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Nonempty core of minimum cost spanning tree games with revenues

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Abstract

A *minimum cost spanning tree* problem analyzes the way to efficiently connect agents to a source when they are located at different places. Once the efficient tree is obtained, the total cost should be allocated among the involved agents in a fair and stable manner. It is well known that there always exist allocations in the core of the cooperative game associated to the minimum cost spanning tree problem (Bird, 1976; Granot and Huberman, 1981). Estévez-Fernández and Reijnierse (2014) investigate minimum cost spanning tree problems with revenues and show that the *cost-revenue game* may have empty core. They provide a sufficient condition to ensure the non-emptiness of the r-core for *elementary cost* problems; that is, minimum cost spanning tree problems in which every connection cost can take only two values (low or high cost). We show that this condition is not necessary and obtain a family of cost-revenue games (*simple* problems, Subiza et al. (2016)) in which the non-emptiness of the r-core is ensured.

Keywords: Minimum cost spanning tree problem, Elementary cost problem, Simple minimum cost spanning tree problem, Cost-revenue game, Core.

JEL classification: C71, D63, D71.

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1. Introduction

We consider a situation in which some individuals, located at different places, want to be connected to a source in order to obtain a good or service.

Each link joining two individuals, or joining any individual to the source, has a specific fixed cost. Moreover, individuals do not mind being connected directly to the source, or indirectly through other individuals. There are several methods to obtain a way of connecting agents to the source so that the total cost of the selected network is minimum. This situation is known as the *minimum cost spanning tree problem* (hereafter *mcst* problem) and it is used to analyze different real-life issues, from telephone and cable TV to water supply networks. Once the optimal tree is obtained, an important question is how this minimum cost should be allocated among the individuals and many proposals have been defined in the literature (see, for instance, Bergantiños and Vidal-Puga (2008) and Bogomolnaia and Moulin (2010) for definitions and a comparative analysis).

Estévez-Fernández and Reijnierse (2014) analyze the problem arising from a general service by explicitly taking into account the revenues generated by this service. The question is how to share the net revenues obtained from cooperation among the agents. They study the case of *mcst* problems with revenues and show that its core (the r-core) may be empty in general cases. Moreover, they find a sufficient condition that ensures the non-emptiness of the r-core in the particular case in which only two values of the connection costs are possible: *low* and *high* cost (this kind of situations are known as *elementary cost* problems, or 2-*mcst* problems). Their sufficient condition requires that cooperation among the members of the grand coalition grants the use of the service under consideration to all its members.

An alternative interpretation (or an equivalent problem) arises when we consider that (instead of revenues) individuals have an *upper bound* on how much they are willing to pay to be connected. In both cases, and contrary to what happens in *mcst* problems, a coalition may profit from not allowing all of its members to get the service that generates the revenues. That is, it is possible a situation in which some individuals remain unconnected in the *optimal* cost-revenues spanning tree.

Subiza et al. (2016) introduce the class of *simple mcst* problems as the sub-class of elementary problems such that only one individual connects directly to the source and at most one individual uses a high-cost connection in the *minimum cost tree*, the one that directly connects to the source (see the formal definition in Section 3). Working with this kind of problems it is easy to obtain the cost of the minimum cost tree and that paper shows that the *Folk* solution (Feltkamp et al., 1994; Bergantiños and Vidal-Puga, 2007), one of the most popular solutions in *mcst* problems, shares the optimal cost equally among the individuals in the grand coalition. Subiza et al. (2016) also introduce the more general class of *simple-components mcst* problems (that includes elementary problems). Simple-components problems allow to divide the problem in several (simple) independent components, so that we only need to solve the (smaller) *mcst* sub-problems and, in some sense, the grand coalition is not relevant.

In this paper we deal with simple *mcst* problems with revenues. By using a sub-family (*plain* problems) as an intermediate step, we show that the core of simple *mcst* games with revenues is non-empty. Moreover, our proof is constructive, so we provide an allocation in the r-core. The results are also used to discuss the case of elementary cost *mcst* problems with revenues.

2. Preliminaries

2.1. ***Minimum cost spanning tree***

A *minimum cost spanning tree* (*mcst*) problem involves a finite set of agents, $N = \{1, 2, \ldots, n\}$, $n \geqslant 2$, who need to be connected to a source ω. We denote by N_ω the set of agents and the source, $N_\omega = N \cup \{\omega\}$. The agents and the source are connected by edges and for $i \neq j$, $c_{ij} \in \mathbb{R}_+$ represents the cost of the edge $e_{ij} = (i, j)$ connecting $i, j \in N_\omega$. As usual in the literature, we assume throughout this article that:

(i) $c_{ii} = 0$, for all $i \in N_\omega$; and
(ii) $c_{ij} = c_{ji}$, for all $i, j \in N_\omega$ (symmetry).

Let $\mathbf{C} = [c_{ij}]_{(n+1)\times(n+1)}$ be the symmetric cost matrix. The *mcst* problem is represented by the pair $(N_\omega, \mathbf{C})$ and $\mathcal{N}_n$ will denote the class of *mcst* problems involving n agents and a source.

Let us denote by C_ω the cost of the tree in which every individual joins directly the source, $C_\omega = \sum_{i \in N} c_{i\omega}$. And, for any individual $i \in N$, c_{i*} represents the minimum connection cost of such an individual (interpreted as the cost to connect with his nearest partner),

$$c_{i*} = \min\{c_{ij},\ j \in N_\omega,\ j \neq i\}.$$

Note that the nearest partner may be the source ω, in which case $c_{i*} = c_{i\omega}$.

