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An egalitarian approach for sharing the cost of a spanning tree

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Abstract

A *minimum cost spanning tree* problem analyzes the way to efficiently connect individuals to a source when they are located at different places; that is, to connect them with the minimum possible cost. This objective requires the cooperation of the involved individuals and, once an efficient network is selected, the question is how to fairly allocate the total cost among these agents. To answer this question the literature proposes several rules providing allocations that, generally, depend on all the possible connection costs, regardless of whether these connections have been used or not in order to build the efficient network. To this regard, our approach defines a simple way to allocate the optimal cost with two main criteria: (1) each individual only pays attention to a few connection costs (the total cost of the optimal network and the cost of connecting by himself to the source); and (2) an egalitarian criteria is used to share costs or benefits. Then, we observe that the spanning tree cost allocation can be turned into a claims problem and, by using claims rules, we define two egalitarian solutions so that the total cost is allocated trying to equalize either the payments in which agents incur, or the benefit that agents obtain throughout cooperation. Finally, by comparing both proposals with other solution concepts proposed in the literature, we select equalizing payments as much as possible and axiomatically analyze it, paying special attention to coalitional stability (core selection), a central property whenever cooperation is needed to carry out the project. As our initial proposal might propose allocations outside the core, we modify it to obtain a core selection and we obtain an alternative interpretation of the *Folk* solution.

Keywords: Minimum cost spanning tree, Egalitarian, Cost sharing, Core
JEL classification: C71, D63, D71.

1. Introduction

We consider a situation in which some individuals, located at different places, want to be connected to a source in order to obtain a good or a service. Each link connecting any two individuals, or connecting each individual to the source, has a specific fixed cost. This situation is known as the *minimum cost spanning tree problem* (hereafter, the *mcst* problem) and it is used to analyze different actual issues, such as telephone, cable TV or water supply networks.

There are several methods for obtaining a way of connecting agents to the source so that *the total cost of the selected network is minimum.* Once the minimum cost network is built, several different solutions have been proposed to allocate the cost among the individuals, such as *Bird rule* (Bird, 1976), *Kar* (Kar, 2002), *Folk* (Feltkamp et al., 1994; Bergantiños and Vidal-Puga, 2007), *Cycle-complete* (Trudeau, 2012), a family of *strict responsive* rules (Bogomolnaia and Moulin, 2010), etc. Some of these solutions take into account the cost of every link in the network, so all the costs are relevant in order to set the final allocation of the optimal cost, although most of them will never be used. Contrary to this trend, we define a model in which only a few costs in the network are considered.

In doing so, our main assumption is that individuals are not worried about costs of links not being used. They have a *local* vision of the network and only a few information is relevant to them. At this point, it is important to remark that cooperation among all individuals is needed to build the efficient network. That is, if some individual does not agree with the allocation of the cost, this individual might connect to the source on his own and the cheapest network is not built.[1]

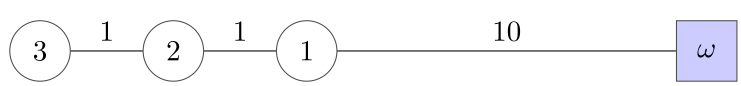
Then, we follow a *reductionist* approach in which some information is not used, or simplified, when obtaining the cost sharing of a *mcst* problem. Specifically, we suppose that each individual is only concerned about two particular costs:

(a) *The cost of the link (or links) used by the individual for efficiently connect to the source by himself.*
(b) *The total cost of the optimal network.*

If there is no cooperation, each agent will connect to the source by himself and the total cost of the network will be (probably) much higher than the one provided by the efficient network (the minimum cost spanning tree).

Our second key point is the use of an *egalitarian* criterion. Taking into account that no one should pay more than the cost of connecting to the source by himself (*individual rationality*), our approach will allocate the cost based on an equal distribution of the benefit of cooperation. This approach leads us to solve the allocation problem by using claims rules. Let us observe some examples that will illustrate our idea.

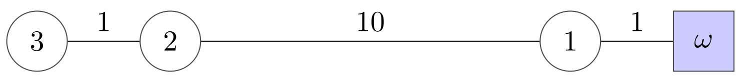
(A) *Consider a set of three houses in a row (at the same distance each one from the other). A water supply ω is located at one end of the row. The cost of each link among the houses is* 1 *monetary unit and the nearest house to the supply may connect directly with a cost of* 10 *units; the second house has a direct cost of* 11 *units; and the cost of directly connecting the farthest house is* 12 *units. The total (minimum) cost of connecting the three houses to the water supply is* 12 *units. If each individual connects directly to the source, the total cost is* 33 *units.*



[1] Actual situations reveal that agents do not necessarily agree on how to distribute this cost, in which case the social optimum is not implemented. Hence, a more expensive network is built (for an example, see Bergantiños and Lorenzo (2004); see also Hernández et al. (2016) for a discussion about individual and social optimality).

In this situation, the cooperation provides a benefit (savings) of 21 monetary units. The question is how to share this benefit. Note that individual 1 is in a "better position" and that he pays less than the other individuals would be a possible option. If the benefit of cooperation is equally shared, each agent obtains a return of 7 unitary units. But an egalitarian cost sharing will propose an allocation of 4 units for each individual.

(B) *Consider a similar situation, but now individual 1 is near to the source, whereas individuals 2 and 3 remain at the same location and the costs vary accordingly (the new situation is as depicted in the following graph):*



As in the previous case, the total (minimum) cost of connecting the three houses to the water supply is 12 units. But the situation is quite different and now cooperation provides a benefit of 12 monetary units. An equal sharing of the benefits originates a negative allocation to individual 1; that is, he obtains a net benefit from his participation in the network. On the other hand, observe that in the new situation an equal allocation of the cost is not admissible for the first individual, since he may connect to the source with a cost of 1 unit, instead of paying 4 units.

