Overlapping generations model

- Time is measured in periods, denoted by t, and indexed by integers: $t \in \{1, 2, 3, \ldots\}$.
- The only agents are consumers.
- Consumers live for two consecutive periods.
- At every period t a new generation of $N(t)$ consumers is born. Members of generation t are young in period t and old in period $t + 1$.
- There is only one good in each period.
- The good is exogenously given (gift of nature).
- The amount of good in period t is $Y(t)$.
- The endowment $Y(t)$ is only available at t.

Demographic structure

	time period			
generation	1	2	3	4
0	*old*			
1	*young* →	*old*		
2		*young* →	*old*	
3			*young* →	*old*
4				*young*
…				
population	$N(0) + N(1)$	$N(1) + N(2)$	$N(2) + N(3)$	$N(3) + N(4)$
amount of good	$Y(1)$	$Y(2)$	$Y(3)$	$Y(4)$

Endowments & consumption

- Member i of generation t has $w_t^i(t)$ units of the good at t and $w_t^i(t+1)$ units at $t + 1$.

- The endowment $Y(t)$ in period t is distributed among the people alive in t:

$$\sum_{i \in N(t)} w_t^i(t) + \sum_{i \in N(t-1)} w_{t-1}^i(t) = Y(t).$$

- Member i of generation t consumes $c_t^i(t)$ units of the good at t and $c_t^i(t+1)$ at $t + 1$.

- The consumption basket of $i \in N(t)$ is a pair $c_t^i = \left(c_t^i(t),\, c_t^i(t+1)\right)$ that establishes i's consumption when young and when old.

Consumption allocations

- A consumption allocation is a sequence $\{c_t^i\}_{t\geq 0, i\in N(t)}$ of consumption baskets of members of all generations (generation 0 consists only of old people).

- A consumption allocation is feasible if, for all $t \geq 1$,

$$\sum_{i\in N(t)} c_t^i(t) + \sum_{i\in N(t-1)} c_{t-1}^i(t) \leq Y(t).$$

- A consumption allocation is efficient if, for all $t \geq 1$,

$$\sum_{i\in N(t)} c_t^i(t) + \sum_{i\in N(t-1)} c_{t-1}^i(t) = Y(t).$$

Preferences & utility functions

- Consumers have preferences over their own consumption vectors.

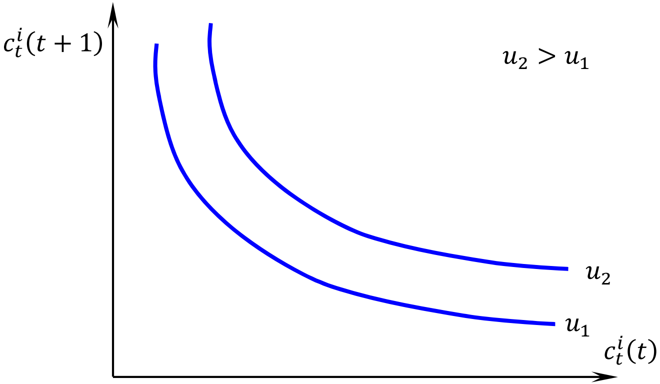
- When young, the preference of consumer $i \in N(t)$ is represented by a utility function u_t^i.

- The value $u_t^i\left(c_t^i(t), c_t^i(t+1)\right)$ is i's utility when he consumes $c_t^i(t)$ now (as young) and consumes $c_t^i(t+1)$ in the future (as old).

- When old, i's utility only depends on $c_t^i(t+1)$, which has already been determined when i was young.

Properties of the utility function

- Each u_t^i is, in general, assumed to satisfy the properties ensuring that <u>indifference curves are differentiable, decreasing, and convex</u>.



Notation

- Period of time and generation t
- Number of members of generation t $N(t)$
- Amount of good available in period t $Y(t)$
- Consumption in period t of individual i of generation t (i young) $c_t^i(t)$
- Consumption in period $t + 1$ of individual i of generation t (i old) $c_t^i(t+1)$
- Endowment in t of $i \in N(t)$ $w_t^i(t)$
- Endowment in $t + 1$ of $i \in N(t)$ $w_t^i(t+1)$
- Utility function of member i of generation t in period t (i young) u_t^i

Pareto efficiency

- A consumption allocation $C = \{c_t^i\}_{t\geq 0, i \in N(t)}$ is Pareto efficient if there does not exist another consumption allocation $\tilde{C} = \{\tilde{c}_t^i\}_{t\geq 0, i \in N(t)}$ such that:

(i) for some $t \geq 1$ and some $i \in N(t)$, $u_t^i(\tilde{c}_t^i) > u_t^i(c_t^i)$, and

(ii) for every $t \geq 1$ and every $i \in N(t)$, $u_t^i(\tilde{c}_t^i) \geq u_t^i(c_t^i)$.

- C Pareto efficient means for no other $\tilde{C}$ some i has more utility and no i has less utility.

Marginal rate of substitution (MRS)

- Define i's marginal rate of substitution as

$$MRS_t^i = \frac{\partial u_t^i / \partial c_t^i(t)}{\partial u_t^i / \partial c_t^i(t+1)}$$

  where i is a member of generation t.

- The MRS_t^i evaluated at $c_t^i = \left(c_t^i(t), c_t^i(t+1)\right)$ is the slope (in absolute value) of the indifference curve containing c_t^i.