Since the cost matrix is nonnegative, it turns out that the graph that connects all individuals to the source with a minimum cost is a spanning tree.

A spanning tree over $(N_\omega, \mathbf{C}) \in \mathcal{N}_n$ is an undirected graph p with no cycles that connects all elements of N_ω. We can identify a spanning tree with a map $p : N \rightarrow N_\omega$ so that $j = p(i)$ is the agent (or the source) whom i connects on his way to the source. This map p defines the edges $e^p_{ij} = (i, p(i))$ in the tree. In a spanning tree each agent is (directly or indirectly) connected to the source ω; that is, for all $i \in N$ there is some $t \in \mathbb{N}$ such that $p^t(i) = \omega$. Moreover, given a spanning tree p, there is a unique path from any i to the source for all $i \in N$, given by the edges $(i, p(i)), (p(i), p^2(i)), \ldots, (p^{t-1}(i), p^t(i) = \omega)$. The cost of building the spanning tree p is the total cost of the edges in this tree; that is, $C_p = \sum_{i=1}^n c_{ip(i)}$. Prim (1957) provides an algorithm which solves the problem of connecting all agents to the source such that the total cost of the network is minimum.[1] The achieved solution, the minimum cost spanning tree, may not be unique. Denote by m a tree with minimum cost and by C_m its cost. That is, for any spanning tree p

$$C_m = \sum_{i=1}^{n} c_{im(i)} \leqslant C_p = \sum_{i=1}^{n} c_{ip(i)}.$$

Once a minimum cost spanning tree is built, the problem at hand is how to allocate the associated cost C_m among the agents.

A cost sharing rule for *mcst* problems is a function that proposes for any *mcst* problem $(N_\omega, \mathbf{C}) \in \mathcal{N}_n$ an allocation $\alpha(N, \mathbf{C}) = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$, such that $\sum_{i=1}^n \alpha_i = C_m$. Many cost sharing rules have been defined in the literature. One prominent solution to solve the allocation of this cost is the so-called *Folk* solution proposed independently by Feltkamp et al. (1994) and Bergantiños and Vidal-Puga (2007), among others. The *Bird* solution (Bird, 1976) allocates to each agent the cost of the link on the (unique) path from this agent to the source in the optimal tree.

As mentioned in Bogomolnaia and Moulin (2010): "To find a fair and in some sense stable division of the optimal cost among the n agents is a challenging question discussed by cooperative game theorists for over thirty

[1] This algorithm has n steps. First, we select the agent i with smallest cost to the source; that is, $c_{i\omega} \leqslant c_{j\omega}$, for all $j \in N$. In the second step, we select an agent in $N \backslash \{i\}$ with the smallest cost either directly to the source or to agent i, who is already connected. We continue until all agents are connected, at each step connecting an agent still not connected to a connected agent or to the source.

years. That literature singles out Stand Alone Core stability as the key incentive compatibility property: no coalition of agents should be charged more than the cheapest cost of connecting all of them to the source, independently of agents outside the coalition." To define the stand alone cost of a coalition $S \subseteq N$, we will denote by $\mathbf{C}|_S$ the cost matrix involving only elements in S; i.e., $\mathbf{C}|_S = [c_{ij}]_{i,j \in S \cup \{\omega\}}$. Then, we associate to any *mcst* problem a *TU* cooperative game (N, c) defined in the following way:

$$c(N) = C_m, \quad c(S) = C_m(S) \quad \forall S \subseteq N, \quad c(\varnothing) = 0,$$

where $C_m(S)$ denotes the minimum cost in the *mcst* sub-problem $(S_\omega, \mathbf{C}|_S)$. This cooperative game is the base of the *Kar* solution (Kar, 2002), which is defined as the *Shapley* value of the cooperative game (N, c); that is, $K(N, \mathbf{C}) = Sh(N, c)$.

The core of the cooperative game (N, c), denoted as $core(N, c)$, is the set of allocations $x \in \mathbb{R}^n$ such that $\sum_{i=1}^{n} x_i = c(N)$ and for any subset $S \subseteq N$, $\sum_{i \in S} x_i \leqslant c(S)$. The aforementioned *Bird* and *Folk* solutions belong to the core of this cooperative game, proving its non-emptiness (see, for instance, Bogomolnaia and Moulin (2010)).

2.2. ***Minimum cost spanning tree with revenues***

Estévez-Fernández and Reijnierse (2014) introduce the minimum cost spanning tree problems with revenues in the following way. Let us consider a *mcst* problem $(N_\omega, \mathbf{C}) \in \mathcal{N}_n$ and suppose now that a group of agents $S \subseteq N$ connected to the source ω can cooperate in order to obtain a higher joint revenue. Whenever a coalition $S \subseteq N$ decides to cooperate, an additive revenue is obtained and a non-additive cost (the cost of connecting individuals in S to the source ω) is generated.

