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On the existence of the Dutta-Ray's egalitarian solution

Francesc Llerena and Llúcia Mauri*

Abstract

A class of balanced games, called exact partition games, is introduced. Within this class, it is shown that the egalitarian solution of Dutta and Ray (1989) behaves as in the class of convex games. Moreover, we provide two axiomatic characterization by means of suitable properties such as consistency, rationality and Lorenz-fairness. As a by-product, alternative characterizations of the egalitarian solution over the class of convex games are obtained.

1 Introduction

On the domain of transferable utility coalitional game (TU-games, for short), several solution concepts have been motivated by the idea of egalitarianism. One of the best known is the weak constrained egalitarian solution (WCES, for short), introduced by Dutta and Ray (1989). This solution is defined in a setting where agents believe in equality as a desirable social goal, but their individual preferences dictate selfish behavior. The WCES yields, whenever it exists, the unique Lorenz-maximal imputation within the Lorenz core, which is a proper extension of the core. Although this is a sharp result because the Lorenz domination generates a partial ranquing, this solution lacks general existence properties. In fact, the class of convex games (Shapley, 1971) is the only standard class of TU-games where its existence is guaranteed. On this domain, Dutta and Ray (1989) describe an algorithm for finding their egalitarian allocation and show that it belongs to the core and Lorenz dominates every other core element. Unfortunatly, several examples in the same paper show that, in a general domain, these assertions are not true: there are games with a nonempty core where the WCES does not exist, and vice-versa; games where both the core and the WCES exist but the latter does not lie in the core, or games where the WCES belongs to the core but does not

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Lorenz dominate every other core element. On the domain of balanced games, an alternative route, already suggested by Dutta and Ray (1989) and latter adopted by Arin and Iñarra (2001) and Hougaard et al. (2001), is to focus on the Lorenz maximal allocations within the core. A problem with this solution concept is that it is not single-valued. To overcome this drawback, Arin and Iñarra (2001) and Arin et al. (2003) propose single-valued solutions which are derived from the application of the Rawlsian criterion on the core. On the domain of convex games all these solution concepts produce the same outcome.

A review of the proofs of Theorems 2 and 3 in Dutta and Ray (1989) shows that weaker conditions than convexity are enough to guarantee that their egalitarian solution behaves as in convex games. With this objective, in Section 3 we introduce a subclasses of balanced games called exact partition games. This class of games is rich enough to include convex games and dominant diagonal assignment games (Solymosi and Raghavan, 2001), but also nonsuperadditive games. Within this class, in Section 4 we provide two axiomatic characterization of the WCES by means of suitable properties such as consistency (à la Davis and Maschler, 1965), rationality and two new properties inspired by the notion of stable sets of von Neumann and Morgensten, but changing the usual order in $\mathbb{R}^N$ by the Lorenz order. As particular cases, we obtain alternative characterizations of the WCES over the domain of convex games, and of the set of Lorenz maximal allocation within the core over the domain of balanced games. Some final remarks conclude the paper. We begin with notation and terminology.

2 Notation and terminology

The set of natural numbers $\mathbb{N}$ denotes the universe of potential players. A **coalition** is a non-empty finite subset of $\mathbb{N}$ and let $\mathcal{N} := \{N \mid \emptyset \neq N \subseteq \mathbb{N}, |N| < \infty\}$ denote the set of all coalitions of $\mathbb{N}$. A **TU-game (a game)** is a pair (N, v) where $N \in \mathcal{N}$ is the set of players and $v : 2^N \longrightarrow \mathbb{R}$ is the characteristic function that assigns to each coalition $S \subseteq N$ a real number $v(S)$, with the convention $v(\emptyset) = 0$. Given $S, T \in \mathcal{N}$, we use $S \subset T$ to indicate strict inclusion, that is, $S \subseteq T$ but $S \neq T$. By $|S|$ we denote the cardinality of the coalition $S \in \mathcal{N}$. By Γ we denote the class of all games.

Given $N \in \mathcal{N}$, let $\mathbb{R}^N$ stand for the space of real-valued vectors indexed by N, $x = (x_i)_{i \in N}$, and for all $S \subseteq N$, $x(S) = \sum_{i \in S} x_i$, with the convention $x(\emptyset) = 0$. For each $x \in \mathbb{R}^N$ and $T \subseteq N$, $x_{|T}$ denotes the restriction of x to T: $x_{|T} = (x_i)_{i \in T} \in \mathbb{R}^T$. Given two vectors $x, y \in \mathbb{R}^N$, $x \geq y$ if $x_i \geq y_i$, for all $i \in N$. We say that $x > y$ if $x \geq y$ and for some $j \in N$, $x_j > y_j$. Let (N, v) be a game and $S \subseteq N$, $S \neq \emptyset$. A coalition S is an **equity coalition** of (N, v) if $S \in \text{Argmax}_{\emptyset \neq R \subseteq N} \left\{\frac{v(R)}{|R|}\right\}$. In addition, S is a **maximal** (w.r.t. inclusion) **equity coalition** of (N, v) if $S \in \text{Argmax}_{\emptyset \neq R \subseteq N} \left\{\frac{v(R)}{|R|}\right\}$ and there is no $T \in \text{Argmax}_{\emptyset \neq R \subseteq N} \left\{\frac{v(R)}{|R|}\right\}$ such that

$S \subset T$. Given N, a set $\pi = \{P_1, \ldots, P_m\}$, where $P_i \subseteq N$ for all $i \in \{1, \ldots, m\}$, with $m \leq |N|$, is a **partition** of N if the following conditions hold: (i) $P_i \neq \emptyset$ for all $i \in \{1, \ldots, m\}$, (ii) $\cup_{i=1}^{m} P_i = N$, and (iii) $P_i \cap P_j = \emptyset$, for all $i, j \in \{1, \ldots, m\}$, $i \neq j$.

The set of **feasible payoff vectors** of a game (N, v) is defined by $X^*(N, v) := \{x \in \mathbb{R}^N \,|\, x(N) \leq v(N)\}$. A **solution** on a class of games $\Gamma' \subseteq \Gamma$ is a mapping σ which associates with each game $(N, v) \in \Gamma'$ a subset $\sigma(N, v)$ of $X^*(N, v)$. Notice that σ is allowed to be empty. A solution on a class of games $\Gamma' \subseteq \Gamma$ is said to be **single-valued** if $|\sigma(N, v)| = 1$ for all $(N, v) \in \Gamma'$. Two games (N, v) and (N, v') are **strategically equivalent** if there is a vector $(d_1, \ldots, d_n) \in \mathbb{R}^N$ and $\alpha > 0$ such that for all coalitions $S \subseteq N$, $v'(S) = \alpha\, v(S) + \sum_{i \in S} d_i$. A solution σ on $\Gamma' \subseteq \Gamma$ satisfies **covariance** if for all two strategically equivalent games $(N, v), (N, v') \in \Gamma'$, $\sigma(N, v') = \alpha\, \sigma(N, v) + \sum_{i \in N} d_i$. The **pre-imputation set** of (N, v) is defined by $X(N, v) := \{x \in \mathbb{R}^N \,|\, x(N) = v(N)\}$, and the set of **imputations** by $I(N, v) := \{x \in X(N, v) \,|\, x_i \geq v(\{i\}),$ for all $i \in N\}$. The **core** of (N, v) is the set of those imputations where each coalition gets at least its worth, that is, $C(N, v) = \{x \in X(N, v) \mid x(S) \geq v(S)$ for all $S \subseteq N\}$. A game (N, v) is **balanced** if it has a non-empty core. By $\Gamma_{\mathcal{B}al}$ we denote the class of balanced games. A game (N, v) is **superadditive** if, for every $S, T \subseteq N, S \cap T = \emptyset$, $v(S) + v(T) \leq v(S \cup T)$. A game (N, v) is **convex** if, for every $S, T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$. By $\Gamma_{\mathcal{C}on}$ we denote the class of convex games. Recall that $\Gamma_{\mathcal{C}on} \subset \Gamma_{\mathcal{B}al}$.

