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Resource allocation with warranties in claims problems

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Abstract

The establishing of warranties in order to ensure some minimum awards to each agent involved in an allocation (claims) problem has been analyzed in the body of literature by introducing *lower bounds*. When focusing on claims problems, four main lower bounds on awards have been defined: the *minimal right* (Curiel et al., 1987), the *fair lower bound* (Moulin, 2002), *securement* (Moreno-Ternero and Villar, 2004) and the *min lower bound* (Dominguez, 2013). The current approach analyzes the effect of requiring the aforementioned lower bounds in an allocation mechanism. We compare the mechanisms thus obtained together with the use of some additional properties. By doing so, we show that there is a correspondence between lower bounds and claims rules, i.e., associated to each particular lower bound, we find a particular claims rule. Consequently, we provide new characterizations for the *constrained equal awards* rule, as well as the *Ibn Ezra* proposal. Finally, a dual analysis, by using *upper bounds* in awards, provides characterizations of the dual rules of the previous ones: the *constrained equal losses* rule and the dual of the *Ibn Ezra* rule.

Keywords: Claims problems, Warranties, Lower bounds, Claims rules
JEL classification: C71, D63, D71.

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1. Introduction

The so-called claims problem reflects a situation where the aggregate claim of a group of individuals is greater than the resources to be distributed among them. The way of rationing the endowment among the agents, taking into account their claims, is prescribed by a rule. In the present paper, we analyze how to distribute any increment of the endowment in terms of two general concepts: *equal treatment* and *warranty* (which is determined by a lower bound on the awards an agent should receive).

The concern of ensuring some minimum individual rights has figured in a large number of contexts. Specifically, the *Universal Basic Income* is a classical issue that has attracted most of the attention in the social policy literature and the political agenda during the last two decades (Noguera, 2010). The establishment of a minimum wage in the labor market, or the debate of ensuring a universal minimum health coverage in the U.S. Senate, are further examples.

From a theoretical point of view, the idea of establishing warranties underlies the analysis of claims problems from its beginning (O'Neill, 1982) up to the present day (Giménez-Gómez and Marco-Gil, 2014). The impact of requiring that a claims rule fulfills a lower bound was first analyzed by Dominguez and Thomson (2006) and Yeh (2008). Furthermore, the recursive application of a lower bound is analyzed and some claims rules are characterized by using some additional properties (see, Dominguez, 2013; and Giménez-Gómez and Marco-Gil, 2014).

Our present approach continues these previous works, but instead of applying recursively a lower bound, we combine the requirement that allocation mechanism should fulfill the lower bound, besides some additional properties that depend on the lower bound being used. Accordingly, we require that an allocation mechanism (i) warrants to each individual at least the amount determined by the particular lower bound (*respect of the lower bound*); and, (ii) fulfills *conditional equal treatment*, *conditional resource monotonicity*, *conditional group solidarity*, or *priority*. The introduced properties compare the warranties associated to the individuals and determine the allocation when the lower bounds of two individuals coincide.

It is noteworthy that a key point in our study is the selection of a particular lower bound with respect to which the above-mentioned properties are applied. Since we are interested in comparing lower bounds among agents, we need to choose a *significant* bound, in the sense that it should be different from zero, whenever the claim is different from zero. Specifically, we analyze four lower bounds that are defined in the literature: the *minimal right* (Curiel et al., 1987), the *fair lower bound* (Moulin, 2002), *securement* (Moreno-Ternero and Villar, 2004) and the *min lower bound* (Dominguez, 2013). Our main results show how these properties provide characterizations of well-known allocation rules in claims problems: the *constrained equal awards* and the *Ibn Ezra* rules.

Finally, note that when facing a claims problem, each individual has a claim on the endowment that represents the maximum amount she can receive and, at the same time, the maximum amount she can lose from her claim. By focusing on losses (which is known as the *dual approach*), we require a warranty on the maximum amount that individual can loss; that is, we consider an upper bound on losses. By analyzing the implications of the existence of upper bounds, we straightforwardly obtained from the previous results characterizations of some dual rules such as the *constrained equal losses* and the *dual Ibn Ezra* rules.

The remainder of the paper is organized as follows. The next section presents the model and introduces the lower bounds. Section 3 introduces the axioms and Section 4 provides our main results. Section 5 presents upper bounds on awards and analyzes duality results. Finally, Section 6 mentions some possible future research. The proof of some auxiliary results is relegated to the Appendix.

2. Preliminaries

2.1. *Claims problems and allocation rules*

Throughout the paper we will consider a set of agents $N = \{1, 2, ..., n\}$. Each agent, $i \in N$, is identified by her *claim*, c_i, on some *endowment* $E > 0$. The *aggregate claim*, C, is given by $C = \sum_{i=1}^{n} c_i$. A *claims problem* appears whenever the endowment is not enough to satisfy the aggregate claim, that is, $C > E$. Without loss of generality, we assume that the agents are indexed according to their claims, $c_1 \leqslant c_2 \leqslant \ldots \leqslant c_n$. The pair $(E, c) \in \mathbb{R}_{++} \times \mathbb{R}^n_+$ represents the claims problem, and $\mathcal{B}$ denotes the set of all claims problems. A *rule* is a single-valued function $\varphi : \mathcal{B} \to \mathbb{R}^n_+$ such that for each problem

$(E, c) \in \mathcal{B}$, and each $i \in N$, fulfills $0 \leqslant \varphi_i(E, c) \leqslant c_i$ (*non-negativity* and *claim-boundedness*), and $\sum_{i=1}^{n} \varphi_i(E, c) = E$ (*efficiency*).

Two of the most important rules proposed by the literature are the uniform rules (Maimonides, 12th century): the *constrained equal awards* rule (that recommends an equal distribution of the endowment subject to no one receiving more than her claim) and the *constrained equal losses* rule (that recommends an equal loss from the claim subject to no one receiving a negative amount). Another classical rule (analyzed in Alcalde et al. (2005)) is the *Ibn Ezra* solution, attributed to Rabbi Abraham Ibn Ezra in the 12th century. This rule suggests that each individually claimed unit should be divided equally among all agents that claim it. Formally, these rules are defined as follows.[1]

The **constrained equal awards** rule, φ^{CEA}:
For each (E, c) in $\mathcal{B}$ and each $i \in N$, $\varphi_i^{CEA}(E, c) \equiv \min\{c_i, \lambda\}$,
where λ is chosen so that $\sum_{i \in N} \min\{c_i, \lambda\} = E$.

The **constrained equal losses** rule, φ^{CEL}:
For each (E, c) in $\mathcal{B}$ and each $i \in N$, $\varphi_i^{CEL}(E, c) \equiv \max\{0, c_i - \mu\}$,
where μ is chosen so that $\sum_{i \in N} \max\{0, c_i - \mu\} = E$.

The **Ibn Ezra** rule, φ^{IE}:
For each (E, c) in $\mathcal{B}$, such that $E \leqslant c_n$, $c_i \leqslant c_{i+1}$, and each $i \in N$,

$$\varphi_i^{IE}(E, c) \equiv \sum_{k=1}^{i} \frac{\min\{c_k, E\} - \min\{c_{k-1}, E\}}{n - k + 1},$$

where, for notational convenience, we consider $c_0 = 0$.