Formally, let $b_i \in \mathbb{R}_+$ the revenue that agent $i \in N$ generates if i gets the service under consideration (that is, if agent i is directly or indirectly connected to the source ω). We denote by $b = (b_1, b_2, \ldots, b_n)$ the vector of revenues. Then, the total revenue that a coalition $S \subseteq N$ can obtain by cooperation is $\sum_{i \in S} b_i$, whereas the net revenue of this coalition is

$$\pi(S) = \sum_{i \in S} b_i - c(S).$$

We denote this kind of problems by $(N_\omega, \mathbf{C}, b)$. Due to the revenue structure of the game, it may be more profitable for coalition S not to form as a whole, so that some agents $i \in S$ do not connect to the source ω. Under such assumption, the *worth* of a coalition S is

$$v_b(S) = \max\{\pi(R), \quad R \subseteq S\}.$$

A new *TU* cooperative game is thus defined by the pair (N, v_b). The problem at hand is how to share the worth $v_b(N)$ obtained by the grand coalition among the agents. A first idea is to allocate to each individual his net revenue, once the cost is shared accordingly to some *mcst* solution. But this sharing may end with some agent obtaining a negative amount, which is a counter-intuitive proposal (individuals can chose not to connect, and then their net revenue is null; so they are not willing to pay any amount). The following example illustrates the situation.

Example 1. *Let us consider the mcst problem with revenues represented in Figure 2, in which the number at the edges represent the connection costs and the boldfaced numbers at the nodes represent the revenue that the corresponding agent obtains.*

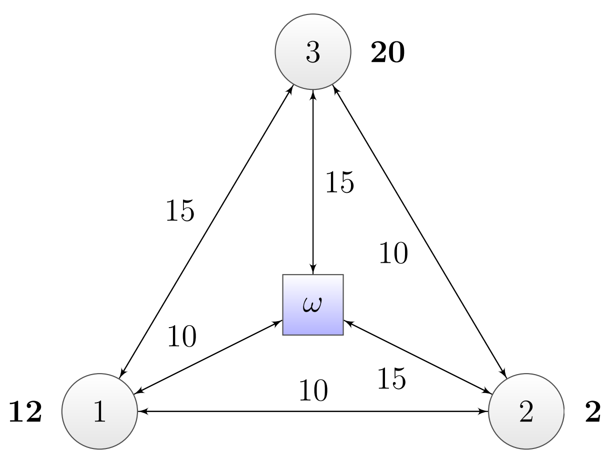
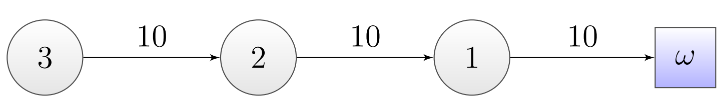


Figure 1: The *mcst* problem with revenues in Example 1.

The minimum cost spanning tree is:



Then, $c(N) = 30$ and the net revenues are $\pi(N) = \sum_{i=1}^{3} b_i - c(N) = 4$. Note that it is profitable that only agents 1 and 3 connect the source with a cost of $c(\{1,3\}) = 25$ and a net revenue $\pi(\{1,3\}) = b_1 + b_3 - c(\{1,3\}) = 7$. That is, the worth of the gran coalition is $\upsilon_b(N) = 7$.

If we use the Bird solution to share the cost of the mcst, then each agent pays the edge he uses in the optimal tree; that is, $y_1 = 10$, $y_2 = 10$, $y_3 = 10$ and the net revenues are $x_1 = 2$, $x_2 = -8$, $x_3 = 10$. This allocation is not efficient (the worth is 7 and, in aggregate, the individuals receive 4 units) and it is not acceptable for agent 2 that receives a negative allocation (that is, this agent should pay 8 monetary units).

In order to share the worth $\upsilon_b(N)$, it is possible to pay attention only to the TU cooperative revenues game (N, υ_b) and therefore to apply TU games solution concepts (Shapley value, nucleolus). In this example, the characteristic function υ_b is given in Table 1 (we also indicate the cost of building the tree, $c(S)$, the revenue $\sum_{i\in S} b_i$, and the net revenue $\pi(S)$, for each coalition S).

S	$\varnothing$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$N = \{1,2,3\}$
$\upsilon_b(S)$	0	2	0	5	2	7	5	7
$c(S)$	0	10	15	15	20	25	25	30
$\sum_{i\in S} b_i$	0	12	2	20	14	32	22	34
$\pi(S)$	0	2	-13	5	-6	7	-3	4

Table 1: Coalitional values of the *mcst* game with revenues in Example 1.

If we compute the Shapley value of the TU game (N, υ_b), we get the sharing $x = \mathrm{Sh}(N, \upsilon_b) = (2, 0, 5)$, which is a natural solution of the mcst problem with revenues (in this case, it coincides with the nucleolus).

A different approach to obtain a solution consists on building the efficient tree with respect to the mcst problem with revenues; that is, only connecting to the source the agents in the coalition $S \subseteq N$ such that $\upsilon_b(N) = \pi(S)$; in this case, only connecting agents 1 and 3 to the source ω. We can now divide the cost by using mcst sharing rules. Then, for all $i \in S$,

$$x_i = b_i - \alpha_i, \; i \in S,$$

where α is a cost sharing rule for mcst problems, provides a sharing of the obtained worth among the involved agents. We complete the allocation by defining $x_j = 0$, for all $j \notin S$. In this example, the connection cost is $c(\{1,3\}) = 25$ and the Bird and Folk solutions agree on the sharing $c_1 = 10$, $c_3 = 15$ ($c_2 = 0$, since this agent is not connected). Then, the worth of the mcst game with revenues is shared as

$$x_1 = b_1 - c_1 = 2; \quad x_2 = 0; \quad x_3 = b_3 - c_3 = 5$$

that coincides with the Shapley value of the TU game (N, v_b).