As aforementioned, our main criterion in allocating the cost of the optimal tree is that of *egalitarian sharing* of benefits of cooperation (the difference between the cost of connecting each individual by himself and the cost of the optimal spanning tree). Equal sharing of a common cost or benefit is one of the main criterion supported in the literature. In Moulin (1987) the egalitarian methods appear, jointly with the proportional method, as the most important (and simple) ways of sharing a joint cost or benefit.

The rest of the paper is organized as follows. Section 2 presents the formal minimum cost spanning tree problem. In Section 3 we relate minimum cost spanning tree problems with claims problem and introduce our solution concept. Its properties are analyzed in Section 4. Section 5 deals with the stability of the proposed solution. We show that, in general, our proposal may lie outside the core in some situations. When looking for a suitable modification fulfilling core stability, we obtain a new interpretation of the *Folk* solution.

2. Preliminaries: Minimum cost spanning tree problem

A *mcst* problem involves a finite set of individuals, $N = \{1, 2, \ldots, n\}$, who want to be connected to a *source* ω. Let $N_\omega = N \cup \{\omega\}$. The agents are connected by edges and for $i \neq j$, $c_{ij} \in \mathbb{R}_+$ represents the cost of the edge e_{ij} connecting agents $i, j \in N$. Following the notation in Kar (2002), c_{ii} represents the cost of the edge connecting agent $i \in N$ to the source ω. Let $\mathbf{C} = [c_{ij}]_{n \times n}$ the $n \times n$ symmetric cost matrix. The *mcst* problem is represented by the pair $(N_\omega, \mathbf{C})$. We denote by $\mathcal{N}_n$ the set of all *mcst* problems with n individuals.

A *spanning tree* over $(N_\omega, \mathbf{C}) \in \mathcal{N}_n$ is an undirected graph p with no cycles, which connects all elements of N_ω. We can identify a spanning tree with a function $p : N \to N_\omega$ so that $p(i)$ is the agent (or the source) to whom i connects in his path to the source, and defines the edges $e^p_i = (i, p(i))$. In a spanning tree each agent is (directly or indirectly) connected to the source

ω. Moreover, given a spanning tree p, there is a single path from any $i \in N$ to the source ω, given by the edges $(i, p(i)), (p(i), p^2(i)), \ldots, (p^{t-1}(i), p^t(i) = \omega)$, for some integer $t < n$. Let us denote by $\mathcal{S}(N_\omega)$ the set of all spanning trees in the problem $(N_\omega, \mathbf{C})$. The cost of building a spanning tree $p \in \mathcal{S}(N_\omega)$ is the sum of the costs of all the edges in this tree; that is,[2]

$$C_p = \sum_{i=1}^{n} c_{ip(i)} = \sum_{i=1}^{n} c\left(e_i^p\right)$$

Given a spanning tree $p \in \mathcal{S}(N_\omega)$, we denote by $p(i,j)$ the set of edges in the (unique) path in p joining i and j.

Prim (1957) provides an algorithm that solves the problem of connecting all the agents to the source at the *minimum cost*.[3] We denote by m a tree with minimum cost and by C_m its cost. That is, for all spanning tree p,

$$C_m = \sum_{i=1}^{n} c_{im(i)} \leqslant C_p = \sum_{i=1}^{n} c_{ip(i)}.$$

Once a network is built, an important issue is how to allocate the associated cost among the agents. A *cost sharing rule* for *mcst* problems is a function $\alpha : \mathcal{N}_n \to \mathbb{R}^n$ that proposes for any *mcst* problem $(N_\omega, \mathbf{C})$ an allocation $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$, such that

$$\sum_{i=1}^{n} \alpha_i = C_m.$$

Remark 1. *In some contexts the non-negativity of the cost α_i allocated to each individual is required. This question is related to the assumption of property or non-property rights on the locations that individuals occupy (see, for instance, Bogomolnaia and Moulin (2010) for a discussion). In the second case, non-property rights approach, the allocations must be necessarily non-negative.*

In what follows we will consider the *non-property* approach, so the allocations will be required to be non-negative.

Bird (1976) proposes a cost allocation so that each individual pays the cost of the edge he directly uses to be connected in the minimum cost spanning tree. In case there are several networks providing the (same) minimum cost, the *Bird* solution allocates to each individual the average of the cost of the connections he uses in these networks. Since then, several authors have proposed other solution concepts in the *mcst* literature: for instance, Kar (2002), Dutta and Kar (2004), Feltkamp et al. (1994); Bergantiños and Vidal-Puga (2007), Bogomolnaia and Moulin (2010), Trudeau (2012), etc.[4]

[2] With some abuse of notation, when $p(i) = \omega$, $c_{ip(i)} = c_{ii}$.

[3] This algorithm has n steps. First, we select the agent i with the lowest connection cost to the source. In the second step, we select an agent in $N \backslash \{i\}$ with the smallest cost, either directly to the source or to agent i, who is already connected. We continue by this way until all agents are connected, i.e., at each step, connecting an agent who is still not connected to one who already is, or directly to the source.

[4] See Bergantiños and Vidal-Puga (2008) for definitions and a comparative analysis of most of these solutions.

Some of these solutions take all the possible connections in the graph into account (all c_{ij} costs), although most of these connections are not used in the optimal tree. Nevertheless, other solutions use a *reductionist approach* and they are obtained by only considering some of the connection costs. Specifically, Bird's solution only considers the cost of the link each individual uses in the optimal network while the costs of other edges are ignored. The *Folk* and *Cycle-complete* solutions also take a reductionist approach. As the reductionist approach ignores some of the available information it reduces the parameters of the problem (and the complexity in computation).