Ibn Ezra's recommendation can be understood as follows (see Alcalde et al. (2005)): Let us consider that from the total amount to share $[0, E]$, each agent i demands the specific parts of the endowment $[0, c_i]$; once the claims are arranged on specific units of the endowment in this way, Ibn Ezra recommends that each unit undergoes equal division among all agents demanding it. Ibn Ezra introduces a four-agent example and the proposed rule performs as follows:

[1] See Thomson (2003, 2015) for a complete and updated survey on claims problems.

First, each agent receives an equal division of the lowest claim, $\frac{c_1}{n}$, and individual 1 does not receive any additional amount; the second and successive agents additionally receive $\frac{c_2-c_1}{n-1}$; moreover, the third and fourth agents additionally receive $\frac{c_3-c_2}{n-2}$; and, finally, the last agent additionally receives the remainder.

The next numerical example illustrates this procedure.

Example 1. *Let us consider a claims problem where an endowment* $E = 120$ *must be divided among four agents with respective claims* $20, 40, 60, 120$. *The Ibn Ezra rule provides the following sharing of the endowment:*

$$\varphi^{IE}(E,c) = \left(\tfrac{20}{4}, \tfrac{20}{4} + \tfrac{20}{3}, \tfrac{20}{4} + \tfrac{20}{3} + \tfrac{20}{2}, \tfrac{20}{4} + \tfrac{20}{3} + \tfrac{20}{2} + \tfrac{720}{12}\right) =$$

$$= \left(\frac{60}{12}, \frac{140}{12}, \frac{260}{12}, \frac{980}{12}\right).$$

In order to compare the three solutions that we have introduced, we obtain the result of applying the constrained equal awards and the constrained equal losses rules to this numerical example.

$$\varphi^{CEA}(E,c) = \left(20, \frac{100}{3}, \frac{100}{3}, \frac{100}{3}\right) \qquad \varphi^{CEL}(E,c) = \left(0, \frac{20}{3}, \frac{80}{3}, \frac{260}{3}\right).$$

2.2. ***Lower bounds***

To conclude this section, it is noteworthy that the concept of lower bound has been always present as a key point in claims problems. Indeed, the idea of pretending to ensure for each agent a minimal amount already appears in the formal definition of a rule, by the non-negativity condition. In general, a lower bound (warranty) is a function such that, for each claims problem (E,c) and each agent $i \in N$, $b_i(E,c)$, denotes the minimal amount that agent i should receive in this claims situation, according to such a warranty. A lower bound should fulfill two compulsory conditions:

(i) *Rationality*: the guaranteed minimum is non-negative and lower than the agent's claim.
(ii) *Feasibility*: the endowment allows the assigning of these amounts to the agents.

A general formal definition is given in Dominguez (2013).

A **lower bound** is a function $b : \mathcal{B} \to \mathbb{R}^n_+$ which maps each claims problem $(E,c) \in \mathcal{B}$, and each $i \in N$, to a real number $b_i(E,c)$ such that

(i) $0 \leqslant b_i(E,c) \leqslant c_i$

(ii) $\sum_{i=1}^{n} b_i(E,c) \leqslant E$

Curiel et al. (1987) introduced a lower bound, the so-called *minimal right*, which requires that each agent receives what is available whenever the other agents have already received their claim, or zero if this is not possible. Moulin (2002) introduces a lower bound, the *fair lower bound*, which establishes that all agents should receive at least the amount assigned to each of them in an equal division, or their full claim. Moreno-Ternero and Villar (2004) propose a lower bound called *securement*, that guarantees (if possible) the *n-th* part of each agent's claim (in other case, this bound guarantees an equal division of the endowment). Finally, Dominguez (2013) introduces the *min lower bound*, proposing that each agent receives (if possible) the *n-th* part of the smallest claim (in other case, this bound guarantees an equal division of the endowment). Formally,

Minimal right (Curiel et al., 1987): for each $(E,c) \in \mathcal{B}$ an each $i \in N$,

$$mr_i(E,c) = \max\left\{0, E - \sum_{j \in N \setminus \{i\}} c_j\right\}.$$

Fair lower bound (Moulin, 2002): for each $(E,c) \in \mathcal{B}$ and each $i \in N$,

$$f_i^l(E,c) = \min\left\{c_i, \frac{E}{n}\right\}.$$

Securement (Moreno-Ternero and Villar, 2004): for each $(E,c) \in \mathcal{B}$ and each $i \in N$,

$$s_i(E,c) = \frac{1}{n}\min\{c_i, E\}.$$

Min lower bound (Dominguez, 2013): for each $(E,c) \in \mathcal{B}$ and each $i \in N$,

$$m_i^l(E,c) = \frac{1}{n}\min\left\{\min_{j \in N} c_j, E\right\}.$$

We denote by $\mathcal{L}$ the family of these lower bounds

$$\mathcal{L} = \left\{mr,\ f^l,\ s,\ m^l\right\}.$$

For the sake of comprehension, to observe the behavior of these lower bounds we compute them with the data in Example 1:

$$(E, c) = (120, (20, 40, 60, 120))$$

$$mr(E, c) = (0, 0, 0, 0) \qquad f^l(E, c) = (20, 30, 30, 30)$$
$$s(E, c) = (5, 10, 15, 30) \qquad m^l(E, c) = (5, 5, 5, 5)$$

Henceforth, it is straightforward that, for each claims problem (E, c) and each individual $i \in N$,

$$0 \leqslant m_i^l(E, c) \leqslant s_i(E, c) \leqslant f_i^l(E, c) \leqslant c_i \tag{1}$$

and, therefore, with these three lower bounds, the fair lower bound is the one providing the highest warranties for all of the involved individuals.

On the other hand, there is no inequality relation between the warranty provided by the minimal right lower bound and the others. For instance, if we consider the claims problem $(E, c) = (200, (40, 46, 60, 120))$

$$mr(E, c) = (0, 0, 0, 54) \qquad f^l(E, c) = (40, 46, 50, 50)$$
$$s(E, c) = (10, 13, 15, 30) \qquad m^l(E, c) = (10, 10, 10, 10)\,.$$

It is noteworthy that the warranty that mr always benefits individuals with relatively large claims, hurting those with lower claims.

3. Axioms

In understanding an axiomatic analysis of the aforementioned claims rules, we introduce several properties, which are referred to as particular lower bound b. The first axiom is our basic assumption: the required lower bound (warranty) is satisfied by the claims rule being analyzed.

Axiom [RB]. Respect of a lower bound b: for each $(E, c) \in \mathcal{B}$, and all $i \in N$, $\varphi_i(E, c) \geqslant b_i(E, c)$.

RB requires that each agent should receive at least her lower bound (so that each agent has a guaranteed minimum level of awards: a warranty). Note that this warranty depends on the selected lower bound, so that this property varies with the lower bound b that is under consideration. Note that, from

Equation (1) if a claims rule φ fulfills RB for $b = f^l$, then it also fulfills this property for $b = s$ and $b = m^l$.