Since building a spanning tree needs the cooperation of the involved agents, an essential condition is that agents agree on that cooperation; that is, the proposed sharing of the worth $v_b(N)$ must belong to the core of the *mcst* game with revenues (N, v_b): the set of allocations $x \in \mathbb{R}^n$ such that

$$\sum_{i=1}^{n} x_i = v_b(N) \quad \text{and for any subset } S \subseteq N, \quad \sum_{i \in S} x_i \geqslant v_b(S).$$

We are interested in analyzing the non-emptiness of the above core, that we name r-core and denote by *core* (N, v_b).

A particular family of *mcst* problems with revenues that has been extensively analyzed is that of *elementary cost* problems, in which the cost of any edge can only take two values (low cost, high cost).

Definition 1. *An mcst problem $(N_\omega, \mathbf{C}) \in \mathcal{N}_n$ has* ***elementary cost*** *if for all $i, j \in N$, $c_{ij} \in \{\kappa_1, \kappa_2\}$, with $0 \leqslant \kappa_1 < \kappa_2$. We will denote an elementary cost mcst problem by $(N_\omega, \mathbf{C}^e)$, and $\mathcal{E}_n$ will represent the class of elementary cost mcst problems involving n agents.*

Remark 1. *Usually, whenever we are only concerned with the minimum connection cost, elementary cost mcst problems are defined such that $\kappa_1 = 0$ and $\kappa_2 = 1$. The general case $0 \leqslant \kappa_1 < \kappa_2$, low and high cost, which turns out to be relevant when revenues are considered, is also known as 2-mcst problems.*

In general, and contrary to what happens in the *mcst* cost-game (N, c), the r-core may be empty, as shown in Estévez-Fernández and Reijnierse (2014): They provide an elementary cost *mcst* problem with revenues that has empty r-core (Example 3.1; the example involves 6 individuals and the source). To solve the emptiness of the r-core, Estévez-Fernández and Reijnierse (2014) assume that the grand coalition N is profitable; that is, they analyze 2-*mcst* problems with revenues $(N_\omega, \mathbf{C}^e, b)$ satisfying the following assumption.

Assumption 1. $\upsilon_b(N) = \pi(N) = \sum_{i=1}^{n} b_i - c(N)$.

Note that the above assumption asks that the optimal revenues-tree (the most profitable tree) requires that all agents are (directly or indirectly) connected to the source. Under these conditions, they show that the r-core is non-empty.

Proposition 1. *(Estévez-Fernández and Reijnierse, 2014) Every elementary cost mcst game with revenues (N, υ_b) satisfying Assumption 1 has a nonempty r-core. Moreover,*

$$x \in core\,(N, \upsilon_b) \quad \Leftrightarrow \quad x = b - y, \text{ with } y \in core(N, c),\ y \leqslant b.$$

Remark 2. *The mcst problem with revenues in Example 1 has elementary cost but it does not fulfill Assumption 1; that is, the grand coalition is not profitable. Nevertheless, the r-core of the cost-revenue game is not empty: as can be observed with the data in Table 1, the proposed sharing of the worth in this example, $x^* = (2, 0, 5)$, belongs to the r-core, that only contains this assignment.*

Note that there is no allocation $y \in core(N, c)$ such that $x^* = b - y$. This is due to the fact that agent 2 is not connected to the source in the optimal revenues-tree.

We are interested in problems in which not necessarily all agents need to connect the source, as in the case shown in Example 1. The following section explores some ways of ensuring the non emptiness of the r-core without requiring Assumption 1 in some classes of *mcst* problems with revenues.

3. Simple *mcst* problems

In Subiza et al. (2016) the class of *simple mcst* problems is introduced and it is used as an intermediate step to analyze elementary cost *mcst* problems. First, the notion of *autonomous component* is needed.

Definition 2. *Given the mcst problem $(N_\omega, \mathbf{C}) \in \mathcal{N}_n$, with minimum connecting cost C_m, a subset $S \subseteq N$ is said to be:*

- ***autonomous*** *if $C_m = C_m(S) + C_m(N \backslash S)$;*
- ***an autonomous component*** *if it is autonomous and has no proper subset that is also autonomous (if $T \subsetneq S$, then T is not autonomous).*

Remark 3. *Obviously, the grand coalition N is always autonomous.*

Definition 3. *(Subiza et al., 2016) An mcst problem $(N_\omega, \mathbf{C}^e) \in \mathcal{E}_n$ with elementary cost is said to be **simple** if the grand coalition N is an autonomous component. We will denote a simple mcst problem by $(N_\omega, \mathbf{C}^s)$ and $\mathcal{S}_n$ will represent the class of simple mcst problems with n agents.*

The following result provides the explicit expression of the minimum cost of building a spanning tree in a simple *mcst* problem. The proof follows directly from Subiza et al. (2016).