3. Egalitarian cost sharing

In order to define the cost that any individual will have without cooperation, we introduce the notion of *indirect* cost.

Definition 1. *Given a mcst problem $(N_\omega, \mathbf{C})$, and $i, j \in N$, the* ***indirect*** *cost to connect individuals i and j is*

$$c^*_{ij} = \min_{p \in \mathcal{S}(N_\omega)} \left\{ \sum c(e) \quad e \in p(i,j) \right\}$$

In particular, c^*_{ii} denotes the indirect cost of connecting individual i to the source.

In this context, demanding that the maximum amount to be allocated to any individual cannot exceed his (indirect) cost of connecting to the source (*individual rationality*) is a compulsory requirement since, in other case, the individual would be better off acting by himself and does not cooperating in building the optimal network.

Definition 2. *A cost allocation $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of the minimum cost C_m in a mcst problem $(N_\omega, \mathbf{C})$, is* ***individually rational*** *if for all $i \in N$, $\alpha_i \leqslant c^*_{ii}$.*

We propose an egalitarian treatment of the agents. So, a first attempt to allocate the cost of the optimal network, C_m is to divide it equally among the individuals:

$$\alpha_i^{E_{cost}} = \frac{1}{n} C_m \quad i = 1, 2, \ldots, n.$$

However, as Example (B) depicts, the equal division may be not individually rational: agent 1 is allocated 4 monetary units, whereas his individual cost to the source is $c^*_{11} = 1$ monetary unit. Then, equal cost division is not admissible since agent 1 will directly connect to the source, without taking into account the other individuals, and then the best option for agents 2 and 3 is to build a network with a cost of 12 monetary units, that yields to a non-optimal solution of the *mcst* problem.

*3.1. **Some considerations about individual rationality***

Individual rationality provides a way to address the problem of allocating the optimal cost in a *mcst* problem by transforming it into a surplus sharing problem:

- First, each agent pays his (indirect) cost to connect the source c^*_{ii}. So, the individuals jointly contribute with the amount $C^* = \sum_{i=1}^{n} c^*_{ii}$ to build a network.

- Then, the efficient tree may be built, with cooperation, at a cost $C_m \leqslant C^*$ and there is a benefit from cooperation given by $B = C^* - C_m$.
- Any method used to share this benefit B, $\sum_{i=1}^{n} x_i = B$, $x_i \geqslant 0$, provides a *final allocation* of the cost C_m, $\alpha_i = c^*_{ii} - x_i$, which is individually rational.

For instance, one way is to share equally the benefits obtained from cooperation:

$$\alpha_i^{Eq} = c^*_{ii} - \frac{C^* - C_m}{n} \quad i = 1, 2, \ldots, n.$$

However, as Example (B) shows, equalizing benefits makes individual 1 to end in a negative allocation, that implies that this agent gets a net profit from participating in the network. This also happens with the solution proposed by Kar (2002), or with *Cycle-complete* solution (Trudeau, 2012). As mentioned in Bogomolnaia and Moulin (2010) this possibility only has sense if the individuals have property rights on their location.

It is noteworthy that if we want to avoid this possibility (since we are in the non-property rights approach) a *constrained equal division* should be considered: *no one obtains a benefit greater than his initial contribution.* Then the benefit of cooperation should be shared as in a claims problem in which each individual *claims all his contribution to be returned.*

3.2. ***A formal claims problem approach***

A *claims problem*, which originates in the seminal paper by (O'Neill, 1982), is a situation involving n individuals who claim some amount d_i, so that the aggregate demand exceeds the available endowment B, $\sum_{i=1}^{n} d_i \geqslant B$. A *claims rule* φ divides efficiently the endowment so that no agent receives a negative amount, nor more than his claim

$$0 \leqslant \varphi_i \leqslant d_i, \quad i = 1, 2, \ldots, n, \quad \sum_{i=1}^{n} \varphi_i = B$$

See Thomson (2003) for a survey on claims (*bankruptcy*) problems.

We follow a previous work (Giménez-Gómez et al., 2014), where two claims problems (pessimistic and optimistic) are associated to a *mcst* problem. In that work, each individual's claim is defined as *the difference between his indirect connection to the source and his cheapest connection cost.* Nonetheless, now, in order to define the claim, we are only considering the indirect connection to the source (the amount initially paid).

Definition 3. *Given a mcst problem $(N_\omega, \mathbf{C})$ with minimum cost C_m, let $C^* = \sum_{i=1}^{n} c^*_{ii}$ and $B = C^* - C_m$. For any vector $d = (d_1, d_2, \ldots, d_n) \in \mathbb{R}^n_+$, where d_i represents the amount individual i claims to be returned from his initial contribution c^*_{ii}, $0 \leqslant d_i \leqslant c^*_{ii}$, $\sum_{i=1}^{n} d_i \geqslant B$, the pair (B, d) defines a claims problem.*

If we use a claims rule φ to solve the problem (B, d), then $\alpha_i = c^*_{ii} - \varphi_i$ is an allocation of the cost C_m of the optimal tree. In this case, the non-negativity of the claims rule, $\varphi_i \geqslant 0$, makes the allocations $\alpha_i = c^*_{ii} - \varphi_i$ individually rational, whereas the claim boundedness, $\varphi_i \leqslant d_i$, implies that the allocations $\alpha_i = c^*_{ii} - \varphi_i$ are non-negative, which is coherent with the non-property rights approach.