- MRS_t^i represents the increase in $c_t^i(t+1)$ necessary to keep utility constant given a decrease of $c_t^i(t)$.

Pareto efficiency & MRS

➡ *Let C be a consumption allocation. Then:*

C *Pareto efficient* $\Rightarrow \forall t \; \forall i,j \in N(t) \;\; MRS_t^i = MRS_t^j$.

- Equality of the MRS of all members of a generation is <u>necessary</u> for Pareto efficiency.
- The converse is not true: equality of the MRS is not sufficient for Pareto efficiency.
- <u>Example</u>. All generations identical with n members each: $u_t^i(c_t^i) = c_t^i(t) \cdot c_t^i(t+1)$ and $\left(c_t^i(t), c_t^i(t+1)\right) = (2, 1)$.

- The consumption allocation C is not Pareto efficient, even though the MRS are all equal.

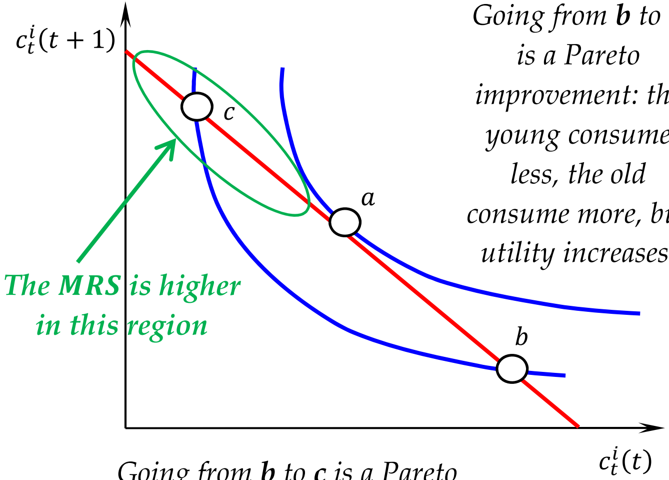
- Consider $\tilde{C}$ obtained from C by letting each young member of generation t give a small $\varepsilon > 0$ to a different old member of $t-1$.

- The old are all better off: each gets an extra ε. Take any young member of generation t. He gives ε to some old in period t but receives ε when old from some young of generation $t+1$. His utility in C is $u_t^i(2,1)=2$. In $\tilde{C}$ it is higher: $u_t^i(2-\varepsilon,1+\varepsilon)=2+\varepsilon(1-\varepsilon)>2$. Key: <u>the number of generation is infinite</u>.

MRS must be high for Pareto efficiency

- Let all generations be alike, with n members, and $u_t^i = c_t^i(t)c_t^i(t+1)$. The allocation a in 13 gives the highest utility to all generations. a marks the beginning of Pareto efficiency.
- Moving from b to a makes generation 0 better off and increases the utility of future generations, so b is not Pareto efficient.
- Moving from a to c makes the old from generation 0 better off but reduces the utility of future generations, so a and c are incomparable according to Pareto efficiency.

a *and* ***c*** *not comparable: generation 0 better off but future generations worse off.*



Going from ***b*** *to* ***a*** *is a Pareto improvement: the young consume less, the old consume more, but utility increases.*

Going from ***b*** *to* ***c*** *is a Pareto improvement: consumers of all generations* $t \geq 1$ *have the same utility but the old from generation 0 are better off.*

Market for lending and borrowing

- Let $r(t) > 0$ designate the (real) interest rate at t: lending 1 unit of the good at t implies receiving $1 + r(t)$ units of the good at $t + 1$. Define the gross interest rate as $R(t) = 1 + r(t)$.
- The market for the asset "lending" is a competitive market, so each i takes $r(t)$ as given.
- Intergenerational lending is not possible: old persons at t cannot pay/collect debts at $t + 1$.
- Thus, lending/borrowing can only take place among members of the same generation.

Budget constraints

- Let $l^i(t)$ be the lending of member i of generation t ($l^i(t)$ is written instead of $l_t^i(t)$ because i does not lend when old: $l_t^i(t+1) = 0$). Then i's budget constraint when young is

$$c_t^i(t) + l^i(t) \leq w_t^i(t).$$

- When old, i's budget constraint is

$$c_t^i(t+1) \leq w_t^i(t+1) + R(t)l^i(t).$$

- If $l^i(t) > 0$, i lends when young and receives $R(t)l^i(t)$ when old. If $l^i(t) < 0$, i borrows when young and pays $R(t)l^i(t)$ when old.

Lifetime budget constraint

- Combining the two constraints yields

$$\underbrace{c_t^i(t) + \frac{c_t^i(t+1)}{R(t)}}_{\text{present value of lifetime consumption}} \leq \underbrace{w_t^i(t) + \frac{w_t^i(t+1)}{R(t)}}_{\text{present value of lifetime endowments}}.$$

- The above inequality gives the consumption basket $\left(c_t^i(t), c_t^i(t+1)\right)$ that are feasible for member i of generation t given endowments $w_t^i = \left(w_t^i(t), w_t^i(t+1)\right)$ and the gross interest rate $R(t)$.

Consumer's decision problem

- Each consumer $i \in N(t)$, $t \geq 1$, is assumed to choose a consumption basket c_t^i that maximizes u_t^i given w_t^i and $R(t)$. This means that the lifetime budget constraint will be satisfied as an equality.