Lemma 1. *Given a simple mcst problem $(N_\omega, \mathbf{C}^s) \in \mathcal{S}_n$*

1) *There is at most one individual $i \in N$ such that $c_{i\omega} = \kappa_1$*
2) *For all $i \in N$, $c_{i*} = \min\{c_{ij},\ j \in N_\omega,\ j \neq i\} = \kappa_1$*
3) $c(N) = \begin{cases} n\kappa_1 & \text{if } c_{i\omega} = \kappa_1 \text{ for some } i \in N \\ & \\ n\kappa_1 + (\kappa_2 - \kappa_1) & \text{otherwise} \end{cases}$

It is clear that if there exists an individual $i \in N$ such that $c_{i\omega} = \kappa_1$, this agent will be the one that directly connects to the source in the minimum cost spanning tree.

Notation. From now on, given a simple *mcst* problem $(N_\omega, \mathbf{C}^s) \in \mathcal{S}_n$ if there is an individual whose cost to connect the source is κ_1 we relabel the agents so that this individual is denoted by 1. Therefore, in a simple *mcst* problem $c_{i\omega} = \kappa_2$, for any $i \geqslant 2$.

From this result we obtain the net revenue of the grand coalition in simple *mcst* problems with revenues.

Corollary 1. *Given a simple mcst problem* $(N_\omega, \mathbf{C}^s) \in \mathcal{S}_n$ *and a vector of revenues* $b = (b_1, b_2, \ldots, b_n)$*, then*

$$\pi(N) = \begin{cases} \displaystyle\sum_{i=1}^{n}(b_i - \kappa_1) & \text{if } c_{1\omega} = \kappa_1 \\ \\ \displaystyle\sum_{i=1}^{n}(b_i - \kappa_1) - (\kappa_2 - \kappa_1) & \text{otherwise} \end{cases}$$

To analyze the net revenue of coalitions other than the grand coalition, $S \subseteq N$, $S \neq N$, it is important to note that although we consider a simple *mcst* problem $(N_\omega, \mathbf{C}^s)$, the sub-problems $(S_\omega, \mathbf{C}^s|_S)$ may no longer be simple for all $S \subseteq N$, as can be observed in Example 2: this *mcst* problem is simple, but if we consider the coalition $S = \{1, 3\}$ the *mcst* sub-problem $(S_\omega, \mathbf{C}^s|_S)$ is not a simple *mcst* problem since it has two autonomous components, $S_1 = \{1\}$, $S_2 = \{3\}$.

In our next definition we restrict the class of simple *mcst* problems by requiring that each sub-problem to be simple.

Definition 4. *Given an mcst problem* $(N_\omega, \mathbf{C}) \in \mathcal{N}_n$*, it is said to be* ***plain*** *if* $(S_\omega, \mathbf{C}|_S)$ *is simple for all* $S \subseteq N$*. We will denote a plain mcst problem by* $(N_\omega, \mathbf{C}^p)$ *and* $\mathcal{P}_n$ *will represent the class of plain mcst problems with* n *agents.*

Plain minimum cost spanning tree problems include situations where the costs are the same for all connections between individuals, and this common cost is strictly lower than the cost of connecting any agent to the source. For example, a group of neighbors who, in order to access to a service platform, could choose to create a network among themselves connecting agents at a price κ_1 for each connection, as long as one of them is connected directly to the platform at a higher price κ_2. Or they could connect individually to the platform directly, each of them at a cost of $\kappa_2 > \kappa_1$.

In the class of plain *mcst* problems we can easily obtain the net revenues of each coalition $S \subseteq N$. The result follows immediately from Corollary 1 and Definition 4.

Corollary 2. *Consider a plain mcst problem* $(N_\omega, \mathbf{C}^p) \in \mathcal{P}_n$ *and a vector of revenues* $b = (b_1, b_2, \ldots, b_n)$*. Then, for each* $S \subseteq N$,

$$\pi(S) = \begin{cases} \sum_{i \in S} (b_i - \kappa_1) & \textit{if } c_{1\omega} = \kappa_1 \\ \\ \sum_{i \in S} (b_i - \kappa_1) - (\kappa_2 - \kappa_1) & \textit{otherwise} \end{cases}$$

4. Non-emptiness of the *r*-core in plain *mcst* problems

The explicit expression of the characteristic function of a *plain* cost revenue game (N, υ_b) and the non-emptiness of the corresponding r-core are straightforwardly obtained from Corollary 2.. First, we define the notion of *effective coalition*.

Definition 5. *Let* $(N_\omega, \mathbf{C})$ *a mcst problem and consider a vector of revenues* $b = (b_1, b_2, \ldots, b_n)$*. For each* $S \subseteq N$*, the* ***effective coalition*** *associated to* S*, that we denote by* S^+*, is the subset of* S *such that:*

a) $\upsilon_b(S) = \pi(S^+)$
b) if $\upsilon_b(S) = \pi(R)$*, for some* $R \subseteq S$*, then* $R \subseteq S^+$

In words, it is the *maximal* subset providing the *worth* $\upsilon_b(S)$. Note that it may be the case that $S^+ = \varnothing$ (in such a case, $\upsilon_b(S) = 0$). Nevertheless, the converse is not true, so it is possible $\upsilon_b(S) = 0$ and $S^+ \neq \varnothing$. Moreover, it is obvious that $\upsilon_b(S) > 0$ implies $S^+ \neq \varnothing$.

The following result shows additional properties for *plain mcst* problems with revenues.