In the literature on *claims problems* the constrained egalitarian solutions (agents are treated as equal as possible, subject to some restrictions) appear in several ways. Among all of them, Maimonides (1135–1204) introduces the two main egalitarian claims rules: the *Constrained Equal Awards* and the *Constrained Equal Losses*, that equalize, respectively, gains and losses satisfying the restrictions of a claims rule. Formally, given a claims problem $(B, d) \in \mathbb{R}_+ \times \mathbb{R}^n_+$

$$CEA_i(B,d) = \min\{\lambda, d_i\} \qquad \lambda \text{ such that } \sum_{i \in N} CEA_i(B,d) = B$$

$$CEL_i(B,d) = \max\{-\lambda + d_i, 0\} \quad \lambda \text{ such that } \sum_{i \in N} CEL_i(B,d) = B$$

Note that these rules are dual of each other (see Aumann and Maschler (1985)), in the sense that CEL allocates losses (the non received claim) in the same way as CEA allocates awards. Formally, duality establishes that:

$$CEA_i(B,d) = d_i - CEL_i(D - B, d) \qquad CEL_i(B,d) = d_i - CEA_i(D - B, d)$$

where D denotes the aggregate claim, $D = \sum_{i=1}^{n} d_i$.

We are assuming the following statements in order to define the set of possible claims rules, as well as the claim of any individual.

Assumption 1. *Each agent's claim is his indirect cost of connecting the source,* $d_i = c^*_{ii}$.

Assumption 2. *Costs are allocated under egalitarian criteria.*

Note that Assumption 1 implies selecting the largest possible claim (according with the non-property approach) for any individual and in this case the conflicting situation obviously appears: the benefit of cooperation is not enough to satisfy the aggregate claim that, in this case, coincides with C^*: $D = C^* > C^* - C_m$. As we want to use an egalitarian criteria (Assumption 2), we initially propose to use the CEA and CEL claims rules to allocate the benefits of cooperation, once Assumption 1 has determined the amount each agent wants to be returned (his claim).

Definition 4. *Given a mcst problem* $(N_\omega, \mathbf{C})$ *such that the cost of the optimal spanning tree is* C_m, $B = C^* - C_m$, $d_i = c^*_{ii}$

a) *the **constrained equal benefits** sharing rule assigns to each individual* $i \in N$ *the amount*

$$\alpha_i^{cea}(N_\omega, \mathbf{C}) = c^*_{ii} - CEA_i(B,d)$$

b) *the **constrained equal costs** sharing rule assigns to each individual $i \in N$ the amount*

$$\alpha_i^{cel}(N_\omega, \mathbf{C}) = c_{ii}^* - CEL_i(B, d)$$

Next example computes these solutions in the *mcst* problems introduced in Section 1 and compares the results with *Bird* and *Folk* proposals.

Example 1. *For the problems (A) and (B), introduced in Section 1, the following cost shares are obtained:*

	α^{cea}			α^{cel}			*Bird*			*Folk*		
Agents / **Problem**	**1**	**2**	**3**	**1**	**2**	**3**	**1**	**2**	**3**	**1**	**2**	**3**
(A)	3	4	5	4	4	4	10	1	1	4	4	4
(B)	0	$\frac{11}{2}$	$\frac{13}{2}$	1	$\frac{11}{2}$	$\frac{11}{2}$	1	10	1	1	$\frac{11}{2}$	$\frac{11}{2}$

Note that the Folk and α^{cel} proposals coincide and that this proposal is the one that most equally distributes the cost of the optimal network.

The following result provides an alternative interpretation of the α^{cea} and α^{cel} sharing rules.

Proposition 1. *For any mcst problem $(N_\omega, \mathbf{C})$ and any $i \in N$,*

$$\alpha_i^{cea}(N_\omega, \mathbf{C}) = CEL_i(C_m, d) \qquad \alpha_i^{cel}(N_\omega, \mathbf{C}) = CEA_i(C_m, d)$$

Proof. It comes immediately from the duality relations between CEA and CEL rules and by noticing that the aggregate claim is $D = C^*$, and then $D - B = C_m$. ■

The above result shows that under the *constrained equal benefits* sharing rule all agents obtain the same savings respect to their initial contribution (the cost of the indirect connection to the source), constrained to no one is allocated a negative amount. In the same way, under the *constrained equal costs* sharing rule all agents pay the same amount, constrained to no one is allocated an amount greater than his indirect cost to the source. So, the solution that better captures the egalitarian criteria (all agents paying the same, if possible) is α^{cel} and that is the one we propose to obtain an egalitarian sharing of the cost of an optimal tree. The following section analyzes its properties.

4. Axiomatic analysis

Taking the *Folk* solution as a benchmark, we analyze if the properties fulfilled by it are satisfied or not by our egalitarian proposal α_i^{cel}. In doing so, next we formally defined them, See Bergantiños and Vidal-Puga (2007) and Bogomolnaia and Moulin (2010) for relationships and interpretations of these properties.

- A solution α for *mcstp* satisfies **positivity (Pos)** if $\alpha(N_\omega, C) \geqslant 0$, for any problem (N_ω, C).
- A solution α for *mcst* problems satisfies **continuity (Cont)** if it is a continuous function of $\mathbf{C}$.
- A solution α for *mcstp* satisfies **symmetry (Sym)** if for any problem (N_ω, C) such that there are $i, j \in N$ with $c_{ik} = c_{jk}$, for all $k \in N$, then $\alpha_i(N_\omega, C) = \alpha_j(N_\omega, C)$.
- A solution α for *mcst* problems satisfies **strong cost monotonicity (StCMon)** if for any pair of problems $(N_\omega, \mathbf{C})$, $(N_\omega, \mathbf{C}')$ such that $\mathbf{C} \leqslant \mathbf{C}'$, then $\alpha_i(N_\omega, \mathbf{C}) \leqslant \alpha_i(N_\omega, \mathbf{C}')$ for all $i \in N$.
- A solution α for *mcst* problems satisfies **cost monotonicity (CMon)** if for any pair of problems $(N_\omega, \mathbf{C})$, $(N_\omega, \mathbf{C}')$ such that $\mathbf{C}$ and $\mathbf{C}'$ coincide except that $c_{ik} < c'_{ik}$, for some $i, k \in N$, then $\alpha_i(N_\omega, \mathbf{C}) \leqslant \alpha_i(N_\omega, \mathbf{C}')$.
- A solution α for *mcst* problems satisfies **population monotonicity (PMon)** if for any problem $(N_\omega, \mathbf{C})$, any subset $S \subseteq N$ and any $i \in S$,

