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Searching for the 'least' and 'most' dictatorial rules*

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Abstract *We derive a least dictatorial social choice function by specifying a plausible metric above the set of social choice functions. Measuring conformity by counting the number of cases a voter believes to be the dictator, we obtain the plurality rule.*

Keywords: Voting rules, dictatorship, plurality rule.
***JEL* Classification Number:** D71.

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1 Introduction

In their seminal paper Farkas and Nitzan (1979) derived the Borda count as the solution of an optimization problem above the set of social choice functions by minimizing the distance from the unanimity principle. Taking other metrics, Nitzan (1981) obtained the plurality rule among other rules. The approach of minimizing the distance from a set of profiles with a clear winner such as the unanimous winner, the majoritarian winner, or the Condorcet winner has been developed further by Lerer and Nitzan (1985), Elkind et al. (2015), and Mahajne et al. (2015) among others.

The common in these works is that they investigate the closeness of a rule to only partially defined rules specifying desirable outcomes. In contrast, we aim to get as far as away from the undesirable dictatorial rule. For a quite simple and natural distance function we find that our goal results on the universal domain in a quite unpleasant rule violating properties like unanimity or monotonicity.

To get partially rid of the unwanted behavior of the least dictatorial rules, we restrict their range to the set of those profiles on which there is no 'consensus winner' (e.g. no unanimous winner, no majoritarian winner, or no Condorcet winner). In addition, from an opposite point of view, we may accept rules which allow the voters to feel themselves as a dictator in as many cases as possible. We call these rules the most dictatorial ones, though they are definitely not worse than the simple dictatorial rule. Using this terminology, we find that the plurality rule is the most dictatorial one.

2 The framework

Let $A = \{1, \ldots, m\}$ be the set of alternatives and $N = \{1, \ldots, n\}$ be the set of voters. We shall denote by $\mathcal{P}$ the set of all linear orderings on A and by $\mathcal{P}^n$ the set of all preference profiles. If $\succ \in \mathcal{P}^n$ and $i \in N$, then $\succ_i$ is the preference ordering of voter i over A.

Definition 1. A mapping $f : \mathcal{P}^n \to A$ that selects the winning alternative is called a *social choice function*, henceforth, SCF.

Note that our definition of an SCF does not allow for possible ties, in which case a fixed tie-breaking rule will be employed.

We will also allow for domain restrictions, since for some preference profiles we may prescribe certain outcomes. Let $\mathcal{S} \subseteq \mathcal{P}^n$ be a subdomain on which the outcome is already prescribed by some externally chosen principle. Then the values of a SCF have to be specified only on $\mathcal{P}^n \setminus \mathcal{S}$. For instance, for profiles with a Condorcet winner, denoted by $\mathcal{S}_c$, we may only consider Condorcet consistent SCFs; or for profiles with a majority supported alternative, denoted by $\mathcal{S}_m$, we may require that the majority winner should be chosen. We consider the following type of domain restriction.

Definition 2. A domain restriction $\mathcal{S} \subseteq \mathcal{P}^n$ is called *anonymous* if for any bijection $\sigma : N \to N$ we have for all $(\succ_1, \ldots, \succ_n) \in \mathcal{P}^n$ that $(\succ_1, \ldots, \succ_n) \in \mathcal{S}$ implies $(\succ_{\sigma^{-1}(1)}, \ldots, \succ_{\sigma^{-1}(n)}) \in \mathcal{S}$.

It can be verified that if $\mathcal{S}$ is anonymous, then also $\overline{\mathcal{S}}$ is anonymous, where $\overline{\mathcal{S}} = \mathcal{P}^n \setminus \mathcal{S}$. If $\mathcal{S} = \emptyset$, we have the case of an unrestricted domain. It is easy to see that $\mathcal{S}_c$ and $\mathcal{S}_m$ are anonymous.

