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Document de treball n.10 - 2018

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Edita:

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www.fcee.urv.es/departaments/economia/public_html/index.html
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Adreçar comentaris al Departament d'Economia / CREIP

ISSN edició en paper: 1576 - 3382
ISSN edició electrònica: 1988 - 0820

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Weak fairness and the Shapley value.

Pedro Calleja and Francesc Llerena *

Abstract

The Shapley value (Shapley, 1953) has been axiomatically characterized from different points of view. Van den Brink (2001) proposes a characterization by means of efficiency, fairness and the null player property. In this paper, we characterize the family of single-valued solutions obtained by relaxing fairness into weak fairness. To point out the Shapley value, we impose the additional axiom of weak self consistency and strengthen the null player property into the dummy player property.

1 Introduction

Probably, the most relevant single-valued solution for cooperative games with transferable utility (games, hereafter) is the *Shapley value* (Shapley, 1953). Many characterizations of this solution, including his original axiomatic approach, use the principle that if a player contributes zero to all coalitions, then she must receive a zero payoff: the *null player property*.

Various authors have proposed alternative foundations of the Shapley value imposing the *null player property*. Particularly, van den Brink (2001) interprets the Shapley value as the unique solution satisfying, additionally, *efficiency* (Pareto optimality) and *fairness*, a property inspired in Myerson's (1977) fairness. For single-valued solutions, *fairness* essentially imposes that if a game suffers an impact consisting in adding another game in which two players are symmetric, then their payoffs should change by the same amount. If we measure the relevance of a player in terms of her marginal contributions to all coalitions, *fairness* is a quite natural requirement since adding such a game does not change the contributions of symmetric players. Recently, Casajus and Yokote (2017) relax *fairness* into *weak*

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We acknowledge F. J. Martinez de Albeniz for his helpful comments and J. M. Izquierdo for the pictures. We also acknowledge the support from research grants ECO2016-75410-P and ECO2017-86481-P.

differential marginality to provide a new axiomatization of the Shapley value, together with *efficiency* and the *null player property*. *Weak differential marginality* says that equal changes in two players' marginal contributions to coalitions containing neither of them should entail that their payoffs change in the same direction. Unfortunately, their characterization does not hold for two-player games. In order to overcome the problem, they strengthen the *null player property* into the *dummy player property*, stating that if a player contributes only her individual worth to all coalitions, then she must receive her individual worth.

In this paper, we study what solutions emerge when weakening *fairness* into *weak fairness* (van den Brink et al., 2016), combined again with *efficiency* and either the *null player property* or the *dummy player property*. *Weak fairness*, a property very much related to *strong aggregate monotonicity* (Arin, 2013), can be viewed as a solidarity axiom in the sense that if only the worth of the grand coalition varies, while the worth of all other coalitions remain unchanged, then players' payoffs should be affected equally.

Another different principle used from Hart and Mas-Colell (1989) to interpret the Shapley value is *self consistency*. Consistency is an outstanding relational property widely used in the axiomatic analysis of solutions imposing that an original agreement should be reconfirmed in the underlying reduced game when some agents leave.¹ In this work, we impose *weak self consistency*, that is, *self consistency* when only one or two agents stay, to select the Shapley value from the set of solutions satisfying *efficiency*, *weak fairness* and the *dummy player property*.

The remainder of this paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, we characterize the family of single-valued solutions satisfying *efficiency*, *weak fairness* and either the *dummy player property* or the *null player property*. Remarkably, these characterizations can be extended to any domain of games. In section 4, we provide a new axiomatization of the Shapley value by means of *weak self consistency*, *weak fairness* and the *dummy player property*. These properties still characterize the Shapley value on the domain of convex games but, unfortunately, they are incompatible on some well-established domains of games, like superadditive or totally balanced games.

2 Preliminaries

The set of natural numbers \mathbb{N} denotes the universe of potential players. A **coalition** is a non-empty finite subset of \mathbb{N} and let \mathcal{N} denote the set of all coalitions of \mathbb{N} . Given $S, T \in \mathcal{N}$, we use $S \subset T$ to indicate strict inclusion, that is, $S \subseteq T$ and $S \neq T$. By $|S|$ we denote the cardinality of the coalition $S \in \mathcal{N}$. A **transferable utility coalitional game** is a pair (N, v) where $N \in \mathcal{N}$ is the set of players and $v : 2^N \rightarrow \mathbb{R}$ is the characteristic function that assigns to each coalition $S \subseteq N$ a

¹See Thomson (2012) for an essay on the consistency property.

real number $v(S)$, representing what S can achieve by agreeing to cooperate, with the convention $v(\emptyset) = 0$. For simplicity of notation, and if no confusion arises, we write $v(i), v(ij), \dots$ instead of $v(\{i\}), v(\{i, j\}), \dots$. By Γ we denote the class of all games.