$$\alpha_i(S_\omega, \mathbf{C}|_S) \geqslant \alpha_i(N_\omega, \mathbf{C}).$$

- A solution α for *mcst* problems satisfies **ranking (Rkg)** if for any problem $(N_\omega, \mathbf{C})$ and $i, j \in N$ such that $c_{ik} \leqslant c_{jk}$ for all $k \in N$, then $\alpha_i(N_\omega, \mathbf{C}) \leqslant \alpha_i(N_\omega, \mathbf{C})$.
- A *mcst* problem (N_ω, C) is **separable** if there are two disjoint subsets $S \cup T = N$, $S \cap T = \varnothing$, such that the *mcst* in N are union of *mcst* in each of the sub-problems, $m(N_\omega, \mathbf{C}) = m^1(S_\omega, \mathbf{C}|_S) \cup m^2(T_\omega, \mathbf{C}|_T)$.

  A solution α for *mcst* problems satisfies **separability (Sep)** if for any separable problem $(N_\omega, \mathbf{C})$, $N = S \cup T$, $S \cap T = \varnothing$,

$$\alpha_i(N_\omega, \mathbf{C}) = \begin{cases} \alpha_i(S_\omega, \mathbf{C}|_S) & \text{if} \quad i \in S \\ \alpha_i(T_\omega, \mathbf{C}|_T) & \text{if} \quad i \in T \end{cases}$$

- A solution α for *mcst* problems satisfies **equal share of extra-costs (EqEx)** if for any pair of problems $(N_\omega, \mathbf{C})$, $(N_\omega, \mathbf{C}')$ such that:
  a) for all $i \in N$, $c_{ii} = c_0$, $c'_{ii} = c'_0$, $c_0 < c'_0$.
  b) for all $i, j \in N$, $i \neq j$, $c'_{ij} = c_{ij} \leqslant c_0$

  then

$$\alpha_i(N_\omega, \mathbf{C}') = \alpha_i(N_\omega, \mathbf{C}) + \frac{c'_0 - c_0}{n} \qquad \forall i \in N$$

- Two *mcst* problems $(N_\omega, \mathbf{C})$ and $(N_\omega, \mathbf{C}')$ are **tree equivalent** if there is tree m such that it is a minimum cost spanning tree for both problems, and moreover $c_{im(i)} = c'_{im(i)}$ for all $i \in N$.

  A solution α for *mcst* problems satisfies **independence of irrelevant trees (IIT)** if for any pair of tree equivalent problems $(N_\omega, \mathbf{C})$ and $(N_\omega, \mathbf{C}')$, then

$$\alpha_i(N_\omega, \mathbf{C}) = \alpha_i(N_\omega, \mathbf{C}'), \quad \forall i \in N$$

Proposition 2. *The α^{cel} solution fulfills positivity, continuity, cost monotonicity, strong cost monotonicity, independence of irrelevant trees, ranking, symmetry and equal share of extra-costs. It does not fulfill population monotonicity, nor separability.*

Proof.
(1) *Positivity* is immediately fulfilled, since the CEL rule satisfies claims boundedness and the maximum amount that can be returned is what each individual has paid.

(2) We know that C_m varies continuously with $\mathbf{C}$ (Bergantiños and Vidal-Puga, 2007), and the indirect costs, $d_i = c^*_{ii}$ are obviously a continuous function of the cost matrix (see Definition 1). On the other hand, CEA is a continuous function on its arguments, which proves that α^{cel} is *continuous.*

(3) We prove the *strong cost monotonicity* property. Since it implies *cost monotonicity* and *independence of irrelevant trees* (Bergantiños and Vidal-Puga, 2007), all three properties are fulfilled.

Consider a pair of problems $(N_\omega, \mathbf{C})$, $(N_\omega, \mathbf{C}')$ such that $\mathbf{C} \leqslant \mathbf{C}'$. Then, $C_m \leqslant C'_m$ and $d_i = c^*_{ii} \leqslant d'_i = (c')^*_{ii}$, for all $i \in N$. Since the claims rule CEA fulfills endowment monotonicity and claims monotonicity (Thomson, 2003), then

$$CEA_i(C_m, d) \leqslant CEA_i(C'_m, d') \text{ that implies } \alpha_i^{cel}(N_\omega, \mathbf{C}) \leqslant \alpha_i^{cel}(N_\omega, \mathbf{C}') \quad \forall i \in N.$$

(4) Given a problem $(N_\omega, \mathbf{C})$ and $i, j \in N$ such that $c_{ik} \leqslant c_{jk}$ for all $k \in N$, then it is obvious that $d_i = c^*_{ii} \leqslant d_j = c^*_{jj}$. Therefore, $CEA_i(C_m, d) \leqslant CEA_j(C_m, d)$ (order preservation, see Thomson (2003)) and α^{cel} fulfills *ranking.*