Let $\mathcal{F} = A^{\mathcal{P}^n}$ be the set of SCFs (plurality, Borda, etc). The subset of $\mathcal{F}$ consisting of the dictatorial rules will be denoted by $\mathcal{D} = \{d_1, \ldots, d_n\}$, where d_i is the dictatorial rule with voter i as the dictator. In order to define a least-dictatorial rule we will employ the following distance function between SCFs:

$$\rho_{\mathcal{S}}(f, g) = \#\{\succ \in \overline{\mathcal{S}} \mid f(\succ) \neq g(\succ)\}, \tag{2.1}$$

where f, g are SCFs and $\rho_{\mathcal{S}}(f, g)$ stands for the number of profiles on which f and g choose different alternatives within $\overline{\mathcal{S}}$.[1] It can be checked that ρ_S specifies a metric above the set of SCFs restricted to $\overline{\mathcal{S}}$. If $S = \emptyset$, we simply write $\rho(f, g)$. Since in case of SCFs we only care about the chosen outcome (and not about a social ranking), and we do not assume any kind of structure on the set of alternatives A, it appears natural that we count for the number of different choices by f and g. We discuss some possible extensions in Section 4.

We specify the set of least dictatorial rules by those ones which are the furthest away from the closest dictatorial rule.

Definition 3. We define the set of *least dictatorial rules* for domain restriction $\mathcal{S}$ by

$$\mathcal{F}_{ld}(\mathcal{S}) = \left\{ f \in \mathcal{F} \mid \forall f' \in \mathcal{F} : \min_{g \in \mathcal{D}} \rho_{\mathcal{S}}(f, g) \geq \min_{g \in \mathcal{D}} \rho_{\mathcal{S}}(f', g) \right\}.$$

When defining least dictatorial rules based on the distance function ρ_S, we could have taken, for instance, the average distance from the dictators. However, we feel that if we would like to be 'least dictatorial', we should be more concerned about the closest dictatorial rule. Anyway, for anonymous SCFs this question does not matter.

A tie-breaking rule $\tau : \mathcal{P}^n \to \mathcal{P}$ maps preference profiles to linear orderings on A, which will be only employed when a formula does not determine a unique winner. From the large set of possible tie-breaking rules, we will restrict ourselves to anonymous tie-breaking rules. If there are more alternatives chosen by a formula 'almost' specifying a SCF, then the highest ranked alternative is selected, based on the given tie-breaking rule among tied alternatives.

When defining $\mathcal{F}_{ld}(\mathcal{S})$, we are looking for SCFs which are the least dictatorial ones. From an opposite point of view, we might believe that a SCF that lets the voters be a dictator in as many cases as possible could result in a desirable SCF. Having this goal in mind, a measure

$$\mu(f, \mathcal{D}) = \sum_{\succ \in \mathcal{P}^n} \# \{i \in N \mid f(\succ) = d_i(\succ)\},$$

appears as a natural candidate, which we call the measure of conformity. Considering all profiles, $\mu(f, \mathcal{D})$ simply counts the number of cases in which a person's top alternative is chosen.

Introducing the notation $\mu(f, g) = \sum_{\succ \in \mathcal{P}^n} \mathbf{1}_{f(\succ) = g(\succ)}$, where $\mathbf{1}_{f(\succ) = g(\succ)}$ indicates whether the two chosen alternatives equal, we can obtain the following relationship between μ and ρ:

$$\mu(f, \mathcal{D}) = \sum_{\succ \in \mathcal{P}^n} \sum_{i \in N} \mathbf{1}_{f(\succ) = d_i(\succ)} = \sum_{i \in N} \mu(f, d_i) = n(m!)^n - \sum_{i \in N} \rho(f, d_i).$$

[1] It is worthwhile emphasizing that we do not require that f and g equal on $\mathcal{S}$.

Definition 4. We define the set of *most dictatorial rules* by

$$\begin{aligned}\mathcal{F}_{md} &= \{f \in \mathcal{F} \mid \forall f' \in \mathcal{F} : \mu(f, \mathcal{D}) \geq \mu(f', \mathcal{D})\} \\ &= \left\{ f \in \mathcal{F} \mid \forall f' \in \mathcal{F} : \sum_{i \in N} \rho(f, d_i) \leq \sum_{i \in N} \rho(f', d_i) \right\}. \end{aligned} \tag{2.2}$$

3 Results

The following rule will play a special role:

Definition 5. Let τ be an anonymous tie-breaking rule. Then the social choice rule f_τ^* is defined in the following way: If there is a unique alternative, being the fewest (including zero) times on the top, then that alternative is the chosen one. If not, disregard those alternatives that are not the fewest times on the top, and select the chosen alternative based on the given tie-breaking rule.