Given $N \in \mathcal{N}$ and $\emptyset \neq N' \subseteq N$, the **unanimity game** $(N, u_{N'})$ associated to N' is defined as $u_{N'}(S) = 1$ if $N' \subseteq S$ and $u_{N'}(S) = 0$ otherwise. Given a game (N, v) and $\emptyset \neq N' \subseteq N$, the **subgame** $(N', v|_{N'})$ is defined as $v|_{N'}(S) = v(S)$ for all $S \subseteq N'$. For any two games (N, v) , (N, w) , and $\alpha \in \mathbb{R}$, we define the game $(N, v+w)$ as $(v+w)(S) = v(S)+w(S)$, and the game $(N, \alpha \cdot v)$ as $(\alpha \cdot v)(S) = \alpha \cdot v(S)$, for all $S \subseteq N$.

Given $N \in \mathcal{N}$, let \mathbb{R}^N stand for the space of real-valued vectors indexed by N , $x = (x_i)_{i \in N}$, and for all $S \subseteq N$, $x(S) = \sum_{i \in S} x_i$, with the convention $x(\emptyset) = 0$. Given $\emptyset \neq S \subseteq N$, $e_S \in \mathbb{R}^N$ is defined as $e_{S,i} := 1$ if $i \in S$ and $e_{S,i} := 0$ otherwise. For each $x \in \mathbb{R}^N$ and $T \subseteq N$, $x|_T$ denotes the restriction of x to T : $x|_T = (x_i)_{i \in T} \in \mathbb{R}^T$.

Let $N \in \mathcal{N}$. The **preimputation set** of (N, v) contains the efficient payoff vectors, that is, $X(N, v) := \{x \in \mathbb{R}^N \mid x(N) = v(N)\}$, and the **core** is the set of preimputations where each coalition gets at least its worth, that is $C(N, v) = \{x \in X(N, v) \mid x(S) \geq v(S) \forall S \subseteq N\}$. A game (N, v) is **balanced** if it has a non-empty core, and it is **totally balanced** if, for all $S \subseteq N$, the subgame $(S, v|_S)$ is balanced. A game (N, v) is **superadditive** if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$. A game (N, v) is **convex** if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subseteq N$.

Player $i \in N$ is called a **dummy player** in (N, v) if $v(S \cup \{i\}) - v(S) = v(i)$ for all $S \subseteq N \setminus \{i\}$, and is called a **null player** in (N, v) if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$. By $D(N, v)$ and $N(N, v)$ we denote the set of dummy and null players in (N, v) , respectively.

We say that players i and j are **symmetric** in (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.

A **single-valued solution** on $\Gamma' \subseteq \Gamma$ is a function $\sigma : \Gamma' \rightarrow \bigcup_{N \in \mathcal{N}} \mathbb{R}^N$ that associates with each game $(N, v) \in \Gamma'$ an $|N|$ -dimensional real vector $\sigma(N, v)$. The **Shapley value**, Sh , is defined by

$$Sh_i(N, v) := \sum_{S \subseteq N \setminus \{i\}} \frac{|S|! (|N| - |S| - 1)!}{|N|!} (v(S \cup \{i\}) - v(S)) \text{ for all } i \in N.$$

3 Efficiency, weak fairness and dummy or null player property

Van den Brink (2001) characterizes the Shapley value on the full domain of games making use of *fairness* together with *efficiency* and the *null player property*.

A single-valued solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Efficiency (E)**: if for all $N \in \mathcal{N}$ and all $(N, v) \in \Gamma'$ it holds that $\sigma(N, v) \in X(N, v)$.
- **Null player property (NP)**: if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$ and all $i \in N$, if i is a null player in (N, v) , then $\sigma_i(N, v) = 0$.
- **Fairness (F)**: if for all $N \in \mathcal{N}$, all $(N, v), (N, v') \in \Gamma'$ with $(N, v + v') \in \Gamma'$ and all $i, j \in N$ such that i and j are symmetric in (N, v') , we have $\sigma_i(N, v + v') - \sigma_i(N, v) = \sigma_j(N, v + v') - \sigma_j(N, v)$.

Casajus and Yokote (2017) relax *fairness* into *weak differential marginality* and strengthen the *null player property* into the *dummy player property* to obtain a new characterization of the Shapley value on the full domain of games. In this section, we investigate what single-valued solutions appear when weakening *fairness* into *weak fairness* combined with *efficiency* and either the *dummy player property* or the *null player property*.

A single-valued solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Weak Fairness (wF)**: if for all $N \in \mathcal{N}$, all $(N, v), (N, v') \in \Gamma'$ such that $v(S) = v'(S)$ for all $S \subset N$ and all $i, j \in N$, we have $\sigma_i(N, v') - \sigma_i(N, v) = \sigma_j(N, v') - \sigma_j(N, v)$.
- **Dummy player property (DP)**: if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$ and all $i \in N$, if i is a dummy player in (N, v) then $\sigma_i(N, v) = v(i)$.