- Formally, i's aim is to

$$maximize_{\{c_t^i(t), c_t^i(t+1)\}} \;\; u_t^i\left(c_t^i(t), c_t^i(t+1)\right)$$

$$subject\ to \;\; c_t^i(t) + \frac{c_t^i(t+1)}{R(t)} = w_t^i(t) + \frac{w_t^i(t+1)}{R(t)}$$

or

$$max_{\{c_t^i(t)\}} \;\; u_t^i\left(c_t^i(t), R(t)[w_t^i(t) - c_t^i(t)] + w_t^i(t+1)\right).$$

Solution to the consumer's problem

- Given that $c_t^i(t+1) = R(t)[w_t^i(t) - c_t^i(t)] + w_t^i(t+1)$, take the total derivative of u_t^i

$$du_t^i = \frac{\partial u_t^i}{\partial c_t^i(t)} dc_t^i + \frac{\partial u_t^i}{\partial c_t^i(t+1)} \frac{\partial c_t^i(t+1)}{\partial c_t^i(t)} dc_t^i \,,$$

that is,

$$\frac{du_t^i}{dc_t^i(t)} = \frac{\partial u_t^i}{\partial c_t^i(t)} + \frac{\partial u_t^i}{\partial c_t^i(t+1)} R(t).$$

- To maximize u_t^i, it must be that $\frac{du_t^i}{dc_t^i(t)} = 0$. As a result,

$$R(t) = \frac{\partial u_t^i / \partial c_t^i(t)}{\partial u_t^i / \partial c_t^i(t+1)} = MRS_t^i \,.$$

Savings

- Using the preceding condition $R(t) = MRS_t^i$ and the budget constraint $c_t^i(t+1) = R(t)[w_t^i(t) - c_t^i(t)] + w_t^i(t+1)$, a demand function for consumption when young is obtained: $c_t^i(t) = C_t^i\big(w_t^i(t), w_t^i(t+1), R(t)\big)$.
- Define the savings $s^i(t)$ of consumer i of generation t as $s^i(t) = w_t^i(t) - c_t^i(t)$.
- Knowing the demand function for consumption C_t^i it is easy to determine the savings function $S^i\big(w_t^i(t), w_t^i(t+1), R(t)\big)$.

A Cobb-Douglas example

- Suppose $u_t^i = c_t^i(t) \cdot c_t^i(t+1)$. It follows from $R(t) = MRS_t^i$ that $R(t) = c_t^i(t+1)/c_t^i(t)$.

- Given that $c_t^i(t+1) = R(t)[w_t^i(t) - c_t^i(t)] + w_t^i(t+1)$, the demand function for consumption is $c_t^i(t) = \frac{1}{2}\left(w_t^i(t) + \frac{w_t^i(t+1)}{R(t)}\right)$. Consumption depends positively on wealth and negatively on the interest rate.

- The savings function is $s^i(t) = w_t^i(t) - \frac{1}{2}\left(w_t^i(t) + \frac{w_t^i(t+1)}{R(t)}\right) = \frac{1}{2}\left(w_t^i(t) - \frac{w_t^i(t+1)}{R(t)}\right)$, so a growing interest rate stimulates savings.

General competitive equilibrium (GCE)

- A GCE is a sequence $\{\hat{R}(t)\}_{t\geq 1}$ of (gross real) interest rates and a consumption allocation $\{\hat{c}_t^i\}_{t\geq 0, i\in N(t)}$ such that:

(i) for all $t \geq 1$ and $i \in N(t)$, $\hat{c}_t^i$ maximizes u_t^i given $\hat{R}(t)$ and i's endowments w_t^i (for $t = 0$, $\hat{c}_t^i$ is just the available wealth of the old); and

(ii) for all $t \geq 1$, $\sum_{i\in N(t)} \hat{c}_t^i(t) + \sum_{i\in N(t-1)} \hat{c}_{t-1}^i(t) = Y(t) = \sum_{i\in N(t)} w_t^i(t) + \sum_{i\in N(t-1)} w_{t-1}^i(t)$ [the goods market clearing condition].

On the equilibrium conditions

- Condition (i) holds that, in every period t and for each consumer i, $\hat{c}_t^i$ is the value of i's demand function for consumption given $\hat{R}(t)$ and i's endowments w_t^i.

- Condition (ii) asserts that the market for the good is in equilibrium at every t.

- There are only two markets: for the good and for loans. Since only the young at t lend or borrow at t, the loan market is in equilibrium when $\sum_{i \in N(t)} l^i(t) = 0$.

Two remarks on GCE

➡ *If* $\{\hat{R}(t)\}$ *and* $\{\hat{c}_t^i\}$ *are a GCE, then, for each* $\hat{R}(t)$*,* $\sum_{i\in N(t)} S^i\big(w_t^i(t), w_t^i(t+1), \hat{R}(t)\big) = 0$.

- Adding up the budget constraints of the young at t, $\sum_{i\in N(t)} c_t^i(t) + \sum_{i\in N(t)} l^i(t) = \sum_{i\in N(t)} w_t^i(t)$. In equilibrium, $\sum_{i\in N(t)} l^i(t) = 0$. Therefore, $0 = \sum_{i\in N(t)} w_t^i(t) - \sum_{i\in N(t)} c_t^i(t) = \sum_{i\in N(t)} s^i(t)$. This proves the previous result.