Axiom [ETEB]. Conditional equal treatment with respect to a lower bound b**:** for each $(E, c) \in \mathcal{B}$, and all $i, j \in N$ such that $c_i \leqslant c_j$, then

$$b_i(E, c) = b_j(E, c) \text{ implies } \varphi_i(E, c) = \varphi_j(E, c), \text{ or } \varphi_i(E, c) = c_i \leqslant \varphi_j(E, c).$$

ETEB demands equal treatment for equal agents (regarding their lower bounds), unless one of them has her demand met in full. This condition implies *anonymity* with respect to the claims. Note that the second part of the axiom is required, since by asking for equal treatment with respect to the lower bound b, it may lead to giving an individual more than her claim, which is not possible in a claims rule.

Axiom [CRM]. Conditional resource monotonicity with respect to a lower bound b**:** if $(E, c), (E', c) \in \mathcal{B}$ are two problems such that the claims vector coincides and $E > E'$, then for all $i \in N$,

$$\varphi_i(E, c) - \varphi_i(E', c) \geqslant b_i(E, c) - b_i(E', c), \text{ or } \varphi_i(E, c) = c_i.$$

CRM asks for a stronger condition: *any change in the awards received by each individual due to a change in the endowment E should be at least equal to the change in her bound.* As before, we need to restrict this idea in order that no individual receives more than her claim.

Axiom [CGS]. Conditional group solidarity for equal changes in a lower bound b**:** if $(E, c), (E', c) \in \mathcal{B}$ are two problems such that the claims vector coincides and $E > E'$, then for all $i, j \in N$ with $c_i \leqslant c_j$,

$$b_i(E, c) - b_i(E', c) = b_j(E, c) - b_j(E', c) \text{ implies}$$
$$\varphi_i(E, c) - \varphi_i(E', c) = \varphi_j(E, c) - \varphi_j(E', c), \text{ or } \varphi_i(E, c) = c_i \leqslant \varphi_j(E, c).$$

CGS requires that if the endowment increases, then this increment should be shared equally among agents who experiment an equal change in their lower bound. As before, this increment needs to be limited by the claim of each individual.

Axiom [PRI]. Priority in allocation with respect to a lower bound b**:** if $(E, c), (E', c) \in \mathcal{B}$ are such that $E > E'$, then for each $i \in N$

$$\varphi_i(E, c) - \varphi_i(E', c) > 0 \text{ implies } b_i(E, c) - b_i(E', c) > 0.$$

PRI endowments that only those agents who increase their lower bound, should increase their allocation.

3.1. ***Relationships***

We analyze the relationships among the previously introduced axioms that, obviously, depend on the selected lower bound. The proofs are relegated to the Appendix.

Lemma 1. *For each lower bound $b \in \mathcal{L}$,*

a) *CRM implies RB.*
b) *CGS implies ETEB.*

Lemma 2. *For $b = m^l$,*

a) *ETEB implies RB.*
b) *CGS implies CRM.*

Lemma 3. *For $b = f^l$ or $b = s$,*

1) *RB and ETEB are independent.*
2) *RB and CGS are independent.*
3) *CRM and ETEB are independent.*
4) *CRM and CGS are independent.*

Lemma 4. *For $b = s$,*

a) *RB and PRI are independent.*
b) *ETEB and PRI are independent.*

Lemma 5. *(Giménez-Gómez and Peris, 2015) For $b = mr$, CGS and PRI are independent.*

4. Main results

In this section we analyze, in terms of the selected lower bound, how some combinations of the aforementioned axioms uniquely determine a claims rule satisfying them. In particular, we provide some characterizations of the *constrained equal awards* and *Ibn Ezra* rules.

Our first results show that the *constrained equal awards* rule fulfills *RB* and *ETEB* with respect to all the lower bounds that we have defined.

Proposition 1. *For each $b \in \mathcal{L}$, φ^{CEA} fulfills RB.*

Proof. First, let us consider a lower bound $b \neq mr$. Then, for each claims problem (E, c), from Equation (1) we know that $b_i(E, c) \leqslant \min\left\{c_i, \frac{E}{n}\right\}$. Moreover, as $\sum_{i=1}^{n} \min\left\{c_i, \frac{E}{n}\right\} \leqslant E$, we deduce that the constant λ in the definition of φ^{CEA} fulfills $\lambda \geqslant \frac{E}{n}$. Hence, $\min\left\{c_i, \frac{E}{n}\right\} \leqslant \min\{c_i, \lambda\}$; that is $b_i(E, c) \leqslant \varphi_i^{CEA}(E, c)$.

Let us now consider the lower bound $b = mr$. If for some claims problem (E, c) and $i \in N$, $mr_i(E, c) > \varphi_i^{CEA}(E, c)$, then

$$E - \sum_{j \neq i} c_j > \min\{c_i, \lambda\} \quad \Rightarrow \quad E > \sum_{j \neq i} c_j + \min\{c_i, \lambda\} \geqslant \sum_{i=1}^{n} \min\{c_i, \lambda\} = E$$

which is a contradiction. Then, $mr_i(E, c) \leqslant \varphi_i^{CEA}(E, c)$, for all $i \in N$. ■

Proposition 2. *For each $b \in \mathcal{L}$, φ^{CEA} fulfills ETEB.*

Proof. Let us consider a lower bound $b \in \mathcal{L}$, a claims problem (E, c) and two individuals $i, j \in N$ such that $c_i \leqslant c_j$ and $b_i(E, c) = b_j(E, c)$. Then, $\varphi_i^{CEA}(E, c) = \min\{c_i, \lambda\} \leqslant \min\{c_j, \lambda\} = \varphi_j^{CEA}(E, c)$, which implies that $\varphi_i^{CEA}(E, c) = \varphi_j^{CEA}(E, c)$, if the minimum is λ in both cases, or whenever the first minimum is c_i, $\varphi_i^{CEA}(E, c) = c_i \leqslant \varphi_j^{CEA}(E, c)$. ■

4.1. ***Minimal right***

Luttens (2010) analyzes the effect of ensuring, for each individual, the warranty given by the minimal right lower bound on the framework of redistribution problems. Nevertheless, his proposal may allocate negative awards to some agents, hence it is not possible to adapt his mechanism as a claims rule. Giménez-Gómez and Peris (2015) introduce a modification of Luttens' mechanism in order to ensure the property of *participation*, which implies non-negative awards for all agents (so-called *respect of minimal right egalitarian mechanism*). The rule obtained by applying this mechanism to claims problems is denoted by φ^{MR}. The following result comes, in a straightforward way, as a consequence of the characterization result obtained in Giménez-Gómez and Peris (2015). It shows that when combining *CGS* and *PRI*, the *respect of minimal right* rule is obtained. Moreover, Proposition 3 shows that this claims rule fulfills the bound on which is based (condition *RB*).