(5) Immediate, since *ranking* implies *symmetry.*

(6) If we consider two problems $(N_\omega, \mathbf{C})$, $(N_\omega, \mathbf{C}')$ such that:

a) for all $i \in N$, $c_{ii} = c_0$, $c'_{ii} = c'_0$, $c_0 < c'_0$.
b) for all $i, j \in N$, $i \neq j$, $c'_{ij} = c_{ij} \leqslant c_0$

then, $C'_m = C_m + (c'_0 - c_0)$. On the other hand, $d'_i = c'_0$ and $d_i = c_0$, for all $i \in N$. So, as all claims are identical for all the individuals, the CEL rule allocates the same amount B/n, B'/n to each individual and

$$\alpha_i^{cel}(N_\omega, \mathbf{C}') = \frac{C'_m}{n} = \frac{C_m}{n} + \frac{c'_0 - c_0}{n} = \alpha_i^{cel}(N_\omega, \mathbf{C}) + \frac{c'_0 - c_0}{n} \quad \forall i \in N$$

which proves that this solution fulfills *equal share of extra-costs.*

(7) We know (Bergantiños and Vidal-Puga, 2007) that *population monotonicity* implies *core selection.* Example 2 shows that α^{cel} may provide allocations outside the core and then our proposal does not fulfill *population monotonicity.*[5]

(8) Example 2 shows that α^{cel} does not fulfill *separability.* ■

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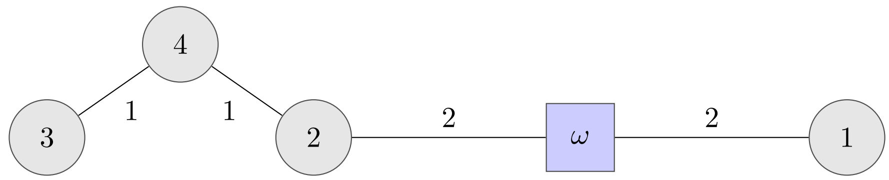
[5] See Section 4.1.

Example 2. *Let us consider the mcst problem defined by the following picture (arcs not depicted have a cost $c_{ij} = 2$):*



There are several spanning trees with minimum cost $C_m = 6$. One of them is given by:

$$m(1) = \omega \quad m(2) = \omega \quad m(3) = 4; \quad m(4) = 2. \qquad .$$

If we denote $S = \{1\}$, $T = \{2,3,4\}$, $m = m^1 \cup m^2$, where m^1 and m^2 are the minimum cost spanning trees in problems $(S, \mathbf{C})$ and $(T, \mathbf{C})$, respectively. On the other hand, $c^*_{ii} = 2$, for all $i \in N$, and then $B = 2$ and $\alpha^{cel} = (3/2, 3/2, 3/2, 3/2)$. We observe that the separability property implies $\alpha_1 = 2$, so α^{cel} does not fulfill this property.

4.1. ***Coalitional stability***

Whenever cooperation is necessary, as in *mcst* situations, the literature on cost sharing singles out stand alone core stability as the key property of any allocation rule: in order the agents want to participate no coalition of agents should be charged more than their cost of connecting to the source. Then, given a coalition $S \subseteq N$, the *stand alone* cost for this coalition to be connected to the source is (in our non-property rights model)

$$v(S) = \min \{C_m(T) : \quad S \subseteq T \subseteq N\}$$

where $C_m(T)$ denotes the cost of the optimal tree connecting coalition T to the source. Note that for any $i \in N$, $v(\{i\}) = c^*_{ii}$. Now we can define our next axiom requiring core stability.[6]

Definition 5. *A cost allocation $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of the minimum cost C_m in a mcst problem $(N_\omega, \mathbf{C})$ is a* ***core selection*** *if for all $\varnothing \neq S \subseteq N$, $\sum_{i \in S} \alpha_i \leqslant v(S)$.*

Example 2 shows that α^{cel} does not fulfill *core selection*, since it allocates a total amount of 9/2 monetary units to agents in $T = \{2,3,4\}$, strictly greater than their cost of connecting the source, $v(T) = 4$.

4.2. ***Axiomatic summary***

Next, Table 1 provides an axiomatic comparison of *Folk* and α^{cel} solutions (see Bergantiños and Vidal-Puga (2007) for *Folk* results).

[6] It is important to remark that the cost function $C_m(S)$, $S \subseteq N$, is not monotonic since the addition of some agents may reduce the cost of the coalition. As we follow the non-property rights approach, any coalition S might use locations of individuals outside S to build their minimum cost spanning tree. So, $v(S)$ represents the minimum cost of connecting all individuals in S to the source ω, possibly using (and paying for) connections of individuals outside S.

	Pos	Cont	Sym	StCMon	CMon	PMon	Rkg	Sep	EqEx	IIT	CS
Folk	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
α^{cel}	✓	✓	✓	✓	✓	×	✓	×	✓	✓	×

Table 1: **Axiomatic analysis.** The table shows which axioms are satisfied by the α^{cel} solutions in comparison with the Folk solution. Each column corresponds with an axiom, whereas each row corresponds with the introduced solutions.

It is clear that *Folk* solution fulfills better properties than α^{cel}. On the other hand, our egalitarian solution is easier to compute and to understand. Moreover, it captures a *solidarity* approach in which the optimal cost (obtained throughout cooperation) is paid as equally as possible among the agents in the project.

The main objection to the α^{cel} solution is that it fails to be a core selection. Then, subsets of agents may have incentives to leave the grand coalition and perform their project by themselves. We analyze this question in the following section.

5. A core egalitarian proposal

In this section we discuss core stability of our egalitarian solution. First we show some classes of *mcst* problems in which α^{cel} is a core selection. Later, we will propose a modification of our solution (maintaining the egalitarian criteria) in order to achieve core selection for all *mcst* situations.

*5.1. **Some mcst problems where α^{cel} is a core-selection***

In some families of *mcst* problems our egalitarian solution always provides core allocations. Let us see two examples of such a kind of families:

1) Let us consider the so-called $2-$*mcst* problems in which the connection cost between two different individuals (houses, villages, ...) can only take one of two possible values (low and high cost).[7]

   Moreover, we assume that $c_{ij} = k_1$, $i \neq j$, $c_{ii} = k_2$, and $0 \leqslant k_1 \leqslant k_2$. Note that, for these family of *mcst* problems the property *equal share of extra-costs* could be applied.