➡ *If* $\{\hat{R}(t)\}$ *is such that, for all* $\hat{R}(t)$*,* $\sum_{i\in N(t)} S^i\big(w_t^i(t), w_t^i(t+1), \hat{R}(t)\big) = 0$*, then, for some* $\{\hat{c}_t^i\}$*,* $\{\hat{R}(t)\}$ *and* $\{\hat{c}_t^i\}$ *constitute a GCE.*

Computing a GCE

- Assume that, for all t: $\left(w_t^i(t), w_t^i(t+1)\right) = (4, 1)$; $u_t^i = c_t^i(t) \cdot c_t^i(t+1)$; and $N(t) = 200$.
- As shown in 20, $s^i(t) = \frac{1}{2}\left(w_t^i(t) - \frac{w_t^i(t+1)}{R(t)}\right)$. Thus, $0 = \sum_{i \in N(t)} s^i(t) = 100\left(4 - \frac{1}{R(t)}\right)$ leads to $R(t) = 1/4$.
- With $R(t) =$ ¼, for all i, $s^i(t) = \frac{4 - 1 \cdot 4}{2} = 0$. This means that <u>no individual saves</u>: there is no lending nor borrowing, and accordingly consumption in each period coincides with the endowment at that period.

Equilibrium $\not\Rightarrow$ Pareto efficiency

- In the previous example, in a GCE, $\left(c_t^i(t), c_t^i(t+1)\right) = (4, 1)$ for all $t \geq 1$ (the old in period 1 consume $w_0^i(1) = 1$). This GCE consumption allocation is not Pareto efficient.

- To see this, suppose the young transfer ε to the old. The old are all clearly better off.

- As regards the young, before the transfer their utility is $4 \cdot 1 = 4$. After the transfer, their new utility is $(4 - \varepsilon)(1 + \varepsilon) = 4 + \varepsilon(3 - \varepsilon) > 4$, for sufficiently small ε (specifically, $\varepsilon < 3$).

Failure of Pareto efficiency

	period						
generation	1	2	3	4	…	*initial utility*	**new utility**
0	*1* **1+ε**					*u(1)*	***u*(1+ε)**
1	*4* **4–ε**	*1* **1+ε**				*4*	**4 + ε(3–ε)**
2		*4* **4–ε**	*1* **1+ε**			*4*	**4 + ε(3–ε)**
3			*4* **4–ε**	*1* **1+ε**		*4*	**4 + ε(3–ε)**
4				*4* **4–ε**		*4*	**4 + ε(3–ε)**
…					…		

Equilibrium consumption allocation in italics (blue).
New consumption allocation in bold face (red).

Taxes

- A government is created that merely taxes endowments (when the tax is negative, it will be called "transfer").

- Individual i of generation t faces the tax scheme $\tau_t^i = \left(\tau_t^i(t), \tau_t^i(t+1)\right)$, where $\tau_t^i(s)$ is the tax that i pays (or receives) in period s.

- The budget constraint on the government when nothing is done with the taxes (taxes are just paid out as transfers) states that, for all $t \geq 1$, $\sum_{i \in N(t)} \tau_t^i(t) + \sum_{i \in N(t-1)} \tau_{t-1}^i(t) = 0$.

GCE with taxes

- To compute GCE, consider the no tax case and replace $w_t^i(s)$ with $w_t^i(s) - \tau_t^i(s)$. The only additional condition to calculate GCE is the government budget constraint.

- In particular, the new lifetime budget constraint of consumer i is

$$c_t^i(t) + \frac{c_t^i(t+1)}{R(t)} = w_t^i(t) - \tau_t^i(t) + \frac{w_t^i(t+1) - \tau_t^i(t+1)}{R(t)}.$$

- The definition of savings becomes

$$s^i(t) = w_t^i(t) - \tau_t^i(t) - c_t^i(t).$$

Government borrowing

- Assume the government can issue one-period bonds, which are (safe) promises of delivering 1 unit of the good at $t + 1$ in exchange for a price $p(t) < 1$ paid at t.

- This means that bonds are issued at discount (price smaller than its face value). The (implicit) rate of return of the bond is $\frac{1-p(t)}{p(t)}$. The gross rate of return is then $\frac{1}{p(t)}$.

- Since the old never lend, the government can only borrow from (sell bonds to) the young.

The government budget constraint

- For $t \geq 1$, let $B(t)$ stand for the units of bonds that the government issues at t.

- The government budget constraint at t holds that $B(t-1)$, the debt to be paid at t, equals

$$\underbrace{\sum_{i \in N(t)} \tau_t^i(t)}_{\textit{taxes on the young}} + \underbrace{\sum_{i \in N(t-1)} \tau_{t-1}^i(t)}_{\textit{taxes on the old}} + \underbrace{p(t)B(t)}_{\textit{new bonds}}.$$

- The constraint shows the ways of redeeming at t bonds issued at $t-1$: tax the young at t; tax the old at t; issue new bonds at t.