Lemma 2. *Consider a plain mcst problem* $(N_\omega, \mathbf{C}^p) \in \mathcal{P}_n$ *and a vector of revenues* $b = (b_1, b_2, \ldots, b_n)$*. Then, for each* $S \subseteq N$*:*

(1) *If* $b_i \geqslant \kappa_2$ *for some* $i \in S$*, then* $i \in S^+$.
(2) *If* $b_i \geqslant \kappa_1$ *for some* $i \in S$*, and* $\upsilon_b(S) > 0$*, then* $i \in S^+$.
(3) *If* $c_{1\omega} = \kappa_1$ *and* $1 \in S^+$*, then*

$$1 \in T^+, \textit{ for any } T \subseteq N, \textit{ such that } 1 \in T \textit{ and } \upsilon_b(T) > 0.$$

(4) *If* $c_{i\omega} = \kappa_2$ *for all* $i \in S$*, then* $b_j < \kappa_1$ *for* $j \in S$ *implies* $j \notin S^+$.

Proof. Let $S \subseteq N$, $S \neq \varnothing$.

(1) Suppose $b_i \geqslant \kappa_2$ for some $i \in S$. If $i \notin S^+$, then $S^+ \cup \{i\} \subseteq S$ and

$$\upsilon_b(S) = \pi(S^+) \leqslant \pi(S^+) + (b_i - \kappa_2) \leqslant \pi(S^+ \cup \{i\})$$

a contradiction.

(2) Now suppose $\upsilon_b(S) > 0$ and $b_i \geqslant \kappa_1$ for some $i \in S$. If $i \notin S^+$, then $S^+ \cup \{i\} \subseteq S$ and since the sub-problem $(S_\omega, \mathbf{C}|_S)$ is simple, individual $i \in S$ may connect with some individual in S^+ at cost κ_1 (note that $S^+ \neq \varnothing$). Therefore,

$$\upsilon_b(S) = \pi(S^+) \leqslant \pi(S^+) + (b_i - \kappa_1) = \pi(S^+ \cup \{i\})$$

a contradiction.

(3) If $c_{1\omega} = \kappa_1$ and $1 \in S^+$, this implies $\upsilon_b(S) > 0$. We distinguish two cases:

(a) If $b_1 \geqslant \kappa_1$, part (2) shows that $1 \in T^+$, for any $T \subseteq N$, such that $1 \in T$ and $\upsilon_b(T) > 0$.

(b) If $b_1 < \kappa_1$, and $1 \in S^+$, then

$$\sum_{i \in S^+} (b_i - \kappa_1) \geqslant \sum_{i \in S^+, i \neq 1} (b_i - \kappa_1) - (\kappa_2 - \kappa_1)$$

from definition of effective coalition, which implies $b_1 - \kappa_1 \geqslant \kappa_1 - \kappa_2$. Then, for any $T \subseteq N$, such that $\upsilon_b(T) > 0$ and $1 \in T$, if $1 \notin T^+$,

$$\upsilon_b(T) = \pi(T^+) = \sum_{i \in T^+} (b_i - \kappa_1) - (\kappa_2 - \kappa_1) \leqslant$$

$$\leqslant \sum_{i \in T^+ \cup \{1\}} (b_i - \kappa_1) = \pi(T^+ \cup \{1\})$$

a contradiction.

(4) If $c_{i\omega} = \kappa_2$ for all $i \in S$, $b_j < \kappa_1$ and $j \in S^+$, then

$$\upsilon_b(S) = \pi(S^+) = \sum_{i \in S^+} (b_i - \kappa_1) - (\kappa_2 - \kappa_1) <$$

$$< \sum_{i \in S^+, i \neq j} (b_i - \kappa_1) - (\kappa_2 - \kappa_1) \leqslant \pi\left(S^+ \setminus \{j\}\right)$$

a contradiction.

This concludes the proof. ■

Remark 4. *As we know (Lemma 1), in simple (and plain) mcst problems there exists at most one individual such that the cost of directly connect the source is κ_1. In the mcst problem this agent always connect the source in order to obtain the efficient tree (minimum cost).*

In the cost-revenue game, the individual fulfilling $c_{1\omega} = \kappa_1$ is not necessarily required to be in the optimal tree (maximum benefit). The proof of the above result shows that this individual belongs to the optimal tree if and only if

$$b_1 \geqslant 2\kappa_1 - \kappa_2$$

and this applies for any coalition $S \subseteq N$ containing this individual. On the other hand, if $c_{1\omega} = \kappa_2$, only agents such that $b_i \geqslant \kappa_1$ may belong to the effective coalition.

Next, we obtain the characteristic function of the cost-revenue game (N, v_b), in the case of plain *mcst* problems. The proof follows directly from Corollary 2.

Lemma 3. *Consider a plain mcst problem $(N_\omega, \mathbf{C}^p) \in \mathcal{P}_n$ and a vector of revenues $b = (b_1, b_2, \ldots, b_n)$. Then, for each $S \subseteq N$, the characteristic function of the cost-revenue game is defined by:*

$$v_b(S) = \begin{cases} 0 & \text{if } S^+ = \varnothing \\ \sum_{i \in S^+} (b_i - \kappa_1) & \text{if } c_{1\omega} = \kappa_1 \text{ and } 1 \in S^+ \\ \sum_{i \in S^+} (b_i - \kappa_1) - (\kappa_2 - \kappa_1) & \text{otherwise} \end{cases}$$

Theorem 1. *For any plain mcst problem $(N_\omega, \mathbf{C}^p) \in \mathcal{P}_n$ and any vector of revenues $b = (b_1, b_2, \ldots, b_n)$, the core of the TU cooperative game (N, v_b) is non-empty.*

Proof. To prove the non-emptiness, we will show that the following allocation belongs to the r-core:

$$x_i = \max\{0, b_i - \kappa_1 - \alpha\} \qquad \forall\, i \in N \tag{1}$$

where $\alpha \geqslant 0$ is chosen so that $\sum_{i=1}^{n} x_i = \upsilon_b(N)$.