Given $N \in \mathcal{N}$, and for any $x \in \mathbb{R}^N$, let us denote by $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)$ the vector obtained from x by rearranging its coordinates in a non-increasing order, that is, $\hat{x}_1 \geq \hat{x}_2 \geq \ldots \geq \hat{x}_n$. In a similar way, for $\emptyset \neq T \subseteq N$, $\widehat{x_{|T}}$ denotes the vector obtained from the restriction of x to T ordering its coordinates in a non-increasing way: $\widehat{x_{|T}}_1 \geq \widehat{x_{|T}}_2 \geq \ldots \geq \widehat{x_{|T}}_t$, where $t = |T|$. For any two vectors $y, x \in \mathbb{R}^N$ with $y(N) = x(N)$, we say that y **Lorenz dominates** x, denoted by $y \succ_{\mathcal{L}} x$, if $\sum_{j=1}^{k} \hat{y}_j \leq \sum_{j=1}^{k} \hat{x}_j$, for all $k \in \{1, \ldots, |N|\}$, with at least one strict inequality. Given a coalition $S \in \mathcal{N}$ and a set $A \subseteq \mathbb{R}^S$, EA denotes the set of allocations that are Lorenz undominated within A. That is, $EA := \{x \in\ A \,|\, \nexists\, y \in A \text{ such that } y \succ_{\mathcal{L}} x\}$. Given a game (N, v), the **Lorenz core** is defined in a recursive way as follows. The Lorenz core of a singleton coalition is $L(\{i\}, v) = \{v(\{i\})\}$. Now suppose that the Lorenz core for all coalitions of cardinality k or less have been defined, where $1 < k < |N|$. The Lorenz core of a coalition $S \subset N$ of size $(k + 1)$ is defined by

$$L(S, v) = \left\{x \in \mathbb{R}^S \mid x(S) = v(S) \text{ and } \nexists\, T \subset S \text{ and } y \in EL(T, v) \text{ such that } y > x_{|T}\right\}.$$

Note that, for all $S \subseteq N$, $C(S, v) \subseteq L(S, v)$.

The **weak constrained egalitarian solution (WCES)** (Dutta and Ray, 1989), denoted by EL, selects the vectors that are Lorenz undominated within the Lorenz core. For all $(N, v) \in \Gamma$, $|EL(N, v)| \leq 1$ (Dutta and Ray,1989). The

constrained egalitarian solution, denoted by CE, is a single-valued solution defined for two person games as follows: let (N, v) be a game with $N = \{i, j\}$ and suppose, without loss of generality, $v(i) \leq v(j)$, then $CE_j(N, v) = \max\left\{\frac{v(N)}{2}, v(j)\right\}$ and $CE_i(N, v) = v(N) - CE_j(N, v)$.

The next two observations will be useful to prove our results.

Remark 1. *(Hougaard et al. 2001 p. 153) Let N be a finite set of players, and let $S \subseteq N$, $S \neq \emptyset$. If x_S, $y_S \in \mathbb{R}^S$, $x_S(S) = y_S(S)$ and $z_{N\setminus S} \in \mathbb{R}^{N\setminus S}$, then x_S Lorenz dominates y_S if and only if $\left(x_S, z_{N\setminus S}\right)$ Lorenz dominates $\left(y_S, z_{N\setminus S}\right)$.*

Remark 2. *Let N be a finite set of players, $c \in \mathbb{R}$ and $(x_1, \ldots, x_n) \in \mathbb{R}^N$. It is well-known that if $\sum_{i\in N} x_i = nc$, then x is Lorenz dominated by $(c, \ldots, c) \in \mathbb{R}^N$. If $\sum_{i\in N} x_i > nc$, let $\epsilon = \sum_{i\in N} x_i - nc$ and define $x^\epsilon = (x_1 - \frac{\epsilon}{n}, \ldots, x_n - \frac{\epsilon}{n})$. Note that $\widehat{x^\epsilon}_i = \widehat{x}_i - \frac{\epsilon}{n} < \widehat{x}_i$, for all $i \in N$. Thus, x^ϵ is Lorenz dominated by $(c, \ldots, c)$ which implies, for all $k = 1, \ldots, n$, $\hat{x}_1 + \ldots + \hat{x}_k > \widehat{x^\epsilon}_1 + \ldots + \widehat{x^\epsilon}_k \geq kc$.*

3 Exact partition games

On the domain of convex games, Dutta and Ray (1989) show that the WCES picks the payoff vector that is obtained by the following algorithm.
Let (N, v) be a convex game and $EL(N, v) = \{x\}$.
Step 1: Define $v_1 = v$. Then find the unique coalition $T_1 \subseteq N$ such that for all $T \subseteq N$, (i) $\frac{v_1(T_1)}{|T_1|} \geq \frac{v_1(T)}{|T|}$, and (ii) if $\frac{v_1(T_1)}{|T_1|} = \frac{v_1(T)}{|T|}$ and $T \neq T_1$, then $|T_1| > |T|$. Uniqueness of such a coalition is guaranteed by convexity of (N, v). For all $i \in T_1$,

$$x_i = \frac{v_1(T_1)}{|T_1|}.$$

Step k: Suppose that $T_1, \ldots, T_{k-1}$ have been defined.
Let $N_k = N \setminus \{T_1 \cup \ldots \cup T_{k-1}\}$ and let (N_k, v_k) be the **marginal game** defined as follows:

$$v_k(S) := v(T_1 \cup \ldots \cup T_{k-1} \cup S) - v(T_1 \cup \ldots \cup T_{k-1}), \tag{1}$$

for all $S \subseteq N_k$.
It can be shown that (N_k, v_k) is convex. Then find the unique coalition $T_k \subseteq N_k$ such that for all $T \subseteq N_k$, (i) $\frac{v_k(T_k)}{|T_k|} \geq \frac{v_k(T)}{|T|}$, and (ii) if $\frac{v_k(T_k)}{|T_k|} = \frac{v_k(T)}{|T|}$ and $T \neq T_k$, then $|T_k| > |T|$. For all $i \in T_k$,

$$x_i = \frac{v_k(T_k)}{|T_k|} = \frac{v(T_1 \cup \ldots \cup T_k) - v(T_1 \cup \ldots \cup T_{k-1})}{|T_k|}.$$

By construction, the WCES satisfies the following conditions: if $\pi = (T_1, \ldots, T_t)$ is the ordered partition of N induced by $EL(N, v) = \{x\}$, then

- **(C1)**: $x_i = x_j$ for all $i, j \in T_k$ and $k = 1, \ldots, t$,
- **(C2)**: $x(T_1 \cup \ldots \cup T_k) = v(T_1 \cup \ldots \cup T_k)$, for all $k = 1, \ldots, t$,
- **(C3)**: $x_i > x_j$ if $i \in T_k$, $j \in T_h$, and $k < h \leq t$.

The idea underlying this procedure is that agents in the unique maximal (w.r.t. inclusion) coalition T_1 maximizing the average worth $\frac{v(T_1)}{|T_1|}$ share equally the amount $v(T_1)$ among them and leave the game. Then, the remaining agents $N \setminus T_1$ play a suitable reduced convex game where, again, agents in the unique maximal coalition with highest average worth divide its worth equally among its members. The process stops when all agents have been paid.

Theorem 2 in Dutta and Ray (1989) states that the output of this algorithm is the WCES and that it belongs to the core. Theorem 3 in the same paper tells us that, for convex games, the WCES Lorenz dominates every other core element. Nevertheless, an analysis of the proofs of the aforementioned results reveals that much weaker conditions than convexity are sufficient to guarantee the same results.

Definition 1. *Let $N = \{1, \ldots, n\}$ be a finite set of players and $x \in \mathbb{R}^N$. We define the ordered partition of N induced by x, $\pi = (N_1, \ldots, N_m)$, as follows:*

$$\begin{aligned} N_1 &= \{i \in N \,|\, x_i \geq x_k \text{ for all } k \in N\}, \\ N_2 &= \{i \in N \setminus N_1 \,|\, x_i \geq x_k \text{ for all } k \in N \setminus N_1\}, \\ &\vdots \\ N_m &= \{i \in N \setminus N_1 \cup \ldots \cup N_{m-1} \,|\, x_i \geq x_k \text{ for all } k \in N \setminus N_1 \cup \ldots \cup N_{m-1}\}. \end{aligned}$$

Theorem 1. *Let (N, v) be a balanced game, $x \in C(N, v)$ and let $\pi = (N_1, \ldots, N_m)$ be the ordered partition of N induced by x. If $x(N_1 \cup \ldots \cup N_k) = v(N_1 \cup \ldots \cup N_k)$, for all $k = 1, \ldots, m$, then $EL(N, v) = \{x\}$ and $x \succ_{\mathcal{L}} y$, for all $y \in C(N, v) \setminus \{x\}$.*

Proof. First we show that $x \succ_{\mathcal{L}} y$, for all $y \in C(N, v) \setminus \{x\}$.