Consumers' budget constraints

- Since only the young buy bonds, a young i of generation t faces the budget constraint

$$c_t^i(t) + l^i(t) + \tau_t^i(t) + p(t)b^i(t) = w_t^i(t).$$

- An old i has the following budget constraint:

$$c_t^i(t+1) + \tau_t^i(t+1) = w_t^i(t+1) + R(t)l^i(t) + b^i(t).$$

- Consumer i's lifetime budget constraint is

$$c_t^i(t) + \frac{c_t^i(t+1)}{R(t)} =$$

$$w_t^i(t) - \tau_t^i(t) + \frac{w_t^i(t+1) - \tau_t^i(t+1)}{R(t)} - b^i(t)\left[p(t) - \frac{1}{R(t)}\right].$$

- In equilibrium, by arbitrage, $p(t) = \frac{1}{R(t)}$.

Equality of returns

- If $1 > p(t)R(t)$, then private lending is more profitable than public lending. Then by borrowing $p(t)$ in the private loan market to purchase one bond, at $t + 1$ the bond pays 1 whereas the refund of the loan requires $p(t)R(t)$. A sure profit of $1 - p(t)R(t)$ is made. But in equilibrium sure profits cannot arise. A growing demand for loans and bonds tends to rise $p(t)$ and $R(t)$.

- Arbitrage opportunities also occur if $1 < p(t)R(t)$ (public lending is more profitable).

General equilibrium with bonds

- The summation of the budget constraints of all the young at t yields (where $i \in N(t)$)

  $$\sum_i c_t^i(t) + \sum_i \tau_t^i(t) + p(t) \sum_i b^i(t) = \sum_i w_t^i(t).$$

- Rearranging, $\sum_i \left[w_t^i(t) - c_t^i(t) - \tau_t^i(t)\right] = p(t) \sum_i b^i(t)$. That is, $\sum_{i \in N(t)} s^i(t) = p(t)B(t)$.

- Defining $S_t(R(t)) = \sum_{i \in N(t)} s^i(t)$ to be the <u>aggregate savings function</u> in period t, where it is emphasized that savings depend on the interest rate, it follows that

  $$S_t(R(t)) = p(t)B(t).$$

- The previous defines the equilibrium in both the private and public loan market. It can be easily verified that this condition implies that the good market is in equilibrium, too.

- Therefore, the <u>general equilibrium condition</u> amounts to $S_t(R(t)) = p(t)B(t)$: total private savings by the young at t equals the total value of the goverment debt <u>at t</u>.

- Since $R(t) = 1/p(t)$, the equilibrium condition can be equivalently expressed as

$$S_t(R(t)) = \frac{B(t)}{R(t)},$$

so <u>savings equal the present value of bonds</u>.

An example

- The government wishes to borrow 25 units of the good at $t = 1$, transfer them to the old at $t = 1$, and pay off the debt by taxing the young at $t = 2$.

- For all t and i, $u_t^i = c_t^i(t) \cdot c_t^i(t+1)$, $N(t) = 100$, $w_t^i = (2, 0)$ if i is odd, and $w_t^i = (1, 1)$ if i is even. Then the savings function is

$$s^i(t) = 1 \text{ if } i \text{ odd} \quad s^i(t) = \frac{1}{2} - \frac{1}{2R(t)} \text{ if } i \text{ even.}$$

- The aggregate savings function is $S_t = 50(1) + 50\frac{1}{2} - \frac{1}{2R(t)} = 75 - \frac{25}{R(t)}$.

- In equilibrium at $t=1$, $S_1 = \frac{B(1)}{R(1)} = 25$. Therefore, $75 - \frac{25}{R(t)} = 25$ and $R(1) = \frac{1}{2}$.

- The savings at $t=1$ are $s^i(1) = 1$ for i odd and $s^i(1) = -\frac{1}{2}$ for i even. This says that the odd lend and the even borrow.

- Total savings by the odd amount to 50. Total borrowing by the even equals 25. The difference is what the government borrows.

- Using $S_1 = B(1)/R(1)$, with $S_1 = 25$ and $R(1) = 1/2$, it follows that $B(1) = 12.5$. This is the amount of <u>bonds issued at $t=1$</u> and the taxes the young at $t=2$ will have to pay.

Rolling over debt

- A goverment rolls over debt when debt is paid off with new debt.

- In the previous example, suppose that the young at $t = 2$ are not taxed: new bonds are issued $t = 2$ to pay off the bonds issued at $t = 1$, $B(1) = 12.5$. Now in equilibrium:

$$S_2 = B(2)/R(2) \text{ and } S_2 = B(1).$$

- Therefore, $R(2) = 0.4$ and $B(2) = 5$. If the same policy is followed at $t = 3$,

$$S_3 = B(3)/R(3) \text{ and } S_3 = B(2).$$

- Accordingly, $R(3) = 0.35$ and $B(3) = 1.78$.