   In this case, if there are n individuals, then $C_m = k_2 + (n-1)k_1$ and $d_i = c^*_{ii} = k_2$. As all claims are identical, then

$$\alpha_i^{cel} = \frac{k_2}{n} + \frac{n-1}{n}k_1 \qquad i = 1, 2, \ldots, n$$

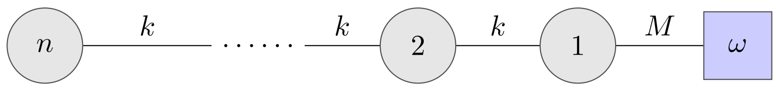
   and, for all $S \subseteq N$, $v(S) = k_2 + k_1\left(|S|-1\right)$. Then, since $k_1 \leqslant k_2$

$$\sum_{i\in S}\alpha_i^{cel} = \sum_{i\in S}\left(\frac{k_2}{n} + \frac{n-1}{n}k_1\right) = |S|\left(\frac{k_2}{n} + \frac{n-1}{n}k_1\right) \leqslant v(S)$$

[7] See, for instance, Estévez-Fernández and Reijnierse (2014); see also Subiza et al. (2016) where this class has been generalized to the so-called *simple mcst* problems.

So the allocation provided by α^{cel} belongs to the core of the cooperative game.

2) Let us consider *linear mcst* problems: a group of individuals $N = \{1, 2, \ldots, n\}$ situated in a row (equally separated) want to connect to a source ω. The cost of connecting one individual with the next one is k monetary units. The cost of connecting individual 1 to the source is M monetary units. If an individual wants to connect to the source, he must do it through all its neighbors on the way towards the source and pay all costs. This is the case of Example **(A)** in Section 1.



Formally, for each $i, j \in N$, $i \neq j$, the connection cost is $c_{ij} = |i - j|k$. For each $i \in N$, the cost to the source is $c_{ii} = M + (i-1)k$.

The *minimum cost spanning tree* m connects each individual to the next, $m(j) = j - 1$, $j \geqslant 2$, and the first one with the source, $m(1) = \omega$, with a total cost $C_m = M + (n-1)k$. For each $i \in N$, $d_i = c^*_{ii} = M + (i-1)k$ and then

$$B = C^* - C_m = (n-1)\left(M + \left(\frac{n}{2} - 1\right)k\right)$$

For any coalition $S \subseteq N$, $v(S) = M + k \max\{i - 1,\ i \in S\}$. To obtain the allocation provided by α^{cel} we distinguish two cases:

a) If $M \geqslant k$, $\alpha_i^{cel} = \dfrac{M}{n} + \dfrac{n-1}{n}k$, for all $i \in N$. Then, for all $S \subseteq N$

$$\sum_{i \in S} \alpha_i^{cel} = \sum_{i \in S}\left(\frac{M}{n} + \frac{n-1}{n}k\right) = |S|\left(\frac{M}{n} + \frac{n-1}{n}k\right) \leqslant$$

$$\leqslant M + (|S| - 1)k \leqslant M + k \max\{i - 1,\ i \in S\} = v(S).$$

b) If $M < k$, $\alpha_1^{cel} = M$, $\alpha_i^{cel} = k$, for all $k \geqslant 2$. Then, for all $S \subseteq N$

i. If $1 \in S$,

$$\sum_{i \in S} \alpha_i^{cel} = M + (|S| - 1)k \leqslant M + k \max\{i - 1,\ i \in S\} = v(S).$$

ii. If $1 \notin S$, $\max\{i - 1,\ i \in S\} \geqslant |S|$ and

$$\sum_{i \in S} \alpha_i^{cel} = |S|k \leqslant M + k \max\{i - 1,\ i \in S\} = v(S).$$

So the allocation provided by α^{cel} belongs to the core of the cooperative game.

5.2. ***A core-egalitarian proposal***

In order to look for a core selection respecting the egalitarian criteria, it would be better to rewrite the α^{cel} solution. Let us denote by $\mathcal{A}$ the set of non-negative individually rational allocations in a *mcst* problem $(N_\omega, \mathbf{C})$

$$\mathcal{A}(N_\omega, \mathbf{C}) = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = C_m \quad 0 \leqslant x_i \leqslant c_{ii}^* \quad i = 1, 2, \ldots, n \right\}$$

then the α^{cel} solution can be expressed as (see, for instance, Vicente (2019) to obtain an expression of the CEA claims rule as a minimization problem; see also Theorem 3, in Thomson (2003)):

$$\alpha^{cel}(N_\omega, \mathbf{C}) = \arg\min \left\{ \sum_{i=1}^{n} \left(x_i - \frac{C_m}{n} \right)^2 \quad x \in \mathcal{A}(N_\omega, \mathbf{C}) \right\}$$

Note that, as the distance function is a continuous and strictly convex function, and the feasible set $\mathcal{A}$ is a compact and convex set, the minimization problem has always a unique solution. This expression provides us a way of obtaining a core allocation that tries to meet our egalitarian criteria, by defining the solution:

$$\beta^{cel}(N_\omega, \mathbf{C}) = \arg\min \left\{ \sum_{i=1}^{n} \left(x_i - \frac{C_m}{n} \right)^2 \quad x \in co(v(N_\omega, \mathbf{C})) \right\}$$

Obviously, this proposal is the most egalitarian core allocation.