Assume, without loss of generality, that $x_1 \geq x_2 \geq \ldots \geq x_n$. Then, the vector obtained from x by rearranging its coordinates in non-increasing order is $\widehat{x} = x$. Let us denote

$$c_k = \begin{cases} \frac{v(N_1)}{|N_1|} & \text{if } k = 1 \\ \\ \frac{v(N_1 \cup \ldots \cup N_{k-1} \cup N_k) - v(N_1 \cup \ldots \cup N_{k-1})}{|N_k|} & \text{if } k > 1 \end{cases}$$

for all $k = 1, \ldots, m$, $(m > 1)$.

Notice that $x_i = c_k$ for all $i \in N_k$ and $k = 1, \ldots, m$. Let $y \in C(N, v)$, $y \neq x$. From Remark 1 we may suppose, without loss of generality, $x_i \neq y_i$ for all $i \in N$. Since $y(N_1) \geq v(N_1) = x(N_1) = c_1|N_1|$, and by Remark 2, we have that for all $t = 1, \ldots, |N_1|$,

$$tc_1 \leq \widehat{y_{|N_1}}_1 + \ldots + \widehat{y_{|N_1}}_t, \tag{2}$$

with at least one strict inequality.

Next we are going to prove that, for all $t = 1, \ldots, |N_2|$,

$$x(N_1) + tc_2 \leq y(N_1) + \widehat{y_{|N_2}}_1 + \ldots + \widehat{y_{|N_2}}_t. \tag{3}$$

If $y(N_2) \geq x(N_2) = |N_2|c_2$, again by Remark 2, $tc_2 \leq \widehat{y_{|N_2}}_1 + \ldots + \widehat{y_{|N_2}}_t$, for all $t = 1, \ldots, |N_2|$. This set of inequalities, together with (2), lead to expression (3).

If $y(N_2) < x(N_2)$, let us denote $\varphi_1 = y(N_1) - x(N_1) \geq 0$ and $\beta_1 = x(N_2) - y(N_2) > 0$. Let $z \in \mathbb{R}^{N_2}$ defined as $z_i = y_i + \frac{\beta_1}{|N_2|}$ for all $i \in N_2$. Since $x(N_2) = y(N_2) + \beta_1 = z(N_2)$, by Remark 2 we have $c_2 \leq \hat{z}_1 = \widehat{y_{|N_2}}_1 + \frac{\beta_1}{|N_2|} \leq \widehat{y_{|N_2}}_1 + \beta_1$, which implies $\beta_1 \geq c_2 - \widehat{y_{|N_2}}_1$. This last inequality, together with $y(N_1 \cup N_2) \geq v(N_1 \cup N_2) = x(N_1 \cup N_2)$, lead to

$$\varphi_1 = y(N_1) - x(N_1) \geq x(N_2) - y(N_2) = \beta_1 \geq c_2 - \widehat{y_{|N_2}}_1. \tag{4}$$

Now from (4) it follows

$$x(N_1) + c_2 \leq y(N_1) + \widehat{y_{|N_2}}_1. \tag{5}$$

If $|N_2| \geq 2$ and $\sum_{i=2}^{|N_2|} \widehat{y_{|N_2}}_i \geq (|N_2| - 1) c_2$, then from Remark 2, $tc_2 \leq \widehat{y_{|N_2}}_2 + \ldots + \widehat{y_{|N_2}}_{t+1}$, for all $t = 1, \ldots, |N_2| - 1$, which leads, together with (5), to (3). Otherwise, if $|N_2| \geq 2$ and $\sum_{i=2}^{|N_2|} \widehat{y_{|N_2}}_i < (|N_2| - 1) c_2$, let us denote

$$\varphi_2 = y(N_1) + \widehat{y_{|N_2}}_1 - x(N_1) - c_2 \text{ and } \beta_2 = (|N_2| - 1) c_2 - \sum_{i=2}^{|N_2|} \widehat{y_{|N_2}}_i > 0.$$

From (4) it follows $\varphi_2 \geq \beta_2 > 0$. Next we show that $\beta_2 \geq c_2 - \widehat{y_{|N_2}}_2$. Choose $k \in N_2$ such that $y_k \geq y_i$ for all $i \in N_2$ and define $z \in \mathbb{R}^{N_2 \setminus \{k\}}$ as $z_i = y_i + \frac{\beta_2}{|N_2|-1}$ for all $i \in N_2 \setminus \{k\}$. Since $z(N_2 \setminus \{k\}) = y(N_2 \setminus \{k\}) + \beta_2 = x(N_2) - c_2$, by Remark 2 we have $c_2 \leq \hat{z}_1 = \widehat{y_{|N_2}}_2 + \frac{\beta_2}{|N_2|-1} \leq \widehat{y_{|N_2}}_2 + \beta_2$, which implies $\beta_2 \geq c_2 - \widehat{y_{|N_2}}_2$. Since $\varphi_2 \geq \beta_2$, we obtain

$$\varphi_2 \geq c_2 - \widehat{y_{|N_2}}_2. \tag{6}$$

Now from (6) it can be checked that $x(N_1) + 2c_2 \leq y(N_1) + \widehat{y_{|N_2}}_1 + \widehat{y_{|N_2}}_2$. Applying the same reasoning for $t = 3, \ldots, |N_2|$ we obtain (3).

Following the same line of argument it can be proved that, for all $k = 3, \ldots, m$ and all $t = 1, \ldots, |N_k|$,

$$x(N_1 \cup \ldots \cup N_{k-1}) + tc_k \leq y(N_1 \cup \ldots \cup N_{k-1}) + \sum_{j=1}^{t} \widehat{y_{|N_k}}_j. \tag{7}$$

Finally, combining (2), (3) and (7) we get

$$\begin{array}{rcl} x_1 = c_1 & \leq & \widehat{y_{|N_1}}_1 \leq \widehat{y}_1 \\ x_1 + x_2 = 2c_1 & \leq & \widehat{y_{|N_1}}_1 + \widehat{y_{|N_1}}_2 \leq \widehat{y}_1 + \widehat{y}_2 \\ & \vdots & \\ x_1 + \ldots + x_{|N_1|} = x(N_1) & \leq & y(N_1) \leq \widehat{y}_1 + \ldots \widehat{y}_{|N_1|} \\ x_1 + \ldots + x_{|N_1|+1} = x(N_1) + c_2 & \leq & y(N_1) + \widehat{y_{|N_2}}_1 \leq \widehat{y}_1 + \ldots \widehat{y}_{|N_1|+1} \\ & \vdots & \\ x_1 + \ldots + x_{|N_1|+|N_2|} = x\left(N_1 \cup N_2\right) & \leq & y\left(N_1 \cup N_2\right) \leq \widehat{y}_1 + \ldots \widehat{y}_{|N_1|+|N_2|} \\ & \vdots & \\ x_1 + \ldots + x_n = x\left(N_1 \cup \ldots \cup N_m\right) & = & y\left(N_1 \cup \ldots \cup N_m\right) = \widehat{y}_1 + \ldots + \widehat{y}_n, \end{array}$$

with at least one strict inequality,[1] which means that $x \succ_{\mathcal{L}} y$.

To see that $EL(N, v) = \{x\}$, we replicate the induction argument used by Dutta and Ray (1989) to prove their Theorem 2 (step 2).[2]

Note first that $EL\left(N_1, v\right) = \left\{x_{|N_1}\right\}$. Next we see that for all $t = 1, \ldots, m-1$, if $EL\left(N_1 \cup \ldots \cup N_t, v\right) = \left\{x_{|N_1 \cup \ldots \cup N_t}\right\}$, then $EL\left(N_1 \cup \ldots \cup N_{t+1}, v\right) = \left\{x_{|N_1 \cup \ldots \cup N_{t+1}}\right\}$.