We need to prove that, for all $S \subseteq N$, $\upsilon_b(S) \leqslant \sum_{i \in S} x_i$.

If $\upsilon_b(S) = 0$ the above condition is obviously fulfilled. Then, we suppose $\upsilon_b(S) > 0$. We distinguish the following cases:

(a) $c_{i\omega} = \kappa_2$ for all $i \in N$
In this case we know (Lemma 2) that $i \in S^+$ if and only if $b_i \geqslant \kappa_1$. On the other hand, the *extra-cost* $\kappa_2 - \kappa_1$ is completely assumed by individuals in this coalition; that is, $\sum_{i \in S^+} \alpha = \kappa_2 - \kappa_1$. Then,

$$\upsilon_b(S) = \sum_{i \in S^+} (b_i - \kappa_1) - (\kappa_2 - \kappa_1) \leqslant \sum_{i \in S^+} ((b_i - \kappa_1) - \alpha) \leqslant \sum_{i \in S^+} x_i \leqslant \sum_{i \in S} x_i$$

(b) $c_{1\omega} = \kappa_1$. If $b_1 < 2\kappa_1 - \kappa_2$ then this individual does not belong to the optimal tree (for any coalition) and the result follows analogously as in the previous case. Otherwise, if $b_1 \geqslant 2\kappa_1 - \kappa_2$, then $\kappa_1 - b_1 \leqslant \kappa_2 - \kappa_1$ and

(b1) If $1 \in S$, $\upsilon_b(S) = \sum_{i \in S^+} (b_i - \kappa_1) \leqslant \sum_{i \in S} x_i$.

(b2) If $1 \notin S$, $\upsilon_b(S) = \sum_{i \in S^+} (b_i - \kappa_1) - (\kappa_2 - \kappa_1) \leqslant$
$\leqslant \sum_{i \in S^+} (b_i - \kappa_1) - (b_1 - \kappa_1) \leqslant \sum_{i \in S} x_i$.

This concludes the proof. ■

5. Simple problems: non-emptiness of the r-core

In order to show the non-emptiness of the r-core of simple *mcst* problems with revenues the first goal is to obtain the net revenues $\pi(S)$ and the characteristic function $\upsilon_b(S)$ of each coalition S. Contrary to what happens with plain problems (sub-problems are also plain), a sub-problem $(S_\omega, \mathbf{C}|_S)$ of a simple *mcst* problem may not be simple (as shown in Example 1). Nevertheless, these sub-problems have elementary cost and can be split in several simple problems (see Subiza et al. (2016)), that we call *simple components*.[2]

[2]In Granot and Huberman (1981) the notion of *simple-components mcst* problem is analyzed with the name of *efficient coalition structure*. We additionally ask the sub-problems to be simple.

The main feature of this kind of problems is that a *mcst* problem with simple components can be solved by solving its simple components.

By using simple components, the following result provides the form of net revenues of simple *mcst* games.

Lemma 4. *Let* $(N_\omega, \mathbf{C}^s) \in \mathcal{S}_n$, $b = (b_1, b_2, \ldots, b_n)$ *the vector of revenues, and consider* $S \subseteq N$. *Then, for each simple component* S_t *in* $(S_\omega, \mathbf{C}|_S)$

$$\pi(S_t) = \begin{cases} \sum_{i \in S_t} (b_i - \kappa_1) & \text{if } c_{1\omega} = \kappa_1 \text{ and } 1 \in S_t \\ \\ \sum_{i \in S_t} (b_i - \kappa_1) - (\kappa_2 - \kappa_1) & \text{otherwise} \end{cases}$$

and

$$\pi(S) = \sum_{t=1}^{q_S} \pi(S_t) =$$

$$= \begin{cases} \sum_{i \in S} (b_i - \kappa_1) - (q_S - 1)(\kappa_2 - \kappa_1) & \text{if } c_{1\omega} = \kappa_1 \text{ and } 1 \in S \\ \\ \sum_{i \in S} (b_i - \kappa_1) - q_S(\kappa_2 - \kappa_1) & \text{otherwise} \end{cases}$$

where q_S *is the number of simple components in* $(S_\omega, \mathbf{C}|_S)$.

Proof. It follows straightforwardly from Corollary 1. Note that, being the *mcst* problem $(N_\omega, \mathbf{C}^s)$ simple, the low and high cost are always the same, $\kappa_1(t) = \kappa_1$ and $\kappa_2(t) = \kappa_2$, for any simple component S_t. To obtain the expression of the revenue of coalition S, $\pi(S)$, observe that the collection $\{S_t : t = 1, 2, \ldots, q_S\}$ is a partition of S, so the individual with low cost to connect the source (if any) belongs at most to one of the subsets $S_t \subseteq S$. ■

Proposition 2. *For any simple mcst problem* $(N_\omega, \mathbf{C}^s) \in \mathcal{S}_n$ *and any vector of revenues* $b = (b_1, b_2, \ldots, b_n)$, *the characteristic function of the TU cooper-*

ative game (N, υ_b) fulfills:

$$\upsilon_b(S) \leqslant \begin{cases} 0 & \textit{if } S^+ = \varnothing \\ \sum_{i \in S^+} (b_i - \kappa_1) & \textit{if } c_{1\omega} = \kappa_1 \textit{ and } 1 \in S^+ \\ \sum_{i \in S^+} (b_i - \kappa_1) - (\kappa_2 - \kappa_1) & \textit{otherwise} \end{cases}$$

where S^+ is the effective coalition associated to S.

