OLG with endogenous production

- The differences with respect the OLG model with exogenous production are listed next.
- People are endowed with labour, not goods.
- Time t good can be stored from t to $t+1$.
- The good stored at $t-1$ is called time t capital.
- Time t good can be produced by using time t labour and time $t-1$ good. This process is represented by a production function.

Labour

- $L_t^i = \left(L_t^i(t), L_t^i(t+1)\right)$ is the lifetime endowment of labour of member i of generation t, $L_t^i(t)$ when young and $L_t^i(t+1)$ when old.
- At each t, there is a competitive labour market where people can sell their labour in exchange for a wage $\omega(t)$ paid in good units.
- People only care about consumption, not leisure. They (inelastically) supply all their labour in both periods of their life. Labour $L(t)$ at t is $\sum_{i\in N(t)} L_t^i(t) + \sum_{i\in N(t-1)} L_{t-1}^i(t)$.

Capital

- Every young individual may save a part $K^i(t)$ of the wage $\omega(t)$.
- Their aggregate savings $\sum_{i\in N(t)} K^i(t)$ at t become the capital stock $K(t+1)$ at $t+1$.
- All capital available at t depreciates (is completely used up) during t.
- At $t=1$, there is an initial endowment $K(1)$.
- $K^i(t)$ represents the capital owned at t (when old) by member i of generation $t-1$.

Production function

- A production function takes the form $Y(t) = G\big(A(t), K(t), L(t)\big)$, where $A(t)$ represents the state of technology at t, $L(t)$ is labour at t, and $K(t)$ is capital at t. For simplicity, $\forall t \;\; Y(t) = A(t)\cdot F\big(K(t), L(t)\big)$.
- F displays constant returns to scale: for $\delta > 0$, $F\big(\delta\cdot K(t), \delta\cdot L(t)\big) = \delta\cdot F\big(K(t), L(t)\big)$.
- Marginal productivities are positive but decreasing: $\frac{\partial F}{\partial K(t)} > 0$, $\frac{\partial F}{\partial L(t)} > 0$, $\frac{\partial^2 F}{\partial K(t)^2} < 0$, and $\frac{\partial^2 F}{\partial L(t)^2} < 0$.

Firms

- There are many profit-maximizing competitive firms with the same production function.
- Competitiveness and constant returns imply that firms emply K and L in the same proportion, so all of them are larger or smaller copies of a given firm.
- As a result, total production $Y(t)$ at t is a function of total capital $K(t)$ and labour $L(t)$ at t. Typical production function: the Cobb-Douglas, $Y(t) = A(t)\cdot K(t)^{\alpha}\cdot L(t)^{1-\alpha}$.

General feasibility condition

- Total production $Y(t)$ at t is: (i) obtained from total labour $L(t)$ and total capital $K(t)$ available at t; and (ii) is either consumed or accumulated for the next period. Formally,

$$\sum_{i\in N(t)} c_t^i(t) + \sum_{i\in N(t-1)} c_{t-1}^i(t) + \sum_{i\in N(t)} K^i(t+1) = A(t)\cdot F\big(K(t), L(t)\big)$$

or

$$C(t) + K(t+1) = A(t)\cdot F\big(K(t), L(t)\big).$$

- Assumptions: $\frac{\partial F}{\partial K(t)} \to \infty$ if $K(t)\to 0$, $\frac{\partial F}{\partial K(t)} \to 0$ if $K(t) \to \infty$, & the same for $L(t)$.

Prices of inputs

- Since the labour market is competitive, the wage rate equals the marginal productivity of labour: $\omega(t) = \partial F/\partial L(t)$.
- The capital market is also assumed to be competitive, so the price $\sigma(t)$ of capital equals the marginal productivity of capital: $\sigma(t) = \partial F/\partial K(t)$.
- Constant returns guarantee that ω and σ depend on the relative, not the absolute, amounts of K and L.