Suppose that $EL\left(N_1 \cup \ldots \cup N_t, v\right) = \left\{x_{|N_1 \cup \ldots \cup N_t}\right\}$ but $EL\left(N_1 \cup \ldots \cup N_{t+1}, v\right) \neq \left\{x_{|N_1 \cup \ldots \cup N_{t+1}}\right\}$, for some t. Since $x\left(N_1 \cup \ldots \cup N_{t+1}\right) = v\left(N_1 \cup \ldots \cup N_{t+1}\right)$ and $x \in C\left(N, v\right)$, we have

$$x_{|N_1 \cup \ldots \cup N_{t+1}} \in C\left(N_1 \cup \ldots \cup N_{t+1}, v_{|N_1 \cup \ldots \cup N_{t+1}}\right) \subseteq L\left(N_1 \cup \ldots \cup N_{t+1}, v\right),$$

and thus there exists $y \in L\left(N_1 \cup \ldots \cup N_{t+1}, v\right)$ with $y \succ_{\mathcal{L}} x_{|N_1 \cup \ldots \cup N_{t+1}}$. Then,

$$\begin{array}{lcl} \hat{y}_1 & \leq & x_1 \\ \hat{y}_1 + \hat{y}_2 & \leq & x_1 + x_2 \\ & \vdots & \\ \hat{y}_1 + \ldots + \hat{y}_{|N_1 \cup \ldots \cup N_{t+1}|} & = & x_1 + \ldots + x_{|N_1 \cup \ldots \cup N_{t+1}|} \end{array} \tag{8}$$

with at least one strict inequality.

Since $y\left(N_1 \cup \ldots \cup N_{t+1}\right) = x\left(N_1 \cup \ldots \cup N_{t+1}\right)$, if $y_j \geq x_j$ for all $j \in N_1 \cup \ldots \cup N_{t+1}$ then we would have $y = x_{|N_1 \cup \ldots \cup N_{t+1}}$, in contradiction with $y \succ_{\mathcal{L}} x_{|N_1 \cup \ldots \cup N_{t+1}}$. As a consequence, the set $\mathcal{J} := \{j \in N_1 \cup \ldots \cup N_{t+1} \mid y_j < x_j\}$ must be non-empty. Take then $q^* = min\left\{k \in \{1, \ldots, t+1\} \mid \mathcal{J} \cap N_k \neq \emptyset\right\}$. We claim that,

$$y_i \leq x_i \text{ for all } i \in N_{q^*}.$$

[1]This strict inequality follows from expression (2).

[2]We describe in detail the induction argument for the convenience of the reader.

Indeed, if $q^* = 1$, for all $i \in N_1$ it follows from (8) that $y_i \le \hat{y}_1 \le \widehat{x}_1 = x_i$. If $q^* > 1$, from $y_i \ge x_i$ for all $i \in N_1$ and expression (8) we have $y_i = x_i$ for all $i \in N_1$.Then, again from (8), we obtain $\hat{y}_{|N_1|+1} \le x_{|N_1|+1}$. The repetition of the same argument leads to $y_i = x_i$ for all $i \in N_1 \cup \ldots \cup N_{q^*-1}$. Now, taking into account (8) and the definition of π we obtain, for all $i \in N_{q^*}$,

$$y_i \le \hat{y}_{|N_1 \cup \ldots \cup N_{q^*-1}|+1} \le \widehat{x}_{|N_1 \cup \ldots \cup N_{q^*-1}|+1} = x_i.$$

Note that $q^* \le t$, since otherwise $y\left(N_1 \cup \ldots \cup N_{t+1}\right) < x\left(N_1 \cup \ldots \cup N_{t+1}\right)$.

So, denote $T = N_1 \cup \ldots \cup N_{q^*}$. By hypothesis, $EL(T, v) = \left\{x_{|T}\right\}$. But then, since $y_i \le x_i$ for all $i \in T$ and there exists $j^* \in N_{q^*}$ such that $y_{j^*} < x_{j^*}$, we conclude that $y \notin L\left(N_1 \cup \ldots \cup N_{t+1}, v\right)$, getting a contradiction. This means that $EL(N, v) = \{x\}$. □

Remark 3. *Under some conditions of positivity, a similar result was stated by Sánchez-Soriano et al. (2014). In that paper, Proposition 2 says the following: The vector* $a = (1_{n_1}a_1, 1_{n_2}a_2 \ldots, 1_{n_t}a_t)$ *such that* $a_1 \ge a_2 \ge \ldots \ge a_t > 0$ *and* $\sum_{i=1}^{t} n_i = n$, *where* $1_{n_i} = (1, \ldots, 1) \in \mathbb{R}^{n_i}$ *for all* $i = 1, \ldots, t$, *Lorenz dominates each other element* $x \in \mathbb{R}^n$ *satisfying* $\sum_{i=1}^{n_1} x_i \ge n_1 a_1$, $\sum_{i=1}^{n_1+n_2} x_i \ge \sum_{i=1}^{2} n_i a_i$, $\ldots$, $\sum_{i=1}^{n-n_t} x_i \ge \sum_{i=1}^{t-1} n_i a_i$, *and* $\sum_{i=1}^{n} x_i = \sum_{i=1}^{t} n_i a_i$.

In our context, this implies $v(N_1 \cup \ldots \cup N_i) > 0$, *for all* $i = 1, \ldots, m$, *being* $(N_1, \ldots, N_m)$ *a partition of* N *as described in Definition 1. At this point, it is important to pointed out that the WCES fails to satisfies covariance (see Dutta and Ray, 1989) and so the problem of existence of the WCES and the properties of Lorenz domination cannot be solved just by looking at positive games.*

Let us show an example to illustrate this point. Let (N, v) *be a game with* $N = \{1, 2, 3\}$ *and* $v(\{1\}) = 0.8$, $v(\{2\}) = -1$, $v(\{3\}) = -2$, $v(\{12\}) = -0.1$, $v(\{13\}) = -0.8$, $v(\{23\}) = -3.5$ *and* $v(\{123\}) = -1.5$. *Let* $x = (0.8, -0.9, -1.4) \in C(N, v)$. *Then, the ordered partition of* N *induced by* x *is* $\pi = (\{1\}, \{2\}, \{3\})$, *with* $x_1 = v(\{1\}) > 0$, $x_1 + x_2 = v(\{1\} \cup \{2\}) < 0$ *and* $x_1 + x_2 + x_3 = v(\{1\} \cup \{2\} \cup \{3\}) < 0$. *From Theorem 1,* $EL(N, v) = \{x\}$ *and* x *Lorenz dominates every other core element. However, this last assertion can not be derived from Proposition 2 in Sánchez-Soriano et al. (2014).*

Theorem 1 generalizes both Theorem 2 and Theorem 3 in Dutta and Ray (1989), and it can be useful to check that a core element is the WCES.

Let us introduce the class of games that satisfies the conditions stated in Theorem 1.

Definition 2. *A game* (N, v) *is an exact partition game if there exists a core element* x *such that the ordered partition of* N *induced by* x, $\pi = (N_1, \ldots, N_m)$, *satisfies* $x(N_1 \cup \ldots \cup N_k) = v(N_1 \cup \ldots \cup N_k)$, *for all* $k = 1, \ldots, m$.

Let Γ_{EP} denote the class of exact partition games. This class is large enough to include convex games and dominant diagonal assignment games,[3] but also non-superadditive games.

Example 1. *Let (N, v) be a balanced game with set of players $N = \{1, 2, 3\}$ and characteristic function:*

S	*v(S)*	*S*	*v(S)*	*S*	*v(S)*
$\{1\}$	1	$\{12\}$	0	$\{123\}$	9
$\{2\}$	1	$\{13\}$	7		
$\{3\}$	1	$\{23\}$	0		

This games is not supperadditive since $v(\{12\}) < v(\{1\}) + v(\{2\})$, but $(N, v) \in \Gamma_{EP}$. Indeed, take $x = (3.5, 2, 3.5) \in C(N, v)$. The ordered partition of N induced by x, $\pi = (\{13\}, \{2\})$, satisfies $x_1 + x_3 = v(\{12\})$ and $x(N) = v(N)$. Hence, $EL(N, v) = \{x\}$ and $(N, v) \in \Gamma_{EP}$.

In Section 4, we will axiomatize the WCES on Γ_{EP}.

4 Axiomatic characterizations

The main concern of this section is to characterize the WCES over the domain of exact partition games, Γ_{EP}. As particular cases, we obtain new axiomatic characterizations over the class of convex games.