Theorem 1. *(Giménez-Gómez and Peris, 2015) Let us consider the bound $b = mr$. Then, φ^{MR} is the only rule satisfying CGS and PRI.*

Proposition 3. *(Giménez-Gómez and Peris, 2015) If we consider the lower bound $b = mr$, then φ^{MR} fulfills RB.*

4.2. ***Fair bound***

Now we analyze the case in which the required warranty is provided by the fair lower bound. Theorem 2 shows that, when using this lower bound, by requiring *RB* and *ETEB* we retrieve the constrained equal awards rule.

Theorem 2. *Let us consider the lower bound $b = f^l$. Then, φ^{CEA} is the only rule satisfying RB and ETEB.*

Proof. By Propositions 1 and 2 we know that φ^{CEA} satisfies *RB* and *ETEB*.

Let us now consider a rule φ satisfying axioms *RB* and *ETEB*. For each $(E, c) \in \mathcal{B}$, as $E < \sum_{i=1}^{n} c_i < nc_n$, there is some $k \in N$ such that $E < nc_k$.

If $E < nc_1$, then $f_i^l(E, c) = \frac{E}{n} \leqslant c_i$, for all $i \in N$. Condition *RB* and *efficiency* imply $\varphi_i(E, c) = \frac{E}{n} = \varphi_i^{CEA}(E, c)$ for all $i \in N$.

Otherwise, there is some $k \in N$ such that $nc_{k-1} \leqslant E < nc_k$. For all $i \leqslant k-1$, $f_i^l(E, c) = c_i$, and for all $i > k-1$, $f_i^l(E, c) = \frac{E}{n}$. By *RB* and *claim-boundedness*, for all $i \leqslant k-1$, $\varphi_i(E, c) = c_i$. *ETEB* and *efficiency* will imply an equal sharing of $E' = E - (c_1 + c_2 + \ldots + c_{k-1})$, among agents $i = k, \ldots, n$, unless some of those agents get more than her claim.

If $\frac{E'}{n-(k-1)} > c_k$, then *ETEB* and *claim-boundedness* implies $\varphi_k(E, c) = c_k$. Now, by *ETEB*, $\varphi_i(E, c) = \varphi_j(E, c)$, for all $i, j > k$, and *efficiency* imply $\varphi_i(E, c) = \frac{E - \sum_{i=1}^{k} c_i}{n-k}$ for all $i > k$, unless some of these amounts are greater than the respective claims.

If $\frac{E''}{n-k} > c_{k+1}$, $E'' = E - (c_1 + c_2 + \ldots + c_k)$, *ETEB* and *claim-boundedness* imply $\varphi_{k+1}(E, c) = c_{k+1}$ and the remainder must be distributed equally by *ETEB*, unless some of these amounts are greater than the respective claims. This argument is repeated until no one gets more than her claim, and we observe that the result is $\varphi(E, c) = \varphi^{CEA}(E, c)$. ■

Our next result establishes that, when the used warranty is the fair lower bound, axioms *CRM* and *ETEB* also characterize the φ^{CEA} rule.

Corollary 1. *Let us consider the bound $b = f^l$. Then, φ^{CEA} is the only rule satisfying ETEB and CRM.*

Proof. By Proposition 2, φ^{CEA} satisfies *ETEB*. In order to prove that it also fulfills *CRM*, let us consider two claims problems (E, c), (E', c) such that $E' < E$, and each individual $i \in N$. If $\varphi_i^{CEA}(E, c) < c_i$, then $\min\{c_i, \lambda\} = \lambda < c_i$, so $\varphi_i^{CEA}(E', c) = \min\{c_i, \lambda'\} = \lambda' < c_i$ since $E' < E$. Therefore,

$$\varphi_i^{CEA}(E, c) - \varphi_i^{CEA}(E', c) = \lambda - \lambda', \qquad f_i^l(E, c) = \frac{E}{n}, \quad f_i^l(E', c) = \frac{E'}{n}.$$

From the definition of φ^{CEA},

$$\lambda = \frac{E - (c_1 + c_2 + \ldots + c_r)}{n - r} \qquad r = \max_k \left\{\varphi_k^{CEA}(E, c) = c_k\right\},$$

$$\lambda' = \frac{E' - (c_1 + c_2 + \ldots + c_s)}{n - s} \qquad s = \max_k \left\{\varphi_k^{CEA}(E', c) = c_k\right\}.$$

As $E' < E$, $s \leqslant r$ and

$$\lambda' \leqslant \frac{E' - (c_1 + c_2 + \ldots + c_r)}{n - r} \quad \Rightarrow$$

$$\Rightarrow \quad \lambda - \lambda' \geqslant \frac{E - E'}{n - r} \geqslant \frac{E - E'}{n} = f_i^l(E, c) - f_i^l(E', c).$$

Hence, *CRM* is fulfilled in this case. On the other hand, if $\varphi_i^{CEA}(E, c) = c_i$, the property is obviously fulfilled.

Consider now a rule φ satisfying axioms *ETEB* and *CRM*. From Lemma 1, φ fulfills *RB*, so that Theorem 2 implies $\varphi = \varphi^{CEA}$. ■

If we now combine the axioms *RB* and *CGS*, again the constrained equal awards rule is characterized.

Corollary 2. *Let us consider the bound $b = f^l$. Then, φ^{CEA} is the only rule satisfying RB and CGS.*

Proof. By Proposition 1, φ^{CEA} satisfies *RB*. In order to prove that it also fulfills *CGS*, let us consider two claims problems (E, c), (E', c) such that $E' < E$, and two individuals $i, j \in N$ with $c_i \leqslant c_j$. We suppose that $f_i^l(E, c) - f_i^l(E', c) = f_j^l(E, c) - f_j^l(E', c)$. We distinguish several possible cases:

a) If $f_i^l(E,c) = \frac{E}{n}$, then $f_i^l(E',c) = \frac{E'}{n}$. In this case, either

   (i) $\varphi_i^{CEA}(E,c) = \varphi_j^{CEA}(E,c) = \lambda < c_i$, in which case (being $E' < E$) $\varphi_i^{CEA}(E',c) = \varphi_j^{CEA}(E',c) = \lambda' < c_i$, and

   $$\varphi_i^{CEA}(E,c) - \varphi_i^{CEA}(E',c) = \varphi_j^{CEA}(E,c) - \varphi_j^{CEA}(E',c) = \lambda - \lambda'; \text{ or}$$

   (ii) $\varphi_i^{CEA}(E,c) = c_i \leqslant \varphi_j^{CEA}(E,c)$.

b) If $f_i^l(E,c) = c_i$, then $\varphi_i^{CEA}(E,c) = c_i \leqslant \varphi_j^{CEA}(E,c)$.

Hence, the φ^{CEA} rule satisfies *CGS*.

Consider now a rule φ satisfying axioms *RB* and *CGS*. From Lemma 1, φ fulfills *ETEB*, so that Theorem 2 implies $\varphi = \varphi^{CEA}$. ■

As we have shown in Lemma 1, *CRM* implies *RB* and *CGS* implies *ETEB*. Moreover, as shown in Theorems 1 and 2, and Corollary 2, the constrained equal awards rule fulfills the four axioms. So, by combining *CRM* and *CGS* we obviously obtain a new characterization result.