- The acumulation of bonds and the dynamics of the interest rate are determined by

$$R(t) = \frac{25}{75 - B(t-1)} \text{ and } B(t) = \frac{25B(t-1)}{75 - B(t-1)}.$$

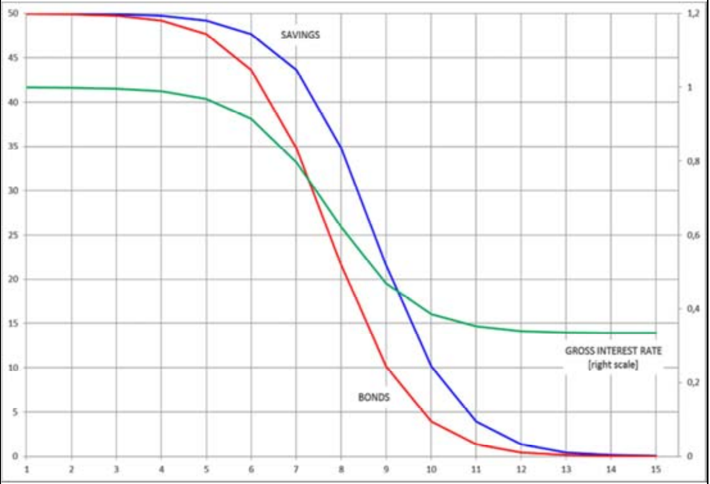
- In a <u>steady state</u>, equilibrium variables take the same value each t. That $B(t-1) = B(t)$ occurs in two cases: (i) $B = 50$ and $R = 1$; (ii) $B = 0$ and $R = 1/3$ (the equilibrium rate).

- The formulas hold when the govt raises at most 50, so $S(1) \leq 50$. If $S(1) < 50$, $B(t)$ goes to 0 and $R(t)$ to $1/3$. If $S(1) > 50$, borrowing becomes unfeasible for some t: a <u>bubble</u> arises (unsustainable price path for the bonds).

The government raises <50 at $t = 1$

t	$S(t)$	$R(t)$	$B(t)$	%
1	49,99	0,9996	49,97001	
2	49,97001	0,998802	49,91014	-0,11981
3	49,91014	0,996419	49,7314	-0,35814
4	49,7314	0,98937	49,20276	-1,06299
5	49,20276	0,969096	47,68218	-3,09042
6	47,68218	0,915154	43,63652	-8,48464
7	43,63652	0,797105	34,78291	-20,2895
8	34,78291	0,621626	21,62197	-37,8374
9	21,62197	0,468358	10,12681	-53,1642
10	10,12681	0,385367	3,902542	-61,4633
11	3,902542	0,35163	1,372251	-64,837
12	1,372251	0,339546	0,465942	-66,0454
13	0,465942	0,335417	0,156285	-66,4583
14	0,156285	0,334029	0,052204	-66,5971
15	0,052204	0,333566	0,017413	-66,6434

Dynamics when $S(1) < 50$



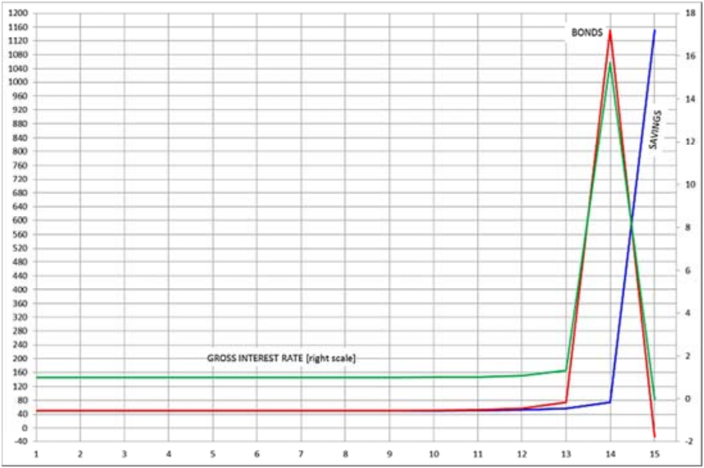
Monotonic convergence to equilibrium ($S = 0$ & $R = \frac{1}{3}$)

The government raises >50 at $t = 1$

t	$S(t)$	$R(t)$	$B(t)$	%
1	50,00001	1	50,00003	
2	50,00003	1,000001	50,00009	0,00012
3	50,00009	1,000004	50,00027	0,00036
4	50,00027	1,000011	50,00081	0,00108
5	50,00081	1,000032	50,00243	0,00324
6	50,00243	1,000097	50,00729	0,009721
7	50,00729	1,000292	50,02188	0,029173
8	50,02188	1,000876	50,0657	0,087595
9	50,0657	1,002635	50,19761	0,263477
10	50,19761	1,007967	50,59755	0,796729
11	50,59755	1,024487	51,83654	2,448716
12	51,83654	1,079286	55,94645	7,928594
13	55,94645	1,312091	73,40684	31,20912
14	73,40684	15,69205	1151,904	1469,205
15	1151,904	-0,02321	-26,7411	-102,321

Impossible

Dynamics when $S(1) > 50$



At $t = 14$ the govt asks for more good (1151) than is available (200). This requires a negative R, which cannot be in equilibrium. The bubble bursts.

Debt sustainability

- If $r(t) > 0$ ($R(t) > 1$), then, by lending L at t, you get more than L at $t + 1$.

- If $-1 \leq r(t) \leq 0$ ($0 \leq R(t) \leq 1$), by lending L at t, you get less than L at $t + 1$.

- If $r(t) < -1$ ($R(t) < 0$), by lending L at t, you have to pay at $t + 1$. In this case, in equilibrium, no one lends.

- To sustain a growing govt debt, population must grow or endowments must grow.

