty$.

3.2 Information structure: deviation increases the noise

In this information structure a deviation causes an instantaneous jump in the variance of the process. For instance, we could have $\mu_t = c \geq 0$ and $\sigma_t = \sigma(k - e_{1t} - e_{2t})$ with $\sigma > 0$, $k > 2$.

Players optimal provision of incentives is achieved with a two-side threshold strategy that distinguish realizations suggesting equilibrium play, i.e., $\{\underline{s} < s_{t+\tau} < \overline{s}\}$, from realizations suggesting defection, i.e., the most extreme ones: $S = \{s_{t+\tau} \leq \underline{s} \cup s_{t+\tau} \geq \overline{s}\}$. Figure 1 (left panel) provides an illustration. The extreme observations are the ones that suggest defection, i.e., above and below the upper and lower curves, respectively.

This information structure is proposed by Fudenberg and Levine (2007). Information quality is maximal in the limit but decays with time. Consequently, full efficiency is possible in the limit.

For small time intervals, incentives are provided through a large quantity of very informative signals. However, we found that as the time interval increases and information deteriorates, the monitoring technology removes the less informative signals. We observe a selective reduction in the employed number of signals. Information quantity and quality move in the same direction.

However, since the discounting incentives weaken with time, at a certain point the decrease in quality must be compensated with an increase in the quantity of signals used to monitor players actions. Information quantity attempts to substitute the falling information quality.

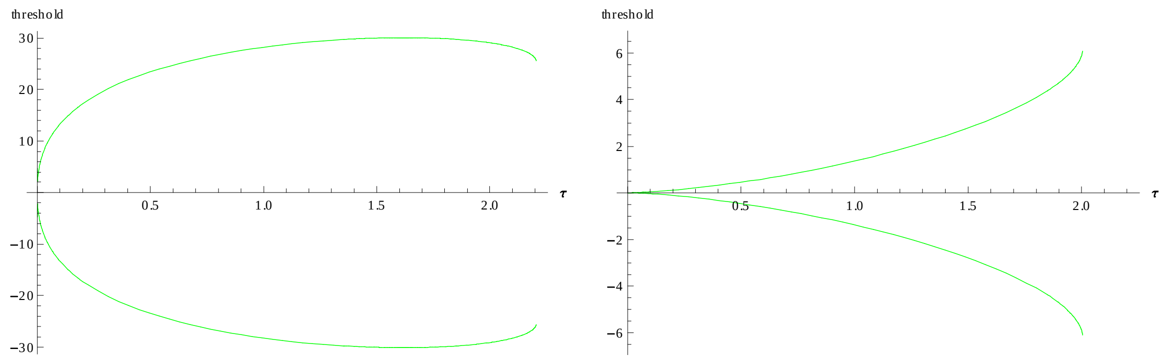

Figure 1: The optimal two-sided threshold strategy for varying time: If a deviation increases the noise (left) the set S is the union of the signals above and below the top and bottom curves, respectively. If a deviation decreases the noise (right) the set S is the intersection of the signals below and above the top and bottom curves, respectively.

For large time intervals the equilibrium degenerates. Payoffs decrease monotonically for any feasible frequency of play.

The following result resumes our findings.

Proposition 5 *If a deviation increases the noise:*

$$\begin{array}{lcl} 0 < \tau \leq \tau^1 & : & \textit{information quantity and quality are complements} \\ \tau^1 < \tau \leq \tau^2 & : & \textit{information quantity and quality are substitutes} \\ \tau^2 < \tau < \infty & : & \textit{the equilibrium degenerates} \end{array}$$

for some cutoffs $0 < \tau^1 < \tau^2 < \infty$.

3.3 Information structure: deviation decreases the noise

In this information structure a deviation causes a discontinuous fall in the variance of the process. For instance, we could have $\mu_t = c \geq 0$ and $\sigma_t = \sigma(k + e_{1t} + e_{2t})$ with $\sigma > 0$, $k > 0$.

The optimal provision of incentives requires a two-sided threshold strategy. In this case, the most extreme observations are the ones that suggest cooperation, i.e., $\{s_{t+\tau} \leq \underline{s} \cup s_{t+\tau} \geq \overline{s}\}$, while the observations around the drift suggest defection, i.e., the less extreme ones: $S = \{\underline{s} < s_{t+\tau} < \overline{s}\}$. Figure 1 (right panel) provides an illustration. The mean observations suggest defection, i.e., the ones in the area below and above the upper and lower curves, respectively.