Proof. We know that $\upsilon_b(S) = \pi(S^+)$ and the result follows from Lemma 4 by observing that $q_{S^+} \geqslant 1$. ■

From this result we obtain the non-emptiness of simple *mcst* games with revenues. The proof runs parallel to that in Theorem 1, so we omit it.[3]

Theorem 2. *For every $(N_\omega, \mathbf{C}^s) \in \mathcal{S}_n$, and every $b = (b_1, b_2, \ldots, b_n)$ the core of the TU cooperative game (N, υ_b) is nonempty.*

6. Final comments

We have shown a sub-class of problems (simple *mcst* games with revenues) in which the core is always not-empty. Moreover, this result cannot be extended to the class of elementary cost *mcst* games with revenues since we know that the core in this case may be empty (Estévez-Fernández and Reijnierse, 2014). These authors provide a sufficient condition to ensure the non-emptiness of the core: the grand coalition must be an effective coalition.

Can we apply our results to discuss the core of elementary cost *mcst* games with revenues? We know that these problems have simple components, and that each simple component is a simple *mcst* problem with not-empty

[3]The main difference is that we now use Proposition 2 instead of Lemma 3, and we get an inequality when analyzing the value of $\upsilon_b(S)$. As in Theorem 1, it can be shown that the allocation

$$x_i = \max\{0, b_i - \kappa_1 - \alpha\} \qquad \forall\, i \in N$$

belongs to the r-core, where $\alpha \geqslant 0$ is chosen so that $\sum_{i=1}^{n} x_i = \upsilon_b(N)$.

core (although the core of the whole problem may be empty). So, we can pick up allocations in the core of such sub-problems and propose them as a possible solution. Let us see the result of applying this procedure in the example provided by Estévez-Fernández and Reijnierse (2014) with empty core, which is represented in Figure 2.

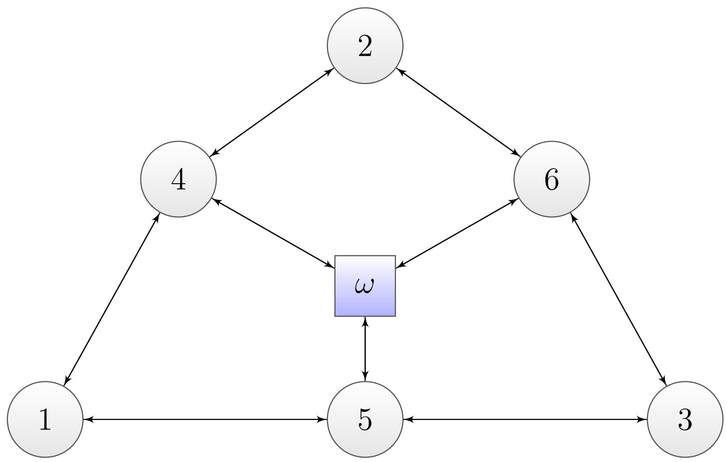


Figure 2: The elementary cost *mcst* problem with revenues in Example 3.1, from Estévez-Fernández and Reijnierse (2014). The cost of each link appearing in the picture is $c_{ij} = 1$, whereas the undrawn links have cost $c_{ij} = 2$. The revenues vector is $b = (2, 2, 2, 0, 0, 0)$.

This elementary cost *mcst* problem can be split into three simple components, as many as individuals with $c_{i\omega} = 1$ (low cost to the source), in the following different ways:

- $S_1 = \{1, 2, 4\}$ $\quad S_2 = \{3, 5\}$ $\quad S_3 = \{6\}$
- $S_1 = \{1, 2, 4\}$ $\quad S_2 = \{3, 6\}$ $\quad S_3 = \{5\}$
- $S_1 = \{1, 3, 5\}$ $\quad S_2 = \{2, 4\}$ $\quad S_3 = \{6\}$
- $S_1 = \{1, 3, 5\}$ $\quad S_2 = \{2, 6\}$ $\quad S_3 = \{4\}$
- $S_1 = \{2, 3, 6\}$ $\quad S_2 = \{1, 4\}$ $\quad S_3 = \{5\}$
- $S_1 = \{2, 3, 6\}$ $\quad S_2 = \{1, 5\}$ $\quad S_3 = \{4\}$

By selecting, in each possible configuration as a simple components problem, the core allocation provided in the proof of Theorem 1 for any simple *mcst* sub-problem we obtain:

$$x_i = 0.5, \text{ if } i \in S_1 \cap \{1,2,3\}, \quad x_i = 0, \text{otherwise}.$$

If all the possible splittings in simple components are considered equally probable, the average of these allocations is

$$x = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0\right)$$

that coincides with the nucleolus of the cost revenues game (N, v_b).

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