Example with growing endowments

- If endowments double each period, then $S_t = \left(75 - \frac{25}{R(t)}\right)2^{t-1}$, $R(t) = \frac{25}{75 - \frac{B(t-1)}{2^{t-1}}}$, and $B(t) = \frac{25B(t-1)}{75 - \frac{B(t-1)}{2^{t-1}}}$.
- The govt can borrow initially 62 but not 63.

t	*S*(*t*)	*R*(*t*)	*B*(*t*)
1	62	1,92	119,2
2	119,2	1,62	193,7
3	193,7	0,94	182,3
4	182,3	0,47	87,3
5	87,3	0,35	31,3
6	31,3	0,337	10,6
7	10,6	0,334	3,5

S(*t*)	*R*(*t*)	*B*(*t*)
63	2,08	131,2
131,25	2,66	350
350	-2	-700
-700	0,15	-107,6
-107,6	0,3	-32,9
-32,9	0,32	-10,8
-10,8	0,33	-3,6

Equivalence between bonds and taxes

- ➡ *Let C be an equilibrium consumption allocation with bonds. Then, for some tax-transfer scheme (without bonds) that balances the govt's budget at each t (taxes at t equal transfers at t), C is also an equilibrium consumption allocation.*

- With bonds, the equilibrium $\hat{R}(t)$ at t solves $S_t\left(\hat{R}(t)\right) = B(t)/\hat{R}(t)$. Given $\hat{R}(t)$ and the bond holdings $b^i(t)$, the same equilibrium consumption allocation can be obtained with taxes (but without bonds) by setting $\tau_t^i(t) = b^i(t)$ and $\tau_t^i(t+1) = -\hat{R}(t) \cdot b^i(t)$.

Ricardian equivalence proposition

➡ *(attrib. David Ricardo) Consumption allocations & interest rates do not change if the govt borrows now & taxes later instead of just taxing now*.

- Relies on the fact that the new policy should <u>not alter the consumer's present value of endowments</u>.

- The equivalence may fail if, for instance, one policy is to borrow from generation 1 & tax generation 2 while the other is to tax generation 1 (different generations involved).

Example of Ricardian equivalence

- Suppose members of generation t are all identical, with $u_t^i = c_t^i(t) \cdot c_t^i(t+1)$. This implies that there is no private borrowing.

- Policy 1: set tax $\tau_t^i(t) = m$. The consumption basket is $c_t^i = \left(w_t^i(t) - m, w_t^i(t+1)\right)$.

- Policy 2: borrow m from generation t at t & tax generation t at $t+1$ to pay off the bonds issued at t. Now, $c_t^i = \left(w_t^i(t) - b^i(t), w_t^i(t+1) + R(t)b^i(t) - \tau_t^i(t+1)\right)$. Since taxes must only pay off the bonds, $\tau_t^i(t+1) = R(t)b^i(t)$.

- But $b^i(t) = m$, so $\tau_t^i(t+1)$ has present value m. This means that the present value of i's tax liability is not altered: it is m (at t) under policy 1 & $m \cdot R(t)$ (at $t+1$) under policy 2.

- The consumption basket is therefore the same under the two policies.

- Moreover, in equilibrium, $R(t) = MRS_t^i$. As $MRS_t^i = \frac{c_t^i(t+1)}{c_t^i(t)}$, the MRS does not change.

- Accordingly, the interest rate is the same under both policies.

Why fiat money?

- Let all generations be identical, grow at a constant rate n, and old people have nothing.

- Specifically, $N(t) = (1 + n)N(t - 1)$, $u_t^i = u_s^j$, and $w_t^i = w_s^j = (w, 0)$, for all generations t and s, $i \in N(t)$, and $j \in N(s)$.

- If inside money (loans) is not possible, there is no trade (autarky) and consumers must consume their endowments (the old starve). The aim is to show that each generation's welfare can be maximized with fiat money.

Welfare maximizing consumption

- The consumption allocation that maximizes generation *t*'s welfare is obtained by maximizing $u_t^i\left(c_t^i(t), c_t^i(t+1)\right)$ subject to the resource constraint at t

$$N(t)c_t^i(t) + N(t-1)c_{t-1}^i(t) = N(t)w$$

where w is the young person's endowment.

- Since $N(t) = (1+n)N(t-1) > 0$ and $c_{t-1}^i(t) = c_t^i(t+1)$,

$$c_t^i(t) + \frac{c_t^i(t+1)}{1+n} = w$$

where n is a short of "biological interest rate".

- With $u_t^i\left(c_t^i(t), c_t^i(t+1)\right) = c_t^i(t) \cdot c_t^i(t+1)$, the solution satisfies

$$1 + n = MRS_t^i = c_t^i(t+1)/c_t^i(t) \text{ and}$$
$$c_t^i(t+1) = \left[w - c_t^i(t)\right](1+n).$$

- Consequently,

$$c_t^i(t) = w/2 \text{ and } c_t^i(t+1) = (1+n)w/2.$$

- In autarky, utility for the young is $u_t^i(w, 0) = 0$ and for the old, it can be taken to be 0 (since the old do not consume). In the previous solution, the young gets $u_t^i(w/2, (1+n)w/2) > 0$ and the old obtains positive utility because $c_t^i(t+1) > 0$.

The role of fiat money

- The previous solution could be regarded as the one a social planner would choose. Is this solution achievable through money markets?

- Imagine that the old invent fiat money in period 1: a worthless asset intended to be generally accepted in exchange for the good.