Clearly, the above specified rule can also be just taken on a subset of profiles $\overline{\mathcal{S}}$ in case of a domain restriction $\mathcal{S}$ and any other known rule can be employed on $\mathcal{S}$.

Proposition 1. *Assume that $\mathcal{S}$ is an anonymous subdomain of $\mathcal{P}^n$. Then $f_\tau^* \in \mathcal{F}_{ld}(\mathcal{S})$. For any anonymous $f \in \mathcal{F}_{ld}(\mathcal{S})$, there exists a tie-breaking rule τ such that $f = f_\tau^*$ on $\overline{\mathcal{S}}$.*

Proof. First, observe that

$$\begin{aligned}\sum_{i \in N} \rho_{\mathcal{S}}(f, d_i) &= \sum_{i \in N} \# \left\{ \succ \in \overline{\mathcal{S}} \mid f(\succ) \neq d_i(\succ) \right\} \\ &= \# \left\{ (i, \succ) \in N \times \overline{\mathcal{S}} \mid f(\succ) \neq d_i(\succ) \right\} \\ &= \sum_{\succ \in \overline{\mathcal{S}}} \# \{ i \in N \mid f(\succ) \neq d_i(\succ) \} \end{aligned} \tag{3.3}$$

for any SCF f.

By the definition of f_τ^* we have

$$\forall \succ \in \mathcal{P}^n : \# \{ i \in N \mid f_\tau^*(\succ) \neq d_i(\succ) \} \geq \# \{ i \in N \mid f(\succ) \neq d_i(\succ) \}. \tag{3.4}$$

Now taking the sums above $\overline{\mathcal{S}}$ of both the left hand side and the right hand side of equation (3.4) and then combining it with (3.3), we get

$$\sum_{i \in N} \rho_{\mathcal{S}}(f_\tau^*, d_i) \geq \sum_{i \in N} \rho_{\mathcal{S}}(f, d_i), \tag{3.5}$$

from which for any $i \in N$ it follows that

$$\rho_{\mathcal{S}}(f_\tau^*, d_i) = \frac{1}{n} \sum_{i \in N} \rho_{\mathcal{S}}(f_\tau^*, d_i) \geq \frac{1}{n} \sum_{i \in N} \rho_{\mathcal{S}}(f, d_i) \geq \min_{i \in N} \rho_{\mathcal{S}}(f, d_i) \tag{3.6}$$

since f_τ^* and $\mathcal{S}$ are anonymous and the average is larger than the minimum; meaning that $f_\tau^* \in \mathcal{F}_{ld}(\mathcal{S})$.

For the second statement observe that if f selects for at least one profile in $\overline{\mathcal{S}}$ an alternative that is not the fewest times on the top, then the inequality in (3.5), and therefore also the inequality in (3.6) will be strict. Finally, an anonymous tie-breaking rule can be chosen in line with f. □

Denote by $\overline{d}_i \in \mathcal{F}$ the SCF which selects the bottom alternative of voter i. For this SCF, call i the inverse dictator.

Definition 6. Assume that the union of the pairwise disjoint sets $\mathcal{C}_1, \ldots, \mathcal{C}_n$ equals $\mathcal{P}^n$. Then the respective combination of inverse dictatorial functions, defining a SCF, selects for profiles in $\mathcal{C}_i$ the bottom alternative of voter i.

Let $\mathcal{F}_{id}$ be the set of inverse dictatorial functions. It might be surprising that none of the combinations of inverse dictatorial functions are least dictatorial ones.

Remark 1. Assuming that $n \geq 2$ and $m \geq 3$, the combinations of inverse dictatorial functions are not least dictatorial.