On the domain of convex games, the first characterization was provided by Dutta (1990) by means of *constrained egalitarianism* and *consistency* with respect to both the max reduced game (Davis and Maschler, 1965) and the self reduced game (Hart and Mas-Colell,1989). *Constrained egalitarianism* is a prescriptive property that imposes to select, for two person games, the Lorenz maximal allocation within the core. Consistency is a sort of internal stability requirement that relates the solution of a game to the solution of the game when some players leave the game.

A solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Constrained egalitarianism** if for all $N \in \mathcal{N}$ with $|N| = 2$, and all $(N, v) \in \Gamma'$, it holds $\sigma(N, v) = CE(N, v)$.

Note that any two person exact partition game is convex. Thus, the WCES satisfies *constrained egalitarianism* on Γ_{EP}.

To define consistency, we need to introduce the notion of reduced game.

[3]Using different arguments, Llerena (2012) shows that on the class of dominant diagonal assignment games, the τ-value (Tijs, 1981) satisfies the requirements of Theorem 1.

Definition 3. *(Davis and Maschler, 1965) Let (N,v) be a game, $\emptyset \neq N' \subset N$ and $x \in \mathbb{R}^K$ where $N \setminus N' \subseteq K \subseteq N$. The max reduced game relative to N' at x is the game $\left(N', r_{M,x}^{N'}(v)\right)$ defined by*

$$r_{M,x}^{N'}(v)(S) = \begin{cases} 0 & \text{if } S = \emptyset, \\ \max_{Q \subseteq N \setminus N'} \{v(S \cup Q) - x(Q)\} & \text{if } \emptyset \neq S \subset N', \\ v(N) - x(N \setminus N') & \text{if } S = N'. \end{cases} \tag{9}$$

Remark 4. *The max reduction operation is transitive (See, for instance, Chang and Hu, 2007). That is, $r_{M,x_{|N'}}^{N''}\left(r_{M,x}^{N'}(v)\right) = r_{M,x}^{N''}(v)$, for all $N \in \mathcal{N}$, all $(N,v) \in \Gamma$, all coalitions $\emptyset \neq N'' \subset N' \subset N$ and all payoff vector $x \in \mathbb{R}^K$ with $N \setminus N'' \subseteq K \subseteq N$.*

In the max reduced game (relative to N' at x), the worth of a coalition $S \subset N'$ is determined under the assumption that S can choose the best partners in $N \setminus N'$, provided they are paid according to x. *Max consistency* says that in this max reduced game, the original agreement should be confirmed.

A solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Max consistency** if for all $N \in \mathcal{N}$, all $(N,v) \in \Gamma'$, all $N' \subset N$, $N' \neq \emptyset$, and all $x \in \sigma(N,v)$, then $\left(N', r_{M,x}^{N'}(v)\right) \in \Gamma'$ and $x_{|N'} \in \sigma\left(N', r_{M,x}^{N'}(v)\right)$.
- **Weak max consistency** if for all $N \in \mathcal{N}$, all $(N,v) \in \Gamma'$, all $N' \subset N$ with $1 \leq |N'| \leq 2$ and all $x \in \sigma(N,v)$, then $\left(N', r_{M,x}^{N'}(v)\right) \in \Gamma'$ and $x_{|N'} \in \sigma\left(N', r_{M,x}^{N'}(v)\right)$.
- **Rich player max consistency** if for all $N \in \mathcal{N}$, all $(N,v) \in \Gamma'$ and all $x \in \sigma(N,v)$, if $N_1 \subseteq N, N_1 \neq N$, is the set of players with highest payoff (w.r.t. x), then $\left(N \setminus N_1, r_{M,x}^{N \setminus N_1}(v)\right) \in \Gamma'$ and $x_{|N \setminus N_1} \in \sigma\left(N \setminus N_1, r_{M,x}^{N \setminus N_1}(v)\right)$.

Weak max consistency applies the condition of *max consistency* to reduced games with at most two players. *Rich player max consistency* weakens *max consistency* just requiring this condition when rich players leave the game. Clearly, *max consistency* implies both *weak* and *rich player max consistency.*

Proposition 1. *The WCES satisfies max consistency on Γ_{EP}.*

Proof. For two person games, *max consistency* clearly holds. Let $(N,v) \in \Gamma_{EP}$ and $x = EL(N,v)$ with $|N| > 2$. Since the max reduction operation is transitive (see Remark 4), it is enough to see that, for all $i \in N$, $\left(N \setminus \{i\}, r_{M,x}^{N \setminus \{i\}}(v)\right) \in \Gamma_{EP}$ and $x_{|N \setminus \{i\}} = EL\left(N \setminus \{i\}, r_{M,x}^{N \setminus \{i\}}(v)\right)$.

Let $\pi = (N_1, \ldots, N_m)$ be the ordered partition of N induced by x. We distinguish two cases:

1) If $m = 1$, then $x = \left(\frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|}\right) \in C(N, v)$. Let $i \in N$. By *max consistency* of the core (Peleg, 1986), $x_{|N \setminus \{i\}} \in C\left(N \setminus \{i\}, r_{M,x}^{N\setminus\{i\}}(v)\right)$. Hence, $\left(N \setminus \{i\}, r_{M,x}^{N\setminus\{i\}}(v)\right) \in \Gamma_{EP}$ and $x_{|N \setminus \{i\}} = EL\left(N \setminus \{i\}, r_{M,x}^{N\setminus\{i\}}(v)\right)$.

2) If $m > 1$, take $k \in \{1, \ldots, m\}$ and $i \in N_k$. The ordered partition of $N \setminus \{i\}$ induced by $x_{|N \setminus \{i\}}$ is either $\pi' = (N_1, \ldots, N_{k-1}, N_k \setminus \{i\}, N_{k+1}, \ldots, N_m)$, if $|N_k| > 1$, or $\pi' = (N_1, \ldots, N_{k-1}, N_{k+1}, \ldots, N_m)$, otherwise.

From the *max consistency* of the core, the definition of max reduced game and the fact that $x(N_1 \cup \ldots \cup N_k) = v(N_1 \cup \ldots \cup N_k)$ for all $k \in \{1, \ldots, m\}$, we have

- For $h \in \{1, \ldots, k-1\}$,

$$\begin{aligned} x(N_1 \cup \ldots \cup N_h) &\geq r_{M,x}^{N\setminus\{i\}}(v)(N_1 \cup \ldots \cup N_h) \\ &\geq v(N_1 \cup \ldots \cup N_h) \\ &= x(N_1 \cup \ldots \cup N_h), \end{aligned}$$

  which means that

$$x(N_1 \cup \ldots \cup N_h) = r_{M,x}^{N\setminus\{i\}}(v)(N_1 \cup \ldots \cup N_h). \tag{10}$$

- For $h \in \{k, \ldots, m\}$,

$$\begin{aligned} x(N_1 \cup \ldots \cup N_k \setminus \{i\} \cup \ldots \cup N_h) &\geq r_{M,x}^{N\setminus\{i\}}(v)(N_1 \cup \ldots \cup N_k \setminus \{i\} \cup \ldots \cup N_h) \\ &\geq v(N_1 \cup \ldots \cup N_k \cup \ldots \cup N_h) - x_i \\ &= x(N_1 \cup \ldots \cup N_k \cup \ldots \cup N_h) - x_i \\ &= x(N_1 \cup \ldots \cup N_k \setminus \{i\} \cup \ldots \cup N_h), \end{aligned}$$

  which means that

$$x(N_1 \cup \ldots \cup N_k \setminus \{i\} \cup \ldots \cup N_h) = r_{M,x}^{N\setminus\{i\}}(v)(N_1 \cup \ldots \cup N_k \setminus \{i\} \cup \ldots \cup N_h). \tag{11}$$

From (10) and (11) it follows that $x_{|N \setminus \{i\}}$ satisfies the conditions stated in Theorem 1 (w.r.t. π'). Hence, we conclude that $\left(N \setminus \{i\}, r_{M,x}^{N\setminus\{i\}}(v)\right) \in \Gamma_{EP}$ and $x_{|N \setminus \{i\}} = EL\left(N \setminus \{i\}, r_{M,x}^{N\setminus\{i\}}(v)\right)$.

□

To prove that *max consistency* together with *constrained egalitarianism* characterize the WCES over the class of convex games, Dutta (1990) invokes *converse max consistency*, which is the dual property of *max consistency*. This property is crucial in his proof of uniqueness.

A solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Converse max consistency** if for all $N \in \mathcal{N}$ with $|N| \geq 3$, all $(N, v) \in \Gamma'$ and all $x \in \mathbb{R}^N$ with $x(N) = v(N)$, if for all $N' \subset N$ with $|N'| = 2$, $\left(N', r^{N'}_{M,x}(v)\right) \in \Gamma'$ and $x_{|N'} \in \sigma\left(N', r^{N'}_{M,x}(v)\right)$, then $x \in \sigma(N, v)$.

Converse max consistency says that if the projection of an efficient allocation x is chosen for every two player max reduced game, then x should be chosen for the original game.

Unfortunatly, Example 2 bellow reveals that the WCES is in conflict with *converse max consistency* on Γ_{EP}.

Example 2. *(Arín and Iñarra, 2001) Let (N, v) be a balanced game with set of players $N = \{1, 2, 3, 4\}$ and characteristic function:*

S	$v(S)$	S	$v(S)$	S	$v(S)$	S	$v(S)$
$\{1\}$	0	$\{12\}$	0	$\{123\}$	0	$\{1234\}$	4
$\{2\}$	0	$\{13\}$	2	$\{124\}$	0		
$\{3\}$	0	$\{14\}$	2	$\{134\}$	0		
$\{4\}$	0	$\{23\}$	2	$\{234\}$	0		
		$\{24\}$	2				
		$\{34\}$	0				

Take $x = (1, 1, 1, 1) \in C(N, v)$. The ordered partition of N induced by x is $\pi = (\{N\})$ and $x(N) = v(N)$. Hence, $EL(N, v) = \{x\}$ and $(N, v) \in \Gamma_{EP}$. Now choose $y = (2, 2, 0, 0) \in C(N, v)$. Below, we describe the max reduced games $\left(N', r^{N'}_{M,y}\right)$ relative to $N' \subset N$ at y with $|N'| = 2$,

S	$r^{\{12\}}_{M,y}(v)$	S	$r^{\{12\}}_{M,y}(v)$
$\{1\}$	2	$\{12\}$	4
$\{2\}$	2		

S	$r^{\{13\}}_{M,y}(v)$	S	$r^{\{13\}}_{M,y}(v)$
$\{1\}$	2	$\{13\}$	2
$\{3\}$	0		

S	$r^{\{14\}}_{M,y}(v)$	S	$r^{\{14\}}_{M,y}(v)$
$\{1\}$	2	$\{14\}$	2
$\{4\}$	0		

S	$r^{\{23\}}_{M,y}(v)$	S	$r^{\{23\}}_{M,y}(v)$
$\{2\}$	2	$\{23\}$	2
$\{3\}$	0		

S	$r^{\{24\}}_{M,y}(v)$	S	$r^{\{24\}}_{M,y}(v)$
$\{2\}$	2	$\{24\}$	2
$\{4\}$	0		

S	$r^{\{34\}}_{M,y}(v)$	S	$r^{\{34\}}_{M,y}(v)$
$\{3\}$	0	$\{34\}$	0
$\{4\}$	0		

The corresponding constrained egalitarian solution are:

$$CE\left(\{12\}, r^{\{12\}}_{M,y}(v)\right) = (2, 2) = y_{|\{12\}} \qquad CE\left(\{13\}, r^{\{13\}}_{M,y}(v)\right) = (2, 0) = y_{|\{13\}},$$

$$CE\left(\{14\}, r^{\{14\}}_{M,y}(v)\right) = (2, 0) = y_{|\{14\}} \qquad CE\left(\{23\}, r^{\{23\}}_{M,y}(v)\right) = (2, 0) = y_{|\{23\}},$$

$$CE\left(\{24\}, r^{\{24\}}_{M,y}(v)\right) = (2, 0) = y_{|\{24\}} \qquad CE\left(\{34\}, r^{\{34\}}_{M,y}(v)\right) = (0, 0) = y_{|\{34\}}.$$

However, $y \neq EL(N,v)$*.*

To be precise, Dutta (1990) only uses *bilateral max consistency*, that is, *max consistency* for only two person games, together with *constrained egalitarianism*, to characterize the WCES on Γ_{Con}. Let us see that on Γ_{EP}, these two properties do not characterize the WCES. To do this, we introduce the egalitarian core (Arin and Iñarra, 2001).

Definition 4. *The egalitarian core of a balanced game* (N,v)*, denoted by* E_gC*, is the set* $E_gC(N,v) = \{x \in C(N,v) \mid x_i > x_j \Rightarrow S_{ij}(x) = 0\}$*, where* $S_{ij}(x) = \max\{v(S) - x(S) \mid i \in S, j \notin S, S \subset N\}$*.*

Arín and Iñarra (2001) show that the egalitarian core satisfies *max consistency* and *constrained egalitarianism* on Γ_{Bal}. Note that a two person balanced games is an exact partition game since the constrained egalitarian solution is a core element satisfying the conditions stated in Theorem 1. Thus, the egalitarian core satisfies *bilateral max consistency* and *constrained egalitarianism* on Γ_{EP}. In Example 2, $EL(N,v) = \{(1,1,1,1)\}$ and $(2,2,0,0) \in E_gC(N,v)$, which means that $EL(N,v) \neq E_gC(N,v)$. The same example also illustrates that the egalitarian core is not *max consistent* on Γ_{EP}. Indeed, consider the max reduced game $\left(N \setminus \{4\}, r_{M,y}^{N\setminus\{4\}}(v)\right)$ with $y = (2,2,0,0)$. As the reader can easily check, $E_gC\left(N \setminus \{4\}, r_{M,y}^{N\setminus\{4\}}(v)\right) = \{(2,2,0\}$ and $\left(N \setminus \{4\}, r_{M,y}^{N\setminus\{4\}}(v)\right) \notin \Gamma_{EP}$.

The second characterization of the WCES provided by Dutta (1990) uses *self consistency* (Hart and Mas-Collel, 1989). This property is defined for single-valued solutions.

A single-valued solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Self consistency** if for all $N \in \mathcal{N}$, all $(N,v) \in \Gamma'$ and all $N' \subset N$, $N' \neq \emptyset$, then $\left(N', r_{S,\sigma}^{N'}(v)\right) \in \Gamma'$ and, for all $i \in N'$, $\sigma_i(N,v) = \sigma_i\left(N', r_{S,\sigma}^{N'}(v)\right)$, where $\left(N', r_{S,\sigma}^{N'}(v)\right)$ is the **self reduced game** of (N,v) relative to N' and σ definded as follows:

$$r_{S,\sigma}^{N'}(v)(R) = \begin{cases} 0 & \text{if } R = \emptyset, \\ v(R \cup (N \setminus N')) - \sum_{i \in N \setminus N'} \sigma_i\left(R \cup (N \setminus N'), v_{|R \cup (N \setminus N')}\right) & \text{if } \emptyset \neq R \subseteq N'. \end{cases} \tag{12}$$

In the self reduced game (relative to N' at σ), the worth of a coalition $R \subseteq N'$ is the worth of $R \cup (N \setminus N')$ in the original game minus the sum of the payoffs that the solution assigns the members of $N \setminus N'$ for the subgame faced by the group $R \cup (N \setminus N')$. *Self consistency* states that in this self reduced game, the original agreement should be accepted. The next example shows that the WCES fails to satisfies *self consistency* on Γ_{EP}.

Example 3. *Let (N,v) be a balanced game with set of players $N=\{1,2,3\}$ and characteristic function:*

S	*v(S)*	*S*	*v(S)*	*S*	*v(S)*
{1}	2	{12}	4	{123}	4
{2}	1	{13}	2		
{3}	0	{23}	1.5		

Take $x=(2,2,0)\in C(N,v)$. The ordered partition of N induced by x, $\pi=(\{12\},\{3\})$, satisfies $x_1+x_2=v(\{12\})$ and $x(N)=v(N)$. Hence, from Theorem 1 we have that $EL(N,v)=\{x\}$ and $(N,v)\in\Gamma_{EP}$.