By taking *weak fairness* a large family of single-valued solutions emerge. In order to describe such a family when considering the *dummy player property* for any domain of games, we first introduce some concepts. In the remaining of this section we deal with a fixed player set N and, consequently, a game (N, v) is only described by its characteristic function v .

Let $\Gamma' \subseteq \Gamma$ be a certain domain of games with player set N . Given $v \in \Gamma'$, a player $i \in N$ is called a **potential dummy player** in v if $v(S \cup \{i\}) - v(S) = v(i)$ for all $S \subset N \setminus \{i\}$. By $PD(v)$ we denote the set of potential dummies in v . Notice that $D(v) \subseteq PD(v)$ and, moreover, any player $i \in PD(v) \setminus D(v)$ will become into a dummy player in game $w \in \Gamma'$ with $w(S) = v(S)$ for all $S \subset N$ and $w(N) = v(i) + v(N \setminus \{i\})$. Moreover, from Lemma 3 in Calleja et al. (2012), it turns out that $PD(v) \neq \emptyset$ and $v(N) = \sum_{i \in PD(v)} v(i) + v(N \setminus PD(v))$ if and only if

$PD(v) = D(v) \neq \emptyset$. Hence, if $PD(v) \neq \emptyset$ then either $PD(v) = D(v)$ or $D(v) = \emptyset$.

We next define the **equivalence relation** \mathcal{R} on Γ' as follows: for all $v, w \in \Gamma'$

$$v \mathcal{R} w \text{ if and only if } v(S) = w(S) \text{ for all } S \subset N.$$

The **equivalence class** containing $v \in \Gamma'$ is denoted by $[v] = \{w \in \Gamma' : w\mathcal{R}v\}$. Let $\Gamma'/\mathcal{R} = \{[v] : v \in \Gamma'\}$ be the **quotient set**. For every equivalence class $[v] \in \Gamma'/\mathcal{R}$ we fix a **representative element**, denoted by v_* . If there is $w \in [v]$ such that $D(w) \neq \emptyset$ then $v_* = w$; notice that w is unique. Otherwise, choose an arbitrary $v_* \in [v]$.

Let Γ'_* stand for the **set of representative games**, one for each equivalence class. Notice that any $v \in [v_*]$ can be expressed as

$$v = v_* + (v(N) - v_*(N)) \cdot u_N.$$

Moreover, $PD(v) = PD(v_*)$ and $v_*(N) = \sum_{i \in PD(v)} v(i) + v(N \setminus PD(v))$, whenever $D(v_*) \neq \emptyset$.

Definition 1. A dummy-adapted Γ'_* -selection is a function $F : \Gamma'_* \rightarrow \mathbb{R}^N$ such that $\sum_{i \in N} F_i(v_*) = v_*(N)$ and $F_i(v_*) = v_*(i)$ for all $i \in D(v_*)$.

A dummy-adapted Γ'_* -selection associates an efficient vector to every representative game, with the particularity that if dummy players appear they receive exactly their individual worth.

Let \mathcal{F}'_D denote the class of dummy-adapted Γ'_* -selections. Given $F \in \mathcal{F}'_D$, we can define the associated single-valued solution σ^F as follows: for all $v \in \Gamma'$, $v \in [v_*]$,

$$\sigma^F(v) = F(v_*) + \frac{v(N) - v_*(N)}{|N|} \cdot e_N. \quad (1)$$

Let $F \in \mathcal{F}'_D$. The interpretation of σ^F is as follows: given $v \in [v_*]$, σ^F first distributes $v_*(N)$ among players according to F , and then it distributes what is left of the gains of cooperation equally. Geometrically, σ^F is the set of straight lines (one for every element of Γ'/\mathcal{R}) going through $F(v_*)$ with direction vector $\frac{e_N}{|N|}$.

Let us consider an example.

Example 1. Let $\Gamma' = \Gamma$. An example of dummy-adapted Γ'_* -selection is the Shapley value, that is, $F(v_*) = Sh(v_*)$ for all $v_* \in \Gamma'_*$. Alternatively, define $F \in \mathcal{F}'_D$ as follows: for all $v_* \in \Gamma'_*$, $F_i(v_*) = v_*(i)$ if $i \in D(v_*)$ and $F_i(v_*) = \frac{v_*(N) - \sum_{i \in D(v_*)} v_*(i)}{|N \setminus D(v_*)|}$ if $i \notin D(v_*)$. Denote by σ^{Sh} and σ^F the corresponding single-valued solutions. Notice that $\sigma^{Sh} = Sh$, since the Shapley value satisfies efficiency and weak fairness. To illustrate these solutions we consider two different cases, depending on the existence or not of potential dummy players.