Corollary 3. *Let us consider the bound* $b = f^l$*. Then,* φ^{CEA} *is the only rule satisfying CRM and CGS.*

4.3. ***Securement***

Throughout this section, we will consider that the minimum amount guaranteed to each individual is provided by the securement lower bound. From the results in the previous subsections, one might ask if for every lower bound b, the introduced axioms characterize the φ^{CEA} rule. We prove that this is not true: Theorem 3 shows that requiring *RB*, *CGS* and *PRI* with the securement lower bound retrieves the Ibn Ezra proposal. It must be noticed that the problems being considered in this section are in the class

$$\mathcal{B}^* = \{(E,c) \in \mathcal{B} : c_i \leqslant c_{i+1},\ E \leqslant c_n\}$$

Theorem 3. *Let us consider the bound* $b = s$*. Then,* φ^{IE} *is the only rule satisfying RB, CGS and PRI.*

Proof. We first prove that φ^{IE} satisfies the required axioms. Let us consider two claims problems (E,c) and (E',c) in $\mathcal{B}^*$, such that $E' < E$.

(RB) If $c_1 \geqslant E$, then $c_i \geqslant E$ and $\varphi_i^{IE}(E,c) = \frac{E}{n} = s_i^l(E,c)$, for all $i \in N$.

Otherwise, $c_1 < E$, we will show that $\varphi_i^{IE}(E,c) \geqslant s_i^l(E,c)$, for each $i \in N$. In this case, $\varphi_1^{IE}(E,c) = \frac{c_1}{n}$ and for $i \geqslant 2$,

$$\varphi_i^{IE}(E,c) = \varphi_{i-1}^{IE}(E,c) + \frac{\min\{c_i, E\} - \min\{c_{i-1}, E\}}{n-(i-1)}. \tag{2}$$

Then,

$$\varphi_2^{IE}(E,c) = \frac{c_1}{n} + \frac{\min\{c_2, E\} - c_1}{n-1} \geqslant \frac{\min\{c_2, E\}}{n}.$$

If we now suppose that $\varphi_i^{IE}(E,c) \geqslant \frac{\min\{c_i,E\}}{n}$, then by applying Equation (2), we obtain

$$\varphi_{i+1}^{IE}(E,c) = \varphi_i^{IE}(E,c) + \frac{\min\{c_{i+1}, E\} - \min\{c_i, E\}}{n-i} \geqslant \frac{\min\{c_{i+1}, E\}}{n}$$

and, by induction, is satisfied for all $i \in N$. Hence, RB is fulfilled.

(CGS) Let us consider $i, j \in N$, such that $c_i \leqslant c_j$ and $s_i(E,c) - s_i(E',c) = s_j(E,c) - s_j(E',c)$. It is easy to observe that only the two following possibilities for the values of the securement lower bound are compatible with the above condition:

a) $s_i(E,c) = s_j(E,c) = \dfrac{E}{n}$, $s_i(E',c) = s_j(E',c) = \dfrac{E'}{n}$.
This case corresponds with $E' < E \leqslant c_i \leqslant c_j$, which implies that $\varphi_i^{IE}(E,c) = \varphi_j^{IE}(E,c)$ and $\varphi_i^{IE}(E',c) = \varphi_j^{IE}(E',c)$. Then, CGS is satisfied.

b) $s_i(E,c) = s_i(E',c) = \dfrac{c_i}{n}$, $s_j(E,c) = s_j(E',c) = \dfrac{c_j}{n}$.
This case corresponds with $c_i \leqslant c_j \leqslant E' < E$, which implies that $\varphi_i^{IE}(E,c) = \varphi_j^{IE}(E,c) = \frac{E}{n}$ and $\varphi_i^{IE}(E',c) = \varphi_j^{IE}(E',c) = \frac{E'}{n}$. Then, CGS is also satisfied.

(PRI) Let us consider $i \in N$ such that $\varphi_i^{IE}(E,c) > \varphi_i^{IE}(E',c)$. We distinguish two cases:

a) If $E' < c_i$, then $s_i(E,c) = \min\left\{\frac{E}{n}, \frac{c_i}{n}\right\} > \frac{E'}{n} = s_i(E',c)$.

b) If $c_i \leqslant E' < E$, then the definition of the Ibn Ezra rule implies $\varphi_i^{IE}(E,c) = \varphi_i^{IE}(E',c)$, a contradiction.

Hence, PRI is fulfilled.

To check the uniqueness, we shall prove that a rule φ satisfying axioms *RB*, *CGS* and *PRI* with respect to the lower bound $b(E,c) = s(E,c)$ coincides with φ^{IE}. Consider a claims problem $(E,c) \in \mathcal{B}^*$. We distinguish several cases:

a) If $E \leqslant c_1$, then $s_i(E,c) = \frac{E}{n}$ for all $i \in N$. By *RB* and efficiency,

$$\varphi_i(E,c) = \frac{E}{n} = \varphi_i^{IE}(E,c).$$

b) If $c_1 < E \leqslant c_2$, $s_1(E,c) = \frac{c_1}{n}$ and, for all $j \geqslant 2$, $s_j(E,c) = \frac{E}{n}$. By *RB*, $\varphi_1(E,c) \geqslant \frac{c_1}{n}$, and $\varphi_j(E,c) \geqslant \frac{E}{n}$. Now, we consider the claims problem (E',c), with $E' = c_1$. Then, $s_j(E,c) = \frac{c_1}{n}$, for all $j \in N$, and this problem is in case a), so $\varphi_i(E',c) = \frac{c_1}{n} = \varphi_i^{IE}(E',c)$. By *PRI* and *CGS*, only agents j, who have increased their lower bound, should receive an equal increase of their allocation, i.e., $\varphi_1(E,c) = \frac{c_1}{n}$, and $\varphi_j(E,c) = \frac{c_1}{n} + \frac{E'-c_1}{n-1}$, that coincides with $\varphi_i^{IE}(E,c)$.

c) If $c_i < E \leqslant c_{i+1}$, we repeat the previous argument, by considering the claims problem (E',c), with $E' = c_i$.

Hence, $\varphi(E,c) = \varphi^{IE}(E,c)$. ■

Finally, since *CRM* implies *RB* and φ^{IE} fulfills the stronger property, Corollary 4 provides a new characterization result for the Ibn Ezra proposal.