Let $N'=\{13\}$. Then,

$$\begin{array}{rcl} r^{N'}_{S,EL}(v)(\{1\}) & = & v(\{12\})-EL_2\left(\{12\},v_{|\{12\}}\right)=4-2=2, \\ r^{N'}_{S,EL}(v)(\{3\}) & = & v(\{23\})-EL_2\left(\{23\},v_{|\{23\}}\right)=1.5-1=0.5 \ \textit{ and} \\ r^{N'}_{S,EL}(v)(\{13\}) & = & v(N)-EL_2(N,v)=4-2=2. \end{array} \tag{13}$$

Note that $\left(N',r^{N'}_{S,EL}(v)\right)$ has no imputations. Thus, $\left(N',r^{N'}_{S,EL}(v)\right)\notin\Gamma_{EP}$ and the WCES is not defined.

In order to characterize the WCES within the domain of exact partition games we will make use, together with consistency, the following properties.

A solution σ on $\Gamma'\subseteq\Gamma$ satisfies

- **Nonemptiness** if for all $N\in\mathcal{N}$ and all $(N,v)\in\Gamma'$, it holds $\sigma(N,v)\neq\emptyset$.
- **Efficiency** if for all $N\in\mathcal{N}$, all $(N,v)\in\Gamma'$ and all $x\in\sigma(N,v)$, then $x(N)=v(N)$.
- **Individual rationality** if for all $N\in\mathcal{N}$, all $(N,v)\in\Gamma'$, all $x\in\sigma(N,v)$ and all $i\in N$, then $x_i\geq v(\{i\})$.
- **Core selection** if for all $N\in\mathcal{N}$, all $(N,v)\in\Gamma'$, all $x\in\sigma(N,v)$ and all $S\subseteq N$, then $x(S)\geq v(S)$.
- **Rich player feasibility** if for all $N\in\mathcal{N}$, all $(N,v)\in\Gamma'$ and all $x\in\sigma(N,v)$, it holds $x(N_1)\leq v(N_1)$, where N_1 denotes the set of players with highest payoff (w.r.t. x).
- **Internal Lorenz stability** if for all $N\in\mathcal{N}$ with $|N|\geq 2$, all $(N,v)\in\Gamma'$ and all $x,y\in\sigma(N,v)$, neither $x\succ_{\mathcal{L}}y$ nor $y\succ_{\mathcal{L}}x$.
- **External Lorenz stability (over the core)** if for all $N\in\mathcal{N}$ with $|N|\geq 2$ and all $(N,v)\in\Gamma'$, if $x\in C(N,v)\setminus\sigma(N,v)$, then there is $y\in\sigma(N,v)$ such that $y\succ_{\mathcal{L}}x$.

Efficiency says that all the gains from cooperation should be shared among the players. *Individual rationality* means that the proposed solution can not be improved upon by a single player, while *core selection* extends this impossibility to any coalition. Note that *core selection*, together with the feasibility assumption of a solution, imply *efficiency*. *Rich player feasibility* states that the total amount received by players with the highest payoff can not exceed what they can get for themselves. *Internal Lorenz stability* is a natural requirement in an egalitarian framework. *External Lorenz stability* (*over the core*) gives priority to the social goal of equality in front of particular interests, in the sense that if a core element is not an outcome of the solution is because there is an allocation in the solution which is more egalitarian (w.r.t. the Lorenz criterion).

Next, we state our first characterization result.

Theorem 2. *The WCES is the unique solution on Γ_{EP} that satisfies weak max consistency, individual rationality, internal Lorenz stability and external Lorenz stability (over the core).*

Proof. Proposition 1 implies *weak max consistency* and *individual rationality* comes from the fact that the WCES selects a core element. *Internal Lorenz stability* is because the WCES is single-valued, and *external Lorenz stability* (*over the core*) follows from Theorem 1.

In order to show uniqueness, suppose there is a solution $\sigma \neq EL$ satisfying the above four properties. Let $(N, v) \in \Gamma_{EP}$. Note that *external Lorenz stability* (*over the core*) implies *nonemptiness*. If $|N| = 1$, by *nonemptiness* and *individual rationality* (and feasibility) $\sigma(N, v) = EL(N, v)$. Suppose $|N| \geq 2$. We first show that $\sigma(N, v) \subseteq C(N, v)$. Let $x \in \sigma(N, v)$ and $i \in N$. Then, *weak max consistency* and *efficiency* for one person game imply $x_i = r^{\{i\}}_{M,x}(v)(\{i\}) = v(N) - \sum_{j \in N \setminus \{i\}} x_j$, which proves *efficiency*. To check coalitional rationality, let $\emptyset \neq S \subset N$ and $i \in N \setminus S$. Chose $k \in S$ and consider the max reduced game $\left(\{ik\}, r^{\{ik\}}_{M,x}(v)\right)$. By *weak max consistency*, $x_{|\{ik\}} \in \sigma\left(\{ik\}, r^{\{ik\}}_{M,x}(v)\right)$ and, by *individual rationality*, $x_k \geq r^{\{ik\}}_{M,x}(v)(\{k\}) \geq v(S) - x(S \setminus \{k\})$, which implies $x(S) \geq v(S)$. Hence, $x \in C(N, v)$. Let us denote $x^* = EL(N, v)$. If $x^* \notin \sigma(N, v)$, by *external Lorenz domination* (*over the core*) there is $y \in \sigma(N, v)$ such that $y \succ_{\mathcal{L}} x^*$, a contradiction. Hence, $x^* \in \sigma(N, v)$. Finally, by *internal Lorenz stability* we conclude that $\sigma(N, v) = EL(N, v)$. □

To see that the properties in Theorem 2 are independent we introduce the following solutions:

- Let σ_1 defined as follows: $\sigma_1(N, v) = \emptyset$, for each $(N, v) \in \Gamma_{EP}$. Then, σ_1 satisfies *weak max consistency*, *individual rationality*, *internal Lorenz stability*, but no *external Lorenz stability* (*over the core*).

- Let σ_2 defined as follows: $\sigma_2(N, v) = C(N, v)$, for each $(N, v) \in \Gamma_{EP}$. Then,

σ_2 satisfies *weak max consistency*, *individual rationality*, *external Lorenz stability* (*over the core*), but not *internal Lorenz stability*.

- Let σ_3 defined as follows: $\sigma_3(N, v) = EI(N, v)$, for each $(N, v) \in \Gamma_{EP}$. That is, σ_3 chooses the Lorenz maximal allocations in the imputation set. Llerena and Mauri (2015) show that this solution is single-valued and Lorenz dominates all core elements. Then, σ_3 satisfies *individual rationality*, *internal Lorenz stability*, *external Lorenz stability* (*over the core*), but not *weak max consistency*.

- Let σ_4 defined as follows: $\sigma_4(N, v) = EL(N, v)$ if $|N| \geq 2$, and $\sigma_4(N, v) = X^*(\{i\}, v)$ if $N = \{i\}$, for each $(N, v) \in \Gamma_{EP}$. Then, σ_4 satisfies *weak max consistency*, *internal Lorenz stability*, *external Lorenz stability* (*over the core*), but not *individual rationality*.

It is well-known that the max reduced game of a convex game relative to a core element is also convex (see, for instance, Hokari, 2002). Moreover, on this domain the WCES selects the unique Lorenz maximal allocation within the core (Dutta and Ray, 1989). Thus, Theorem 2 holds on the domain of convex games.

Theorem 3. *The WCES is the unique solution on Γ_{Con} that satisfies weak max consistency, individual rationality, internal Lorenz stability and external Lorenz stability (over the core).*

Defined on the domain of convex games, σ_1, σ_2, σ_3 and σ_4 show the independence of the properties in Theorem 3.

Although the WCES satisfies nice properties on the domain of convex games, and some of them are inherited on the domain of exact partition games, its existence is not linked to the nonemptiness of the core. On the domain of balanced games, an alternative route, already suggested by Dutta and Ray (1989) and latter adopted by Arin and Iñarra (2001) and Hougaard et al. (2001), is to focus on the Lorenz maximal allocations within the core.

Definition 5. *The Lorenz maximal core of a balanced game (N, v), denoted by $EC(N, v)$, is the set* $EC(N, v) = \{x \in C(N, v) \mid \nexists\, y \in C(N, v) \text{ such that } y \succ_{\mathcal{L}} x\}$.

By definition, the Lorenz maximal core satisfies *individual rationality* and *internal Lorenz stability*. *External Lorenz stability* (*over the core*) follows by compactness of the core. Arin and Iñarra (2001) and also Hougaard et al. (2001), show that the Lorenz maximal core satisfies *max consistency*. Since *weak max consistency* and *individual rationality* imply *core selection*, uniqueness follows directly from *internal Lorenz stability* and *external Lorenz stability* (*over the core*). Thus, properties in Theorem 2 also characterize the Lorenz maximal core on the domain of balanced games.