- Let M be the amount of fiat money created at $t = 1$ and, for all t, let $p(t)$ designate the price of the good in terms of money: 1 unit of good at t is worth $p(t)$ units of money.

Money in the budget constraints

- $p(t)$ can be interpreted as the price level in the economy, whereas $1/p(t)$ would be price or value of money (amount of good that one unit of money can purchase).

- The previous solution could be regarded as the one a social planner would choose. Is this solution achievable through money markets?

- If the young at t buy $m^i(t)$ units of money, the constraints for the young and old are

$$c_t^i(t) + \frac{m^i(t)}{p(t)} = w \quad \text{and} \quad c_t^i(t+1) = \frac{m^i(t)}{p(t+1)}.$$

Equilibrium in the money market

- Money supply at t is given by M. Money demand per person at t is $m^i(t) = p(t)(w - c_t^i(t))$. Total demand is then $N(t)m^i(t)$.

- In equilibrium, $N(t)m^i(t) = M$. That is,
$$p(t) = \frac{M}{N(t)\left[w - c_t^i(t)\right]}.$$

- This relationship is also valid for $t + 1$:
$$p(t+1) = \frac{M}{N(t+1)\left[w - c_{t+1}^i(t+1)\right]}.$$

- As all generations are alike, $c_{t+1}^i(t+1) = c_t^i(t)$. Thus, given $N(t+1) = (1+n)N(t)$,

$$\frac{p(t)}{p(t+1)} = \frac{N(t+1)}{N(t)} = 1 + n\,.$$

- The above is the equilibrium condition in the money market.

- $P = p(t)/p(t+1)$ is the gross return of fiat money: is the amount of good earned in $t + 1$ by investing one unit of good in money.

- 1 unit of good at t can get $p(t)$ units of money at t. As 1 unit of money at $t + 1$ buys $1/p(t+1)$ units of good at $t + 1$, $p(t)$ can buy $P = p(t)/p(t+1)$. So 1 unit of good invested in money at t yields P units of good at $t + 1$.

Money demand & consumption

- The young maximize $c_t^i(t) \cdot c_t^i(t+1)$, that is,

$$\left(w - \frac{m^i(t)}{p(t)}\right) \cdot \frac{m^i(t)}{p(t+1)}$$

- After equating to zero the derivative with respect to $m^i(t)$, real money demand is

$$\frac{m^i(t)}{p(t)} = \frac{w}{2}.$$

- Consumption when young and old are

$$c_t^i(t) = w - \frac{m^i(t)}{p(t)} = \frac{w}{2}$$

$$c_t^i(t+1) = \frac{m^i(t)}{p(t+1)} = \frac{w \cdot p(t)/2}{p(t+1)} = \frac{w(1+n)}{2}.$$

Fiat money and welfare

- The preceding results show that fiat money (i) can replicate the consumption patterns that maximize each generation's welfare and (ii) improves upon the no trade situation.

- Defining the inflation rate at t as $\pi(t) = \frac{p(t)-p(t-1)}{p(t-1)}$, it follows that

$$1 + \pi(t) = \frac{1}{1+n}.$$

- Thus, $\pi(t) = -n/(1+n)$: there is a deflation at a constant rate. Also, $c_t^i(t+1) = \frac{w/2}{1+\pi(t+1)}$: the old consume half of the (inflation-based) present value of the young's endowment.

Fully funded pensions

- From 22, the govt taxes the young at t, lends the revenues, and pays out the proceeds to the old at $t+1$ as a pension. Are the young forced to save more than they wish?
- When young, i's budget constraint is $c_t^i(t) + l^i(t) + \tau(t) = w_t^i(t)$; when old, it is $c_t^i(t+1) = w_t^i(t+1) + R(t)\left[l^i(t) + \tau(t)\right]$.
- The pension has no effect since budget constraints coincide with those without the pension ($l^i + \tau$ replaces l^i). Savings are cut to pay taxes so that income remains the same.

Unfunded (pay-as-you-go) pensions

- The pension $p(t)$ to the old at t are paid out from current tax receipts $\tau(t)$ on the young.
- Suppose population grows at rate n. The govt budget constraint at t is $\tau(t)N(t) = p(t)N(t-1)$. That is, $\tau(t)(1+n)N(t-1) = p(t)N(t-1)$. Therefore, $\tau(t)(1+n) = p(t)$.
- When young, i's budget constraint is $c_t^i(t) + l^i(t) + \tau(t) = w_t^i(t)$; when old, it is $c_t^i(t+1) = w_t^i(t+1) + R(t)l^i(t) + p(t) = w_t^i(t+1) + R(t)l^i(t) + \tau(t)(1+n)$.

- The lifetime budget constraint is

$$c_t^i(t) + \frac{c_t^i(t+1)}{R(t)} =$$

$$= w_t^i(t) + \frac{w_t^i(t+1)}{R(t)} + \tau(t)\left(\frac{1+n}{1+r(t)} - 1\right).$$

- Without the pension, the term $\tau(t)\left(\frac{n-r}{1+r(t)}\right)$ is missing. If $n > r(t)$, the budget set with the pension is larger, so a more preferred consumption basket is feasible (pyramid scheme).

- If $n < r(t)$, the budget set with the pension is smaller. As the welfare maximizing basket without the pension is not feasible now, the pension reduces the young's welfare.