Corollary 4. *Let us consider the bound $b = s$. Then, φ^{IE} is the only rule satisfying CGS, PRI and CRM.*

Proof. We only need to prove that φ^{IE} fulfills *CRM*. Let us consider two claims problems (E,c) and (E',c) in $\mathcal{B}^*$, such that $E' < E$, and an individual $i \in N$. We distinguish the following possibilities:

a) If $c_i < E' < E$, then $s_i(E,c) = s_i(E,c) = \frac{c_i}{n}$, so $s_i(E,c) - s_i(E',c) = 0$ and the condition is obviously satisfied, since $E > E'$ implies that $\varphi_i^{IE}(E,c) \geqslant \varphi_i^{IE}(E',c)$.

b) Consider that $c_{i-1} < E' \leqslant c_i < E$, or $E' \leqslant c_{i-1} \leqslant c_i < E$. In this case, $s_i(E,c) = \frac{c_i}{n}$ and $s_i(E',c) = \frac{E'}{n}$. Then,

$$\varphi_i^{IE}(E,c) - \varphi_i^{IE}(E',c) = \varphi_{i-1}^{IE}(E,c) + \frac{c_i - c_{i-1}}{n-(i-1)} - \varphi_{i-1}^{IE}(E',c) - \frac{E' - c_{i-1}}{n-(i-1)}$$

$$\geqslant \frac{c_i - c_{i-1}}{n-(i-1)} - \frac{E' - c_{i-1}}{n-(i-1)} = \frac{c_i - E'}{n-(i-1)} \geqslant \frac{c_i - E'}{n} = s_i(E,c) - s_i(E',c).$$

c) If $c_i \geqslant E > c_{i-1} \geqslant E'$, then $s_i(E,c) = \frac{E}{n}$, $s_i(E',c) = \frac{E'}{n}$ and

$$\varphi_i^{IE}(E,c) - \varphi_i^{IE}(E',c) = \varphi_{i-1}^{IE}(E,c) + \frac{c_i - c_{i-1}}{n-(i-1)} - \varphi_{i-1}^{IE}(E',c) - \frac{E' - c_{i-1}}{n-(i-1)}$$

$$= \left(\varphi_{i-1}^{IE}(E,c) - \varphi_{i-1}^{IE}(E',c)\right) + \frac{E - c_{i-1}}{n-(i-1)} \geqslant \frac{c_i - E'}{n-(i-1)} + \frac{E - c_{i-1}}{n-(i-1)} =$$

$$= \frac{E - E'}{n-(i-1)} + \frac{c_i - c_{i-1}}{n-(i-1)} \geqslant \frac{E-E'}{n} = s_i(E,c) - s_i(E',c).$$

d) If $E' < E \leqslant c_{i-1} \leqslant c_i$, then $s_i(E,c) = s_{i-1}(E,c) = \frac{E}{n}$ and $s_i(E',c) = s_{i-1}(E',c) = \frac{E'}{n}$; so

$$\varphi_i^{IE}(E,c) - \varphi_i^{IE}(E',c) = \varphi_{i-1}^{IE}(E,c) - \varphi_{i-1}^{IE}(E',c) \geqslant$$

$$\geqslant \frac{E-E'}{n} = s_i(E,c) - s_i(E',c).$$

Hence, *CRM* is fulfilled. ■

4.4. *Min lower bound*

Finally, we analyze the case of the min lower bound. Theorem 4 shows that, by using this lower bound, requiring *ETEB* directly characterizes the φ^{CEA} rule.

Theorem 4. *Let us consider the bound* $b = m^l$. *Then,* φ^{CEA} *is the only rule satisfying ETEB.*

Proof. By Proposition 2, φ^{CEA} fulfills *ETEB* with respect to this lower bound.

Now, consider a rule φ satisfying *ETEB* with respect to m^l and a claims problem $(E,c) \in \mathcal{B}$. We suppose, without loss of generality, that $c_1 \leqslant c_2 \leqslant \ldots \leqslant c_n$. Then, $m_i^l(E,c) = \min\left\{\frac{c_1}{n}, \frac{E}{n}\right\}$. We have two possibilities:

1) If $E \leqslant c_1$, then $m_i^l(E,c) = \frac{E}{n}$ for each $i \in N$. By *ETEB* and efficiency, $\varphi_i(E,c) = \frac{E}{n} = \varphi_i^{CEA}(E,c)$, since $\frac{E}{n} \leqslant c_i$ for all i.
2) If $E > c_1$, then $m_i^l(E,c) = \frac{c_1}{n}$ for each $i \in N$. By *ETEB*, all individuals receive the same amount λ unless they receive $c_i \leqslant \lambda$, and this coincides with $\varphi^{CEA}(E,c)$.

Hence, φ coincides with the constrained awards rule. ■

The above result remains valid if we use *CGS* instead of *ETEB*.

Corollary 5. *Let us consider the bound $b = m^l$. Then, the φ^{CEA} rule is the only one satisfying CGS.*

Proof. First, let us observe that the constrained equal awards rule fulfills *CGS* with respect to the min lower bound. We consider two claims problems $(E, c), (E', c) \in \mathcal{B}$, $E' < E$ and two agents $i, j \in N$, with $c_i \leqslant c_j$. As $\varphi_i^{CEA}(E, c) = \min\{\lambda, c_i\}$, we have the following two possibilities:

1) If $\varphi_i^{CEA}(E, c) = c_i$, the condition is fulfilled.
2) If $\varphi_i^{CEA}(E, c) = \lambda$, then $\varphi_j^{CEA}(E, c) = \lambda$ and, as $E' < E$, $\varphi_i^{CEA}(E', c) = \varphi_j^{CEA}(E', c) = \lambda' < \lambda$. Then,

$$\varphi_i^{CEA}(E, c) - \varphi_i^{CEA}(E', c) = \varphi_j^{CEA}(E, c) - \varphi_j^{CEA}(E', c) = \lambda - \lambda'.$$

Hence, φ^{CEA} fulfills *CGS*.

Now, let us consider φ a rule satisfying *CGS*. Then, from Lemma 1 we know that *ETEB* is fulfilled, and, from Theorem 4, $\varphi = \varphi^{CEA}$. ■

Proposition 4. *Let us consider the bound $b = m^l$. Then, the rule φ^{CEA} satisfies CRM with respect to this lower bound.*

Proof. We consider two claims problems $(E, c), (E', c) \in \mathcal{B}$, $E' < E$. Without loss of generality, we suppose that $c_1 \leqslant c_2 \leqslant \ldots \leqslant c_n$. Then, $m_i^l(E, c) = \min\left\{\frac{c_1}{n}, \frac{E}{n}\right\}$, $m_i^l(E', c) = \min\left\{\frac{c_1}{n}, \frac{E'}{n}\right\}$. We distinguish the following possibilities:

1) If $E' < E \leqslant c_1$, then $m_i^l(E, c) = \frac{E}{n}$ and $m_i^l(E', c) = \frac{E'}{n}$, for each $i \in N$. In this case, $\varphi_i^{CEA}(E, c) = \frac{E}{n}$ and $\varphi_i^{CEA}(E', c) = \frac{E'}{n}$, and the property is satisfied.
2) If $E' \leqslant c_1 < E$, then $m_i^l(E, c) = \frac{c_1}{n}$ and $m_i^l(E', c) = \frac{E'}{n}$, for each $i \in N$. Then, $\varphi_i^{CEA}(E', c) = \frac{E'}{n}$ and $\varphi_i^{CEA}(E, c) = \min\{\lambda, c_i\}$. Hence, $\varphi_i^{CEA}(E, c) = \lambda \geqslant \frac{E}{n}$, and the condition is fulfilled, or $\varphi_i^{CEA}(E, c) = c_i$ that also fulfills *CRM*.
3) If $c_1 < E < E'$, then $m_i^l(E, c) = \frac{c_1}{n}$ and $m_i^l(E', c) = \frac{c_1}{n}$, for each $i \in N$. Hence, $m_i^l(E, c) - m_i^l(E', c) = 0$ and the condition is obviously fulfilled, since the constrained equal awards is monotone with respect to the endowment.