Let (N, v) be a game with $N = \{1, 2, 3\}$, $v(i) = 0$ for all $i \in N$, $v(\{1, 2\}) = v(\{1, 3\}) = v(N) = 1$ and $v(\{2, 3\}) = 0$. Since $PD(v) = \emptyset$, as a representative element of $[v]$ we can take the game v_* , being $v_*(N) = 0$ and $v_*(S) = v(S)$ for any

other coalition $S \subset N$. Then,

$$\sigma^{Sh}(v) = Sh(v_*) + \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(\frac{1}{3}, \frac{-1}{6}, \frac{-1}{6}\right) + \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$$

and

$$\sigma^F(v) = F(v_*) + \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = (0, 0, 0) + \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

Figure 1 gives a geometric interpretation of σ^{Sh} and σ^F in this case.

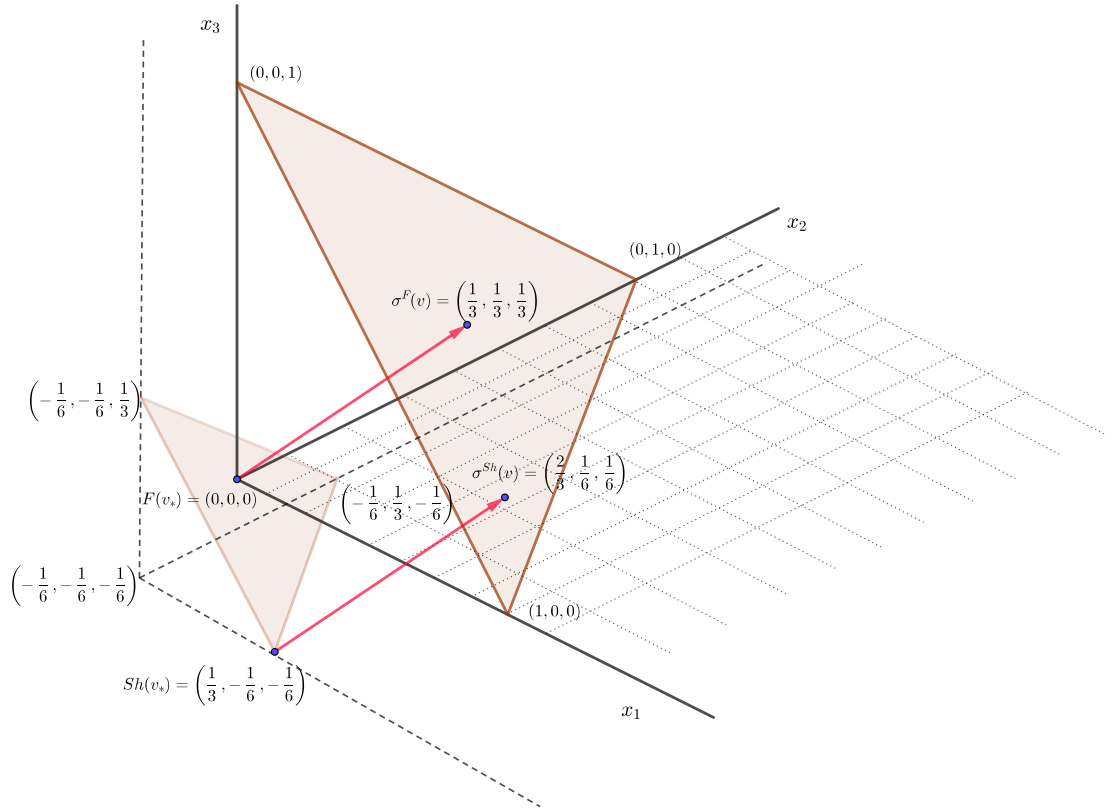


Figure 1

Now consider the game (N, w) with $w(1) = w(\{1, 3\}) = \frac{1}{2}$, $w(2) = w(3) = w(\{2, 3\}) = 0$, $w(\{1, 2\}) = 1$ and $w(N) = 2$. Since $PD(w) = \{3\}$, the representative element $v_* \in [w]$ is given by $v_*(N) = 1$ and $v_*(S) = w(S)$ for any other coalition $S \subset N$. Hence,

$$\sigma^{Sh}(w) = Sh(v_*) + \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(\frac{3}{4}, \frac{1}{4}, 0\right) + \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(\frac{13}{12}, \frac{7}{12}, \frac{1}{3}\right)$$

and

$$\sigma^F(w) = F(v_*) + \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(\frac{1}{2}, \frac{1}{2}, 0\right) + \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(\frac{5}{6}, \frac{5}{6}, \frac{1}{3}\right).$$

Figure 2 gives a geometric interpretation of σ^{Sh} and σ^F in this case.