Proof. Let us focus on profiles $\Pi_{x,y}$ in which all preferences have either $x \in A$ on the top and $y \in A$ at the bottom or $y \in A$ on the top and $x \in A$ at the bottom. From these types of profiles there exists at least n ones (in fact even at least $2^n - 2$) in case of $n \geq 2$ and $m \geq 3$. Picking an arbitrary inverse dictatorial function $\hat{f} \in \mathcal{F}_{id}$, we can devise a rule f' by changing the choice of $\hat{f}$ on profiles $\Pi_{x,y}$ such that it chooses an intermediate alternative $z \in A$, different from both x and y. Observe that then we have

$$\sum_{i \in N} \rho(\hat{f}, d_i) + n \leq \sum_{i \in N} \rho(f', d_i) \leq \sum_{i \in N} \rho(f^*_\tau, d_i), \tag{3.7}$$

where the latter inequality follows from (3.5). By dividing both sides by n and taking into consideration that f^*_τ is anonymous, we obtain

$$\min_{i \in N} \rho(\hat{f}, d_i) + 1 \leq \left(\frac{1}{n} \sum_{i \in N} \rho(\hat{f}, d_i) \right) + 1 < \rho(f^*_\tau, d_i), \tag{3.8}$$

which shows that any combined inverse dictatorial functions is not as far away from the closest dictatorial rule as f^*_τ. □

Though f^*_τ performs well according to our specification of a least dictatorial rule, as it can be easily verified, above the universal domain it can select a Pareto dominated alternative, never selects a unanimous winner, and violates monotonicity among many other desirable properties. Therefore, we have introduced anonymous domain restrictions so that, for instance, on profiles with a unanimous winner, the unanimous winner should be selected, and we are searching for the least dictatorial rules only above the set of profiles which do not have a unanimous winner. However, Proposition 1 shows that even if we restrict our choices above an anonymous subset $\mathcal{S}$ of profiles, f^*_τ has to be employed above $\overline{\mathcal{S}}$, if we would like to be anonymous and least dictatorial according to our definition.

Turning to the most dictatorial rules, the following rules play a central role:

Definition 7. Let τ be an anonymous tie-breaking rule. Then the social choice rule $\tilde{f}_\tau$ is defined in the following way: If there is a unique alternative, being the most times on the top, then that alternative is the chosen one. If not, disregard those alternatives that are not the most times on the top, and select the chosen alternative based on the given tie-breaking rule.

The above specified rule is basically the plurality rule.

Proposition 2. *$\tilde{f}_\tau \in \mathcal{F}_{md}$. For any anonymous $f \in \mathcal{F}_{md}$, there exists a tie-breaking rule τ such that $f = \tilde{f}_\tau$.*

Proof. By the definition of $\tilde{f}_\tau$ we have

$$\forall \succ \in \mathcal{P}^n : \# \left\{ i \in N \mid \tilde{f}_\tau(\succ) = d_i(\succ) \right\} \geq \# \left\{ i \in N \mid f(\succ) = d_i(\succ) \right\} \tag{3.9}$$

for any $f \in \mathcal{F}$. Now summing (3.9) above $\mathcal{P}^n$, we get

$$\mu(\tilde{f}_\tau, \mathcal{D}) \geq \mu(f, \mathcal{D}), \tag{3.10}$$

from which it follows that $\tilde{f}_\tau \in \mathcal{F}_{md}$.

For the second statement observe that if f selects for at least one profile in $\mathcal{P}^n$ an alternative that is not the most times on the top, then the inequality in (3.10) will be strict. The tie-breaking rule τ can be selected in line with f. □

4 Concluding remarks

In Section 3 we considered a metric, which did not take the distribution of preferences in a profile into consideration. A possible extension of the metric given by (2.1), which can be then considered as the special uniform case, may lead to the metric of the functional form specified below:

$$\rho_{\mathcal{S},w}(f,g) = \sum_{\succ \in \overline{\mathcal{S}}} w(\succ) \mathbf{1}_{f(\succ) \neq g(\succ)}, \tag{4.11}$$

where the weight function w could take into account the homogeneity of profile $\succ$, for instance, in case of identical preferences it would be the most disturbing that the alternatives chosen by f and g differ (heavy weight), while in case of 'very heterogeneous' profiles this might seem more natural (light weigth), and $\mathbf{1}_{f(\succ) \neq g(\succ)}$ indicates whether the two chosen alternatives differ.

We could get a more refined picture if we consider social choice rules instead of SCFs, that is, we care about the whole social ranking and not only about the socially best alternative. We plan to address the investigation of metrics given by (4.11) and the case of social choice rules in future research.

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