Fudenberg and Levine (2007) show that the highest payoff is obtained in the limit. However, its value depends on the relative impact of a deviation on the variance. If this is sufficiently strong, we may get arbitrarily close from efficiency. However, if the impact is weak, we may observe degeneracy and the impossibility of establishing incentives for any frequency of play.

Under this information structure the quality of the signals improves with time. For small time intervals the incentives are provided with a small number of low precision signals. In spite of it the quality of information might be enough to sustain a non-trivial equilibrium, i.e., while the most efficient use of information quality and quantity cancels the discounting deviation incentives.

Under this information structure, the monitoring technology employs an increasing number of signals for all frequencies of play because time quality improvements are slower than deviation incentives. Consequently, cooperation requires an increasing quantity of signals of low but increasing quality. This pattern remains until the equilibrium collapses. Payoffs decrease monotonically for all frequencies of play.

The following result resumes our findings.

Proposition 6 *If a deviation decreases the noise:*

$$\begin{array}{lcl} \tau^1 < \tau \leq \tau^2 & : & \textit{information quantity and quality are complements} \\ \tau^2 < \tau < \infty & : & \textit{the equilibrium degenerates} \end{array}$$

for some cutoffs $0 \leq \tau^1 \leq \tau^2 < \infty$.[9]

4 Conclusion

Any conclusion regarding the efficient use of information depends crucially on whether or not the signals' precision improves with time, but the magnitude of these improvements is also relevant. For instance, both information structures of Sections 3.1 and 3.3 improve with time. The former is very noisy for small time intervals but improves faster [with time] than the latter.

It is also important whether the efficient use of information quality and quantity can generate joint benefits that compensate the increasing deviation incentives. If this is the case, the monitoring relaxes and payoffs improve with the time interval. Otherwise, payoffs fall. For instance, in the information structure of Section 3.3 the monitoring tights for

[9]Note that in the limit we have,

$$v \uparrow g - l\sigma(1,0) / (\sigma(1,1) - \sigma(1,0)),$$

where $\sigma(1,0)$ and $\sigma(1,1)$ are the noise under defection and cooperation, respectively. If $\tau^1 > 0$ there is an equilibrium degeneracy in the limit, i.e., for $(g+l)/g > \sigma(1,1) / \sigma(1,0)$.

any frequency of play. However, in the information structures of Sections 3.1 and 3.2 the monitoring tights (relaxes, respectively) only for sufficiently large (small, respectively) time intervals.

As a general pattern, quantity and quality tend to complement each other when signals quality is sufficiently high. Otherwise, information quantity tends to fill the gap left by low information quality.

Other regularity is that for low frequencies of play incentives are based on a large use of information quantity rather than quality. This observation seems natural for time improving information structures as the ones in Sections 3.1 and 3.3. In Section 3.2 the intuition is different because in small time intervals extreme events are unlikely.

Regarding this point, if a deviation increases the noise of the process (Section 3.2), we are able to obtain efficient results but not if a deviation decreases the noise of the process (Section 3.3). The reason is that Brownian motion is a regular events process. Therefore, in small time intervals, monitoring extreme events is easier because they are unlikely to be confused with infinitesimal realizations. However, when actions affect the drift (Section 3.1) the provision of incentives is more difficult because reliable inference about the drift of the process requires a sufficiently large time.[10,11]

A necessary but not sufficient condition for an increase in payoffs is an improvement in the information quality.[12] For instance, in the structure of Section 3.3 quality improves with time but not the payoffs. Unless, if that improvement is strong enough to allow for a reduction on the quantity of signals used for monitoring proposes. Then payoffs increase. For instance, in Section 3.1 a payoff improvement is guaranteed in the information substitution region but not necessarily in the information complementary region (see Proposition 4). In the latter the information quantity and quality are mostly used to sustain incentives through more severe punishments than to create value.

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[10] See Prakasa Rao (1999) for a formal treatment of the statistical inference methods for diffusion type processes.

[11] The way Poisson and Brownian signals provide incentives is different (Fundenberg and Levine, 2007, 2009; Fudenberg and Olszewski, 2011; Osório, 2015).

[12] In this sense our results do not contradict Kandori (1992).

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