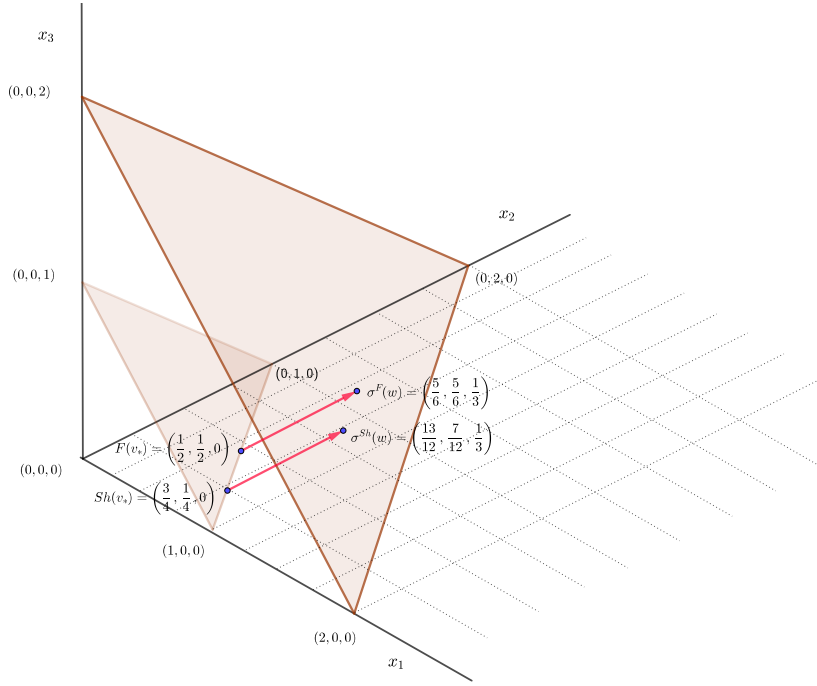


Figure 2

Now, we are in a position to obtain our characterization result.

Theorem 1. *A single-valued solution σ satisfies efficiency, weak fairness and the dummy player property on Γ' if and only if there exists a dummy-adapted Γ'_* -selection $F \in \mathcal{F}'_D$ such that $\sigma = \sigma^F$.*

Proof Let σ be a single-valued solution on Γ' and $F \in \mathcal{F}'_D$ such that $\sigma = \sigma^F$. Then, **E** follows directly from $F \in \mathcal{F}'_D$. To check **DP**, let $v \in \Gamma'$, $v \in [v_*]$, with $D(v) \neq \emptyset$. Then, $v = v_*$. Consequently, for all $i \in D(v)$, $\sigma_i(v) = F_i(v_*) = v(i)$ where the last equality follows from $F \in \mathcal{F}'_D$. To check **wF**, let $v, w \in \Gamma'$ be such that $v(S) = w(S)$ for all $S \subset N$. Hence, $v, w \in [v_*]$ and, for all $i \in N$,

$$\begin{aligned} \sigma_i(w) - \sigma_i(v) &= F_i(v_*) + \frac{w(N) - v_*(N)}{|N|} - F_i(v_*) - \frac{v(N) - v_*(N)}{|N|} \\ &= \frac{w(N) - v(N)}{|N|}. \end{aligned}$$

To show the reverse implication, let σ be a single-valued solution on Γ' satisfying **E**, **wF** and **DP**. Define $F : \Gamma'_* \rightarrow \mathbb{R}^N$ as $F(v_*) = \sigma(v_*)$ for all $v_* \in \Gamma'_*$. By **E** and **DP**, it follows directly $F \in \mathcal{F}'_D$. To finish, let us show that $\sigma = \sigma^F$ with $F \in \mathcal{F}'_D$ as defined above. Let $v \in \Gamma'$, $v \in [v_*]$. Since $v = v_* + (v(N) - v_*(N)) \cdot u_N$, by **E** and **wF**,

$$\begin{aligned} \sigma(v) &= \sigma(v_*) + \frac{v(N) - v_*(N)}{|N|} \cdot e_N \\ &= F(v_*) + \frac{v(N) - v_*(N)}{|N|} \cdot e_N \\ &= \sigma^F(v). \end{aligned}$$

□

The properties in Theorem 1 are non-redundant on Γ' provided that Γ' is rich enough. The **equal division solution**, ED , defined by $ED_i(v) := \frac{v(N)}{|N|}$ for all $v \in \Gamma'$ and all $i \in N$ meets all properties but **DP**. The single-valued solution ρ defined as $\rho_i(v) = v(i)$ for all $v \in \Gamma'$ and all $i \in N$ meets all properties but **E**. Let π be a permutation on N , the **marginal contribution solution** relative to π , mc^π , defined as $mc_i^\pi(v) := v(\{j \in N \mid \pi(j) \leq \pi(i)\}) - v(\{j \in N \mid \pi(j) < \pi(i)\})$ for all $v \in \Gamma'$ and all $i \in N$ meets all properties but **wF**.