On the other hand, as $co(v(N_\omega, \mathbf{C})) \subseteq \mathcal{A}(N_\omega, \mathbf{C})$, we obtain the following result:

$$\text{If } \alpha^{cel}(N_\omega, \mathbf{C}) \in co(v(N_\omega, \mathbf{C})) \text{ then } \beta^{cel}(N_\omega, \mathbf{C}) = \alpha^{cel}(N_\omega, \mathbf{C})$$

In Example 2, $\alpha^{cel}(N_\omega, \mathbf{C}) = (3/2, 3/2, 3/2, 3/2) \notin co(v(N_\omega, \mathbf{C}))$. If we compute the β^{cel} solution, we obtain

$$\beta^{cel}(N_\omega, \mathbf{C}) = (2, 4/3, 4/3, 4/3)$$

that coincides with the *Folk* solution in this example. Example 2 also shows that in general $2-$*mcst* problems (there are only two possible costs: high and low) solution α^{cel} may provide allocations outside the core. The following result analyzes the result of applying our core-egalitarian solution, β^{cel}, to this kind of problems.

Proposition 3. *For any* $2-$*mcst problem* $(N_\omega, \mathbf{C})$,

$$\beta^{cel}(N_\omega, \mathbf{C}) = F(N_\omega, \mathbf{C})$$

where F *stands for the Folk solution.*

Proof. From Subiza et al. (2016) we know that in this class of problems there is a partition of the set of agents, such that in each subset the *Folk* solution proposes the same allocation to all individuals in this group (*simple* components). To simplify the proof we suppose that there are just two components in the $2-$*mcst* problem $(N_\omega, \mathbf{C})$:[8]

$$N = N_1 \cup N_2,\ N_1 \cap N_2 = \varnothing, \quad F_i = c_1,\ \forall i \in N_1, \quad F_j = c_2,\ \forall j \in N_2$$

[8] For more than two components, the reasoning follows an analogous argument.

Let us denote by n_i the cardinality of the subset N_i, $i = 1, 2$, $n = n_1 + n_2$ and, as usually, C_m is the cost of the optimal tree. Then,

$$n_1 c_1 + n_2 c_2 = C_m, \qquad n_1 c_1 = v(N_1),\ n_2 c_2 = v(N_2) \qquad v(N_1) + v(N_2) = v(N)$$

by applying separability. Moreover, we know that this allocation is in the core of the monotonic cooperative game.

On the other hand, after reordering the agents, we can write

$$\beta^{cel}\left(N_\omega, \mathbf{C}\right) = (x, y) \qquad x = (x_1, x_2, \ldots, x_{n_1}) \qquad y = (y_1, y_2, \ldots, y_{n_2})$$

such that (x, y) minimizes

$$\sum_{i=1}^{n_1} \left(x_i - \frac{C_m}{n}\right)^2 + \sum_{j=1}^{n_2} \left(y_j - \frac{C_m}{n}\right)^2 \qquad (x, y) \in co\left(v\left(N_\omega, \mathbf{C}\right)\right)$$

Then, by noticing that

$$\sum_{i=1}^{n_1} x_i = v(N_1) \qquad \sum_{j=1}^{n_2} y_j = v(N_2)$$

the function minimizes when all components x_i are identical for all $i \in N_1$, and y_j are identical for all $j \in N_2$. Then,

$$x_i = \frac{v(N_1)}{n_1} = c_1 = F_i \qquad y_j = \frac{v(N_2)}{n_2} = c_2 = F_j$$

and both solutions coincide. ■

5.3. ***A piece-wise linear extension***

If we denote by $\mathcal{C}_n$ the set of all cost matrices involving n individuals, $\mathbf{C}_{n\times n}$, and by $\mathcal{C}_n^b$ the set of *elementary cost matrices*, i.e., $c_{ij} \in \{0, 1\}$ for all $i, j = 1, 2, \ldots, n$, we know (see Bogomolnaia and Moulin (2010)) that there is a basis

$$\mathbf{C}^1,\ \mathbf{C}^2,\ \ldots,\ \mathbf{C}^p \in \mathcal{C}_n^b \qquad p = \frac{n(n+1)}{2}$$

such that any cost matrix $\mathbf{C} \in \mathcal{C}_n$ can be expressed as

$$\mathbf{C} = \sum_{k=1}^{p} \lambda_k\, \mathbf{C}^k \qquad \lambda_k \in \mathbb{R}$$

Then, given a *mcst* solution ψ^b defined only for elementary cost matrices (a *partial* solution), its *piece-wise linear extension* ψ is defined by

$$\psi\left(N_\omega, \mathbf{C}\right) = \sum_{k=1}^{p} \lambda_k\, \psi^b\left(N_\omega, \mathbf{C}^k\right)$$

As proved in Bogomolnaia and Moulin (2010), piece-wise linear solutions have the advantage that many normative properties automatically extend from elementary to arbitrary cost matrices. In particular, they show that this is the case with the properties of *ranking*, *cost monotonicity*, *polynomial complexity*, *population monotonicity* and *positivity*. Then, we can define the piece-wise linear extension of the β^{cel} solution defined only on elementary problems:

$$\Upsilon\left(N_\omega, \mathbf{C}\right) = \sum_{k=1}^{p} \lambda_k \, \beta^{cel}\left(N_\omega, \mathbf{C}^k\right)$$

As an immediate consequence of Proposition 3 (elementary problems are a particular case of $2-mcst$ problems) we obtain that this extension coincides with the *Folk* solution.

Corollary 1. *For any mcst problem* $(N_\omega, \mathbf{C})$*,* $\Upsilon\left(N_\omega, \mathbf{C}\right) = F\left(N_\omega, \mathbf{C}\right)$*.*

Then, the *Folk* solution appears as the piece-wise linear extension of a solution, β^{cel}, that picks the most egalitarian allocation in the core associated to the *mcst* problem, giving an alternative interpretation to this solution.

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