Therefore, φ^{CEA} fulfills *CRM*. ■

The following example shows that φ^{CEA} is not the only rule satisfying *CRM* with respect to m^l.

Example 2. *Let us consider the rule φ, defined for each claims problem $(E,c) \in \mathcal{B}$ in the following way:*

$$\varphi(E,c) = \begin{cases} \varphi^{CEA}(E,c) & \text{if} \quad E \leqslant c_1 \\ \left(\frac{c_1}{n}, \min\left\{\frac{c_1}{n} + \varepsilon, c_2\right\}, \ldots, \min\left\{\frac{c_1}{n} + \varepsilon, c_n\right\}\right) & \text{if} \quad c_1 < E \end{cases}$$

where we suppose that $c_1 \leqslant c_2 \leqslant \ldots \leqslant c_n$ and ε is such that $\sum_{i=1}^{n} \varphi_i(E,c) = E$. Clearly, this rule differs from the constrained equal awards. To observe that it fulfills CRM, we consider two claims problems $(E,c), (E',c) \in \mathcal{B}$, $E' < E$. Only the following cases are possible:

1) *$E' < E \leqslant c_1$. In this case, $\varphi = \varphi^{CEA}$ and, By Proposition 4, the condition is fulfilled.*
2) *$E' \leqslant c_1 < E$. Then, $m_i^l(E,c) = \frac{c_1}{n}$, $m_i^l(E',c) = \frac{E'}{n}$, for all $i \in N$. If we observe individual $i = 1$,*

$$\varphi_1(E,c) - \varphi_1(E',c) = \frac{c_1}{n} - \frac{E'}{n} = m_1^l(E,c) - m_1^l(E',c).$$

   For other individuals,

$$\varphi_i(E,c) - \varphi_i(E',c) = \min\left\{\frac{c_1}{n} + \varepsilon, c_2\right\} - \frac{E'}{n} \geqslant$$

$$\geqslant \frac{c_1}{n} - \frac{E'}{n} = m_i^l(E,c) - m_i^l(E',c)$$

   hence, the property is satisfied.
3) *$c_1 < E' < E$. In this case $m_i^l(E,c) = \frac{c_1}{n} = m_i^l(E',c)$, for all $i \in N$, and the condition obviously holds.*

5. Upper bounds: the dual approach

An important tool in the analysis of claims problems is the notion of *duality*: we focus on losses incurred by the agents (what they do not receive with respect to their claims) instead of focusing on awards. The *total loss* in a claims problem $(E, c) \in \mathcal{B}$ is defined as the difference between the aggregate claim and the endowment; that is, $L = \sum_{i=1}^{n} c_i - E$. Hence, (L, c) is also a claims problem, which is known as the *dual problem*. Then, given a claims rule φ, its *dual rule*, φ^d, assigns losses in the same way as φ assigns gains (Aumann and Maschler, 1985). Formally, for all $i = 1, 2, \ldots, n$

$$\varphi_i^d(E, c) = c_i - \varphi_i(L, c).$$

It is noteworthy that some rules exist that are self-dual (distributing losses gives the same result as distributing awards). But this is not true in general. It is well known that the constrained equal awards and the constrained equal losses rules are dual of each other (Herrero, 2003). Following the definition, the dual of the Ibn Ezra rule is:

$$\varphi_i^{IE^d}(E, c) = c_i - \varphi_i^{IE}(L, c) \qquad\qquad i = 1, 2, \ldots, n.$$

With the data in Example 1, $(E, c) = (120, (20, 40, 60, 120))$, the dual of the Ibn Ezra rule provides the allocation:

$$\varphi_i^{IE^d}(E, c) = \left(\frac{180}{12}, \frac{340}{12}, \frac{460}{12}, \frac{460}{12}\right).$$

Our final discussion is devoted to the analysis of the existence of *upper bounds*. The idea of an upper bound on awards is to establish the maximum amount that an individual i should obtain in a particular claims problem by taking into account her claim c_i, the endowment E and other individuals' claims. In fact, the definition of a claims rule already requires an upper bound on awards: no one can get more than her respective claim. Formally,

An **upper bound** is a function $B : \mathcal{B} \to \mathbb{R}_+^n$ which maps each claims problem $(E, c) \in \mathcal{B}$, and each $i \in N$, to a real number $B_i(E, c)$, such that

(i) $0 \leqslant B_i(E, c) \leqslant c_i$

(ii) $\sum_{i=1}^{n} B_i(E, c) \geqslant E$

Moulin (2002) introduces the *fair* upper bound f^u: for each $(E, c) \in \mathcal{B}$ and each $i \in N$,

$$f_i^u(E, c) = \max\left\{0, c_i - \frac{L}{n}\right\}, \text{where} \quad L = \sum_{i=1}^{n} c_i - E.$$

If we observe this definition,

$$f_i^u(E, c) = -\min\left\{0, -c_i + \frac{L}{n}\right\} = c_i - \min\left\{c_i, \frac{L}{n}\right\} = c_i - f_i^l(L, c).$$

That is, the fair upper bound can be defined by using the fair lower bound on the *dual* claims problem (L, c); and vice versa: the fair upper bound defines a lower bound on the dual problem that coincides with the fair lower bound.

The same occurs with any upper bound: $c_i - B_i(E, c)$ defines a lower bound of the dual claims problem (L, c), since

(i) $0 \leqslant c_i - B_i(E, c) \leqslant c_i$

(ii) $\sum_{i=1}^{n} (c_i - B_i(E, c)) = C - \sum_{i=1}^{n} B_i(E, c) \leqslant C - E = L$

We use this *duality* to introduce the upper bounds associated to the minimal right, securement and min lower bounds.

Maximal right, mr^u: for each $(E, c) \in \mathcal{B}$ and each $i \in N$,

$$mr_i^u(E, c) = c_i - mr_i(L, c) = c_i - \max\left\{0, L - \textstyle\sum_{j \in N \setminus \{i\}} c_j\right\} =$$
$$= \min\{c_i, C - L\} = \min\{c_i, E\}.$$

Remark 1. *Note that the maximal right, obtained in the above, corresponds with the notion of a truncated claim, introduced in Curiel et al. (1987).*

Securement upper bound, s^u: for each $(E, c) \in \mathcal{B}$ and each $i \in N$,

$$s_i^u(E, c) = c_i - s_i(L, c) = c_i - \tfrac{1}{n}\min\{c_i, L\} =$$
$$= \max\left\{c_i - \tfrac{c_i}{n}, c_i - \tfrac{L}{n}\right\}.$$

Min upper bound, m^u: for each $(E, c) \in \mathcal{B}$ and each $i \in N$,

$$m_i^u(E, c) = c_i - m_i^l(L, c) = c_i - \tfrac{1}{n}\min\{\min_{j \in N} c_j, L\} =$$
$$= \max\left\{c_i - \tfrac{\min_{j \in N} c_j}{n}, c_i - \tfrac{L}{n}\right\}.$$

On the other hand, two properties are dual if whenever a rule satisfies one of the properties, its dual rule satisfies the other. Then, when a rule φ fulfills some axiom with respect to a given lower bound, its dual rule φ^d fulfills the dual axiom with respect to the dual upper bound. Then, the following results are an immediate consequence of the results obtained in Section 4.