A natural observation is to ask for the consequences of weakening the *dummy player property* into the *null player property* in Theorem 1. It is not difficult to extend the notion of potential dummy player to potential null player in order to characterize the family of single-valued solutions satisfying *efficiency*, *weak fairness* and the *null player property*.

Let $v \in \Gamma'$, a player $i \in N$ is called a **potential null player** in v if $v(S \cup \{i\}) = v(S)$ for all $S \subset N \setminus \{i\}$. Let $PN(v)$ be the set of potential null players in v . Clearly, $N(v) \subseteq PN(v)$ and, moreover, $PN(v) \neq \emptyset$ and $v(N) = v(N \setminus PN(v))$ if and only if $PN(v) = N(v) \neq \emptyset$. For every equivalence class $[v] \in \Gamma' / \mathcal{R}$ we choose the **representative element** v_* as follows: if there is $w \in [v]$ such that $N(w) \neq \emptyset$ then $v_* = w$; otherwise, choose an arbitrary $v_* \in [v]$.

Now, we can define a null-adapted Γ'_* -selection analogously to Definition 1. Note that $\mathcal{F}'_D \subset \mathcal{F}'_N$, where \mathcal{F}'_N denotes the class of null-adapted Γ'_* -selections.

Theorem 2. *A single-valued solution σ satisfies efficiency, weak fairness and the null player property on Γ' if and only if there exists a null-adapted Γ'_* -selection $F \in \mathcal{F}'_N$ such that $\sigma = \sigma^F$.*

The single-valued solutions used to prove the independence of the properties in Theorem 1 also show that properties in Theorem 2 are non-redundant on Γ' provided that Γ' is rich enough.

4 Consistency, weak fairness and dummy or null player property

Consistency is a kind of internal stability requirement that relates the solution of a game to the solution of a reduced game that appears when some agents leave. The different ways in which the coalitions of the remaining agents are evaluated give rise to different notions of reduced game. Here we deal with the self reduced game (Hart and Mas-Colell, 1989).

Definition 2. Let σ be a single-valued solution, $N \in \mathcal{N}$, $(N, v) \in \Gamma$, and $\emptyset \neq N' \subset N$. The **self reduced game** relative to N' at σ is the game $(N', r_\sigma^{N'}(v))$ defined by

$$r_\sigma^{N'}(v)(R) := \begin{cases} 0 & \text{if } R = \emptyset, \\ v(R \cup N'') - \sum_{i \in N''} \sigma_i(R \cup N'', v|_{R \cup N''}) & \text{if } \emptyset \neq R \subseteq N', \end{cases}$$

where $N'' = N \setminus N'$.

In the self reduced game (relative to N' at σ), the worth of a coalition $R \subseteq N'$ is determined under the assumption that R joins all members of $N'' = N \setminus N'$, provided they are paid according to σ in the subgame associated to $R \cup N''$.

A single-valued solution σ on $\Gamma' \subseteq \Gamma$ satisfies

- **Self consistency (SC):** if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma$ and all $\emptyset \neq N' \subset N$, then $(N', r_\sigma^{N'}(v)) \in \Gamma'$ and $\sigma(N, v)|_{N'} = \sigma(N', r_\sigma^{N'}(v))$.

The consistency principle states that in the corresponding reduced game the original agreement should be reconfirmed. **Weak self consistency (wSC)** imposes this internal stability requirement for reduced games with at most two players.

Self consistency has been used to characterize the Shapley value by Hart and Mas-Colell (1989). It turns out that imposing *weak self consistency*, in addition to those properties in the statement of Theorem 1, provides a new axiomatic interpretation of the Shapley value on the full domain of games. However, these properties are redundant, what allows us to drop *efficiency* leading to the following non-redundant characterization.

Theorem 3. *The Shapley value is the unique single-valued solution on Γ that satisfies weak self consistency, weak fairness and the dummy player property.*

Proof. Clearly the Shapley value satisfies **wSC**, **wF** and **DP**. Let σ be a single-valued solution on Γ satisfying these properties. To prove that $\sigma = Sh$, we will use an induction argument on the number of players. First, we show that **wSC** and **DP** imply **E**. Let $(\{i\}, v)$ be a one-player game, since player i is a dummy player, by

DP $\sigma(\{i\}, v) = v(i)$, which means that σ satisfies efficiency for one-player games. Let $N \in \mathcal{N}$ with $|N| \geq 2$, $(N, v) \in \Gamma$ and $i \in N$. Then, efficiency for one-player games implies $\sigma_i(\{i\}, r_\sigma^{\{i\}}(v)) = r_\sigma^{\{i\}}(v)(i) = v(N) - \sum_{j \in N \setminus \{i\}} \sigma_j(N, v)$. By **wSC**, $\sigma_i(N, v) = \sigma_i(\{i\}, r_\sigma^{\{i\}}(v))$ and thus $\sigma_i(N, v) = v(N) - \sum_{j \in N \setminus \{i\}} \sigma_j(N, v)$, which proves **E**.