Theorem 4. *The Lorenz maximal core is the unique solution on* Γ_{Bal} *that satisfies weak max consistency, individual rationality, internal Lorenz stability and external Lorenz stability (over the core).*

Solution σ_1, σ_2 and σ_3 defined on Γ_{Bal}, together with solution σ_5 defined bellow, show the independence of the properties in Theorem 4.

- Let σ_5 defined as follows: $\sigma_5(N,v) = EC(N,v)$ if $|N| \geq 2$, and $\sigma_5(N,v) = X^*(\{i\},v)$ if $N = \{i\}$, for each $(N,v) \in \Gamma_{Bal}$. Then, σ_4 satisfies *weak max consistency*, *internal Lorenz stability*, *external Lorenz stability* (*over the core*), but not *individual rationality.*

Our second characterization is by means of *nonemptiness*, *rich player max consistency*, *core selection* and *rich payer feasibility.*

Theorem 5. *The WCES is the unique solution on* Γ_{EP} *that satisfies nonemptiness, rich player max consistency, core selection, and rich player feasibility.*

Proof. Proposition 1 implies *rich player max consistency*, *nonemptiness* and *core selection* follow from the fact that the WCES selects a core element, *rich player feasibility* comes from the structure of the WCES on Γ_{EP}.

In order to show uniqueness, suppose there is a solution $\sigma \neq EL$ satisfying the above four properties. Let $(N,v) \in \Gamma_{EP}$, $EL(N,v) = \{x\}$ and $\pi = (N_1, N_2, \ldots, N_m)$ be the ordered partition of N induced by x. First, we will see that N_1 is the unique maximal equity coalition of (N,v). Let $R \subseteq N$ be an equity coalition. Recall that $x_k = \frac{v(N_1)}{|N_1|}$, for all $k \in N_1$. Since $x \in C(N,v)$, there exists $i \in R$ such that $x_i \geq \frac{v(R)}{|R|}$. Thus, for each $k \in N_1$, it holds $x_k = \frac{v(N_1)}{|N_1|} \geq x_i \geq \frac{v(R)}{|R|} \geq \frac{v(N_1)}{|N_1|}$, which means that $\frac{v(R)}{|R|} = \frac{v(N_1)}{|N_1|}$. Hence, N_1 is an equity coalition. Suppose that $R \setminus N_1 \neq \emptyset$. Then,

$$\begin{aligned} x(R) &= \sum_{i \in N_1 \cap R} x_i + \sum_{i \in R \setminus N_1} x_i = |N_1 \cap R| \frac{v(N_1)}{|N_1|} + \sum_{i \in R \setminus N_1} x_i \\ &< |N_1 \cap R| \frac{v(N_1)}{|N_1|} + |R \setminus N_1| \frac{v(N_1)}{|N_1|} = \frac{v(N_1)}{|N_1|} |R| = v(R), \end{aligned}$$

contradicting $x \in C(N,v)$. Hence, $R \subseteq N_1$.

By *nonemptiness*, $\sigma(N,v) \neq \emptyset$. Let $y \in \sigma(N,v)$ and $\pi' = (R_1, R_2, \ldots, R_k)$ be the ordered partition of N induced by y. By *core selection* and *rich player feasibility*, $y_i = \frac{v(R_1)}{|R_1|}$ for all $i \in R_1$. If $R_1 = N$, by *core selection* $y = x$ and thus $\sigma(N,v) = EL(N,v)$. Otherwise, since $x \succ_{\mathcal{L}} y$, $\hat{x}_1 \leq \hat{y}_1$ which means that $y_i \geq \frac{v(N_1)}{|N_1|}$ for all $i \in R_1$. Hence, $\frac{v(R_1)}{|R_1|} \geq \frac{v(N_1)}{|N_1|}$. This, together with the fact that

N_1 is the unique maximal equity coalition of (N, v), leads to $R_1 \subseteq N_1$. Suppose that $|R_1| < |N_1|$. Then,

$$\begin{array}{lcl} \hat{x}_1 & = & \hat{y}_1 \\ \hat{x}_1 + \hat{x}_2 & = & \hat{y}_1 + \hat{y}_2 \\ & \vdots & \\ \hat{x}_1 + \ldots + \hat{x}_{|R_1|} & = & \hat{y}_1 + \ldots + \hat{y}_{|R_1|} \\ \hat{x}_1 + \ldots + \hat{x}_{|R_1|} + \hat{x}_{|R_1|+1} & > & \hat{y}_1 + \ldots + \hat{y}_{|R_1|} + \hat{y}_{|R_1|+1} \end{array}$$

in contradiction with $x \succ_{\mathcal{L}} y$. Thus, $R_1 = N_1$ and $x_i = y_i$ for all $i \in N_1$, which imply $\left(N \setminus N_1, r_{M,x}^{N\setminus N_1}(v)\right) = \left(N \setminus R_1, r_{M,y}^{N\setminus R_1}(v)\right)$. By *rich player max consistency*, $y_{|N\setminus N_1} \in \sigma\left(N \setminus N_1, r_{M,x}^{N\setminus N_1}(v)\right)$ and $x_{|N\setminus N_1} = EL\left(N \setminus N_1, r_{M,x}^{N\setminus N_1}(v)\right)$, with $\left(N \setminus N_1, r_{M,x}^{N\setminus N_1}(v)\right) \in \Gamma_{EP}$. Applying the same arguments as before, it can be checked that $N_2 = R_2$ and $x_i = y_i$ for all $i \in N_2$. Following this reasoning step by step we reach $x = y$, which means that $\sigma = EL$. □

Solution σ_1 defined on Γ_{EP}, together with the following σ_6, σ_7 and σ_8 show the independence of the properties in Theorem 5.

- Let σ_6 defined as follows: $\sigma_6(N, v) = \{x \in C(N, v) \mid x(N_1) = v(N_1)\}$, for each $(N, v) \in \Gamma_{EP}$, where N_1 denotes the set of players with highest payoff (w.r.t. x). Then, σ_6 satisfies *nonemptiness*, *core selection* and *rich player feasibility*, but not *rich player max consistency*.

- Let σ_7 defined as follows: $\sigma_7(N, v) = \left\{\left(\frac{v(N)}{|N|}, \ldots \frac{v(N)}{|N|}\right)\right\}$, for each $(N, v) \in \Gamma_{EP}$. Then, σ_7 satisfies *nonemptiness*, *rich player max consistency* and *rich player feasibility*, but not *core selection*.

- Let σ_8 defined as follows: $\sigma_8(N, v) = EL(N, v)$ if $|N| \geq 3$, and $\sigma_8(N, v) = C(N, v)$ if $|N| \leq 2$, for each $(N, v) \in \Gamma_{EP}$. Then, σ_8 satisfies *nonemptiness*, *rich player max consistency* and *core selection*, but not *rich player feasibility*.

Theorem 5 also holds on the domain of convex games.

Theorem 6. *The WCES is the unique solution on Γ_{Con} that satisfies nonemptiness, rich player max consistency, core selection and rich player feasibility.*

Definded on the domain of convex games, σ_1, σ_6, σ_7 and σ_8 show the independence of the properties in Theorem 6.

Finally, let us pointed out that on the domain of balanced games, the properties stated in Theorem 5 do not characterize the Lorenz maximal core since it fails to satisfy *rich player feasibility*.

5 Final remarks

We have introduced a subclass of balanced games, called exact partition games Γ_{EP}. This class is large enough to include convex games and dominant diagonal assignment games, but also nonsuperadditive games. On Γ_{EP}, we have shown that the WCES behaves as in convex games, that is, it exists, belongs to the core and Lorenz dominates every other core element. Moreover, we have provided two axiomatic characterizations by means of consistency, rationality, and two properties of fairness based on the Lorenz criterion. Interestingly, both characterizations hold over the domain of convex games. Additionally, one of them could be extended to balanced games characterizing the Lorenz maximal core on this domain. Finally, for future research it could be interesting to study whether the characterizations of the WCES given by Klijn et al. (2000), Hougaard et al. (2001) and Arin et al. (2003) over the domain of convex games can be extended to Γ_{EP}.

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