Remark 2. *The dual axiom of ETEB coincides with itself. The same occurs with CGS. On the other hand, the dual axioms of RB and CRM are obtained by reversing the inequality in the original properties.*

Corollary 6. *Let us consider the bound $b = mr^u$. Then, φ^{MR^d} is the only rule satisfying CGS^d and PRI^d.*

Corollary 7. *Let us consider the bound $b = f^u$. Then, φ^{CEL} is the only rule satisfying:*

1. *RB^d and $ETEB^d$.*
2. *$ETEB^d$ and CRM^d.*
3. *RB^d and CGS^d.*
4. *CRM^d and CGS^d.*

Corollary 8. *Let us consider the bound $b = s^u$. Then, φ^{IE^d} is the only rule satisfying:*

1. *CGS^d, RB^d and PRI^d.*
2. *CRM^d, CGS^d and PRI^d.*

Corollary 9. *Let us consider the bound $b = m^u$. Then, φ^{CEL} is the only rule satisfying:*

1. *$ETEB^d$.*
2. *CGS^d.*

6. Final Remarks

Throughout this paper, we have shown that by asking for some warranties (lower bounds) in claims problems, we can associate each warranty with a particular claims rule: the *fair* and the min lower bounds are linked to the *constrained equal awards* rule; the *securement* lower bound is associated to the *Ibn Ezra* solution.

The analyzed correspondence between lower bounds and claims rules makes us wonder about warranties that are linked to other important claims rules such as the *constrained equal losses*, the *proportional*, the *adjusted proportional*, the *Talmudian*, etc., a question that remains open.

Finally, we put forward for discussion the converse question: if we propose a reasonable warranty (a lower bound) or a maximum award (an upper bound), is it possible to define a unique claims rule satisfying the required axioms?

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APPENDIX: Proof of Lemmas in Section 3

Proof of Lemma 1

a) Let us consider the problems (E, c) and (E', c) with $E' = 0$. Then, for all $b \in \mathcal{L}$, $b'_i = 0$ for all $i \in N$. If φ is a claims rule satisfying CRM, then $\varphi_i(E, c) \geqslant b_i(E, C)$ or $\varphi_i(E, c) = c_i$. Since $b_i(E, c) \leqslant c_i$, RB is fulfilled.

b) Let us consider the problems (E, c) and (E', c) with $E' = 0$. Then, for all $b \in \mathcal{L}$, and each claims rule φ, $b'_i = 0$ and $\varphi_i(E', c) = 0$, for all $i \in N$. Hence, if φ is a claims rule fulfilling *CGS*, then the condition in *ETEB* coincides with the condition in *CGS*, so it is obviously satisfied. ■

Proof of Lemma 2

First, note that the min lower bound ensures an equal warranty for all agents: if we consider, without loss of generality, that the agents in a claims problem (E, c) are ordered, such that $c_1 \leqslant c_2 \leqslant \ldots \leqslant c_n$, then for all $i \in N$

$$(1)\ m_i^l(E,c) = \frac{c_1}{n} \leqslant \frac{E}{n}, \quad \text{or} \quad (2)\ m_i^l(E,c) = \frac{E}{n} \leqslant \frac{c_1}{n}.$$

Now, let us consider a claims rule φ satisfying *ETEB*. This property implies that agents receive the same award, or receive their claim in full; that is, $\varphi_i(E, c) \geqslant \min\left\{c_i, \frac{E}{n}\right\}$, or $\varphi_i(E, c) = \frac{E}{n}$, respectively. In both cases, $\varphi_i(E, c) \geqslant m_i^l(E, c)$ and then *RB* holds. With a similar reasoning, it is easy to obtain that *CGS* implies *CRM*. ■

Proof of Lemma 3

See Examples 3 and 4. ■

Example 3. *Let $n = 3$ and the rule φ^a defined by:*

$$\varphi_i^a(E,(c_1,c_2,c_3)) = \begin{cases} \min\left\{c_i, \frac{E}{3}\right\} & i = 1,2 \\ \\ E - \min\left\{c_1, \frac{E}{3}\right\} - \min\left\{c_2, \frac{E}{3}\right\} & i = 3 \end{cases}$$

It is clear that φ^a satisfies CRM and RB for $b = f^l$ or $b = s$. Consider now the claims problem $(E, c) = (9, (1, 9, 10))$. Then, $\varphi^a(E, c) = (1, 3, 5)$, whereas $b_2(E, c) = b_3(E, c)$ for $b = f^l$ or $b = s$. Therefore, φ^a does not satisfy ETEB, hence neither does CGS.

Example 4. *Let $n = 3$ and the rule φ^* defined by:*

$$\varphi_i^*(E,c) = \begin{cases} \varphi^{CEA}(E,c) & \text{if } f_1^l(E,c) = f_2^l(E,c) = f_3^l(E,c) \\ \\ \varphi^{CEA}(E,c) + (-x,-x,2x) & \text{if } f_1^l(E,c) = f_2^l(E,c) < f_3^l(E,c) \\ \\ \varphi^{CEA}(E,c) + (-2x,x,x) & \text{if } f_1^l(E,c) < f_2^l(E,c) \end{cases}$$

It is clear that φ^* fulfills ETEB and CGS for $b = f^l$ or $b = s$. If we consider the problem $(E, c) = (12, (1, 9, 10))$, then $\varphi^*(E, c) = (0, 6, 6)$, hence RB and CRM are not satisfied.

Proof of Lemma 4

a) We use the claims rule φ^a and the problem (E, c) introduced in Example 3. We now consider the problem $(E', c) = (12, (1, 9, 10))$. Then, if we compare the securement lower bound of both problems, $s(E, c) = \left(\frac{1}{3}, 3, 3\right)$ and $s(E', c) = \left(\frac{1}{3}, 3, \frac{10}{3}\right)$; that is, $s_2(E, c) = s_2(E', c)$. Nevertheless, $\varphi^a(E, c) = (1, 3, 5)$ and $\varphi^a(E', c) = (1, 4, 7)$, contradicting *PRI*.
On the other hand, the constrained equal losses rule, φ^{CEL} fulfills *PRI* and does not satisfy *RB* nor *CRM*.

b) Let $n = 3$ and consider the constrained equal awards rule, φ^{CEA}. It is clear that *ETEB* and *CGS* are fulfilled. If we now consider the problems $(E, c) = (3, (3, 6, 9))$ and $(E', c) = (6, (3, 6, 9))$, then $\varphi^{CEA}(E, c) = (1, 1, 1)$, and $\varphi^{CEA}(E', c) = (2, 2, 2)$. Note that $s(E, c) = (1, 1, 1)$, and $s(E', c) = (1, 2, 2)$, hence *PRI* is not satisfied.
On the other hand, the constrained equal losses rule, φ^{CEL} fulfills *PRI* and does not satisfy *ETEB* nor *CGS*. ■