Thus, from **E**, $\sigma = Sh$ for one-player games. Now, let $N = \{i, j\} \in \mathcal{N}$ and $(N, v) \in \Gamma$. If $v(N) = v(i) + v(j)$, then by **DP** it follows directly that $\sigma(N, v) = (v(i), v(j))$. If $v(N) \neq v(i) + v(j)$, consider the associated game (N, v') defined as follows: $v'(k) = v(k)$ for all $k \in N$, and $v'(N) = v(i) + v(j)$. Clearly $v = v' + (v(N) - v(i) - v(j)) \cdot u_N$. Then, by **wF**, **DP** and **E** we obtain $\sigma_k(N, v) = v(k) + \frac{1}{2}(v(N) - v(i) - v(j))$ for all $k \in N$. Thus, $\sigma = Sh$ for two-player games.

Induction hypothesis: for all $(N, v) \in \Gamma$ with $|N| \leq t$, $t \geq 2$, it holds $\sigma(N, v) = Sh(N, v)$.

Let (N, v) be a game with $|N| = t + 1$. Denote $x = \sigma(N, v)$ and $y = Sh(N, v)$. Let $N' = \{i, j\} \subset N$ and take $k \in N'$. Since $(N', r_\sigma^{N'}(v))$ is a two-player game, then by **E** and **wSC**,

$$x_k = \sigma_k(N', r_\sigma^{N'}(v)) = r_\sigma^{N'}(v)(k) + \frac{1}{2}(x_i + x_j - r_\sigma^{N'}(v)(i) - r_\sigma^{N'}(v)(j)). \quad (2)$$

Moreover,

$$y_k = Sh_k(N', r_{Sh}^{N'}(v)) = r_{Sh}^{N'}(v)(k) + \frac{1}{2}(y_i + y_j - r_{Sh}^{N'}(v)(i) - r_{Sh}^{N'}(v)(j)). \quad (3)$$

By the induction hypothesis, if $N'' = N \setminus N'$, we have

$$Sh(k \cup N'', v|_{k \cup N''}) = \sigma(k \cup N'', v|_{k \cup N''}),$$

which leads to

$$\begin{aligned} r_{Sh}^{N'}(v)(k) &= v(k \cup N'') - \sum_{l \in N''} Sh_l(k \cup N'', v|_{k \cup N''}) \\ &= v(k \cup N'') - \sum_{l \in N''} \sigma_l(k \cup N'', v|_{k \cup N''}) \\ &= r_\sigma^{N'}(v)(k). \end{aligned} \quad (4)$$

Now, combining (2), (3) and (4) we obtain

$$x_k - y_k = \frac{1}{2}(x_i + x_j) - \frac{1}{2}(y_i + y_j).$$

Hence, by taking any possible two-person reduction, we have $x_i - y_i = x_j - y_j$ for all $i, j \in N$. Equivalently, for all $i \in N$ it holds $x_i - y_i = \alpha$ for some $\alpha \in \mathbb{R}$. Finally, by **E** it comes out that $0 = \sum_{i \in N} x_i - \sum_{i \in N} y_i = n\alpha$, which implies $\alpha = 0$ and, consequently, $\sigma(N, v) = Sh(N, v)$. \square

The properties in Theorem 3 are non-redundant on Γ . The **equal division solution** meets all properties but **DP**. The single-valued solution ρ meets all properties but **wSC**, and the **marginal contribution solution** meets all properties but **wF**.

At this point, a natural question is to ask for the applicability of Theorem 3 to some domains of games. Because of the definition of self reduced game, it only makes sense to consider classes of games such that all subgames belong to the class, like the well-established domains of convex games, superadditive games and totally balanced games. Although the domain of convex games is not closed under the self reduction operation for the Shapley value, Hokari and Gellekom (2003) show that it satisfies *weak self consistency* on this domain. This observation allows us to follow the same arguments as in the proof of Theorem 3 to conclude that on the domain of convex games *weak self consistency*, *weak fairness* and the *dummy player property* still characterize the Shapley value. Unfortunately, for superadditive or totally balanced games, these three properties are incompatible.

Proposition 1. *There is no single-valued solution on the domain of superadditive games that satisfies weak self consistency, weak fairness and the dummy player property.*

Proof. Suppose, on the contrary, there exists a single-valued solution σ on the domain of superadditive games satisfying *weak self consistency*, *weak fairness* and the *dummy player property*. Consider the superadditive game (N, v) with $N = \{1, 2, 3\}$ and characteristic function $v(i) = 0$ for all $i \in N$, and $v(S) = 1$ for any other coalition $S \subseteq N$. Let $N' = \{1, 2\}$ and $(N', r_\sigma^{N'}(v))$ be the self reduced game relative to N' at σ . Since **wSC** together with **DP** imply **E** (see proof of Theorem 3), by **E**, **wF** and **DP** we have that $\sigma_3(\{1, 3\}, v_{|\{1,3\}}) = \frac{1}{2}$ and $\sigma_3(\{2, 3\}, v_{|\{2,3\}}) = \frac{1}{2}$. Hence,

$$r_\sigma^{N'}(v)(1) = v(\{1, 3\}) - \sigma_3(\{1, 3\}, v_{|\{1,3\}}) = \frac{1}{2},$$

$$r_\sigma^{N'}(v)(2) = v(\{2, 3\}) - \sigma_3(\{2, 3\}, v_{|\{2,3\}}) = \frac{1}{2}$$

and

$$r_\sigma^{N'}(v)(N') = v(N) - \sigma_3(N, v) = 1 - \sigma_3(N, v).$$

By **wSC**, $(N', r_\sigma^{N'}(v))$ must be a superadditive game, which means that $\frac{1}{2} + \frac{1}{2} \leq 1 - \sigma_3(N, v)$ or, equivalently, $\sigma_3(N, v) \leq 0$. In a similar way, but considering the self reduced games $(\{1, 3\}, r_\sigma^{\{1,3\}}(v))$ and $(\{2, 3\}, r_\sigma^{\{2,3\}}(v))$, it can be checked that $\sigma_2(N, v) \leq 0$ and $\sigma_1(N, v) \leq 0$, in contradiction with *efficiency*. \square

Proposition 2. *There is no single-valued solution on the domain of totally balanced games that satisfies weak self consistency, weak fairness and the dummy player property.*

Proof. Suppose, on the contrary, there exists a single-valued solution σ on the domain of totally balanced games satisfying *weak self consistency*, *weak fairness* and the *dummy player property*. Consider the totally balanced game (N, v) with $N = \{1, 2, 3\}$ and characteristic function $v(i) = 0$ for all $i \in N$, $v(\{1, 2\}) =$

$v(\{1, 3\}) = 1$, $v(\{2, 3\}) = 0$ and $v(N) = 1$. Let $N' = \{1, 2\}$ and $(N', r_\sigma^{N'}(v))$ be the self reduced game relative to N' at σ . Let us recall that **wSC** together with **DP** imply **E** (see proof of Theorem 3). Thus, by **E**, **wF** and **DP** we have that $\sigma_3(\{1, 3\}, v_{|\{1,3\}}) = \frac{1}{2}$ and $\sigma_3(\{2, 3\}, v_{|\{2,3\}}) = 0$. Hence,

$$r_\sigma^{N'}(v)(1) = v(\{1, 3\}) - \sigma_3(\{1, 3\}, v_{|\{1,3\}}) = \frac{1}{2},$$

$$r_\sigma^{N'}(v)(2) = v(\{2, 3\}) - \sigma_3(\{2, 3\}, v_{|\{2,3\}}) = 0$$

and

$$r_\sigma^{N'}(v)(N') = v(N) - \sigma_3(N, v) = 1 - \sigma_3(N, v).$$

By **wSC**, $(N', r_\sigma^{N'}(v))$ must be a totally balanced game, which means that $\frac{1}{2} + 0 \leq 1 - \sigma_3(N, v)$ or, equivalently, $\sigma_3(N, v) \leq \frac{1}{2}$. In a similar way, but considering the self reduced games $(\{1, 3\}, r_\sigma^{\{1,3\}}(v))$ and $(\{2, 3\}, r_\sigma^{\{2,3\}}(v))$, it can be checked that $\sigma_2(N, v) \leq \frac{1}{2}$ and $\sigma_1(N, v) \leq 0$. Thus, by *efficiency*, $\sigma(N, v) = (0, \frac{1}{2}, \frac{1}{2})$, which leads to

$$r_\sigma^{\{1,3\}}(v)(1) = \frac{1}{2}, \quad r_\sigma^{\{1,3\}}(v)(3) = 0$$

and

$$r_\sigma^{\{1,3\}}(v)(\{1, 3\}) = 1 - \frac{1}{2} = \frac{1}{2}.$$

By **E** and **DP**, $\sigma(\{1, 3\}, r_\sigma^{\{1,3\}}(v)) = (\frac{1}{2}, 0) \neq \sigma(N, v)_{|\{1,3\}}$, contradicting **wSC**. \square

To finish, let us remark that *weak self consistency*, *weak fairness* and the *null player property* do not characterize the Shapley value on the full domain of games. Indeed, the **zero solution** $\mathbf{z}(N, v) = (0, \dots, 0) \in \mathbb{R}^N$ for all $(N, v) \in \Gamma$ satisfies these three properties. Notice that the single-valued solution \mathbf{z} does not satisfy *efficiency*. If we additionally impose *efficiency*, the Shapley value meets all these properties but uniqueness still remains open.

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