Cobb-Douglas example /1

- Let $Y(t) = A(t) \cdot K(t)^{\alpha} \cdot L(t)^{1-\alpha}$. Then:
$$\omega(t) = \frac{\partial F}{\partial L(t)} = (1-\alpha) \cdot A(t) \cdot \left(\frac{K(t)}{L(t)}\right)^{\alpha}$$
$$\sigma(t) = \frac{\partial F}{\partial K(t)} = \alpha \cdot A(t) \cdot \left(\frac{L(t)}{K(t)}\right)^{1-\alpha}$$
- By the uniqueness of the input prices, all firms use K and L in the same proportion: firms using more K will be using more L.
- Since all labour is hired, the total wage bill is $\omega(t) \cdot L(t) = (1-\alpha) \cdot A(t) \cdot \left(\frac{K(t)}{L(t)}\right)^{\alpha} \cdot L(t) = (1-\alpha) \cdot Y(t)$. Similarly, $\sigma(t) \cdot K(t) = \alpha \cdot Y(t)$.

Cobb-Douglas example /2

- This says that the total payment to labour is the fraction $1-\alpha$ of output, whereas the total payment to capital is the fraction α. As a result,
$$\omega(t) \cdot L(t) + \sigma(t) \cdot K(t) = Y(t).$$
- Production is distributed between labour & capital in fixed proportions. This holds for production functions with constant returns.
- Another implication of the previous results is that firms earn no profit.

Worker-consumer's budget constraints

- As before, every i aims at maximizing utility subject to the budget constraints. When young and old, i's budget constraints are
$$c_t^i(t) + l^i(t) + K^i(t+1) = \omega(t) L_t^i(t)$$
$$c_t^i(t+1) = R(t) l^i(t) + \sigma(t+1) K^i(t+1) + \omega(t+1) L_t^i(t+1).$$
- By combining the two constraints,
$$c_t^i(t) + \frac{c_t^i(t+1)}{R(t)} = \omega(t) L_t^i(t) + \frac{\omega(t+1) L_t^i(t+1)}{R(t)} + K^i(t+1)\left(\frac{\sigma(t+1)}{R(t)} - 1\right).$$

Equality between σ and R

- If $\sigma(t+1) > R(t)$, everyone would borrow as much of the good to invest in capital. This cannot be in equilibrium: no one lends.
- If $\sigma(t+1) < R(t)$, nobody holds capital, so $K(t+1) = 0$. This makes the marginal productivity of K, arbitrarily large. Hence, $\sigma(t+1)$ is also arbitrarily large, contradicting the assumption that $\sigma(t+1) < R(t)$.
- Therefore, in equilibrium, only $\sigma(t+1) = R(t)$ is possible, so $K^i(t+1)\left(\frac{\sigma(t+1)}{R(t)} - 1\right) = 0$.

Worker-consumer's decision problem

- The decision problem of every $i \in N(t)$ is the same as in 17 (with exogenous production) because the lifetime budget constraints in the two cases are analogous: endowments $w_t^i(s)$ are now the wage incomes $\omega(s) L_t^i(s)$.
- The only qualification to be made is that $\omega(t+1)$ is not known at t (and neither is $\sigma(t+1)$ known). Accordingly, for both problems to be the same, it is necessary to postulate perfect foresight: individuals know at t the market prices prevailing at $t+1$.

General competitive equilibrium

- A general competitive equilibrium (with initial $K(1) > 0$, production function F, labour endowments, and perfect foresight) is a sequence $\{\hat{R}(t), \hat{\sigma}(t), \hat{\omega}(t), \hat{K}(t)\}_{t\geq 1}$ such that, for all $t \geq 1$:

(i) $S_t\left(\hat{R}(t)\right) = \hat{K}(t+1)$, where S_t is the total savings function obtained by maximizing each individual's utility;

(ii) $\hat{\sigma}(t+1) = \hat{R}(t)$;

(iii) $\hat{\sigma}(t) = \partial F/\partial K(t)$; and

(iv) $\hat{\omega}(t) = \partial F/\partial L(t)$.

Cobb-Douglas example

- Let $u_t^i = c_t^i(t) \cdot c_t^i(t+1)$ and $Y(t) = A(t) \cdot K(t)^\alpha \cdot L(t)^{1-\alpha}$. Then, defining $L_t(s) = \sum_{i \in N(s)} L_t^i(s)$ and $L(t) = L_t(t) + L_{t-1}(t)$,

$$S_t = \frac{\omega(t) L_t(t)}{2} - \frac{\omega(t+1) L_t(t+1)}{2}$$

$$\omega(t) = (1-\alpha) A(t) \left(\frac{K(t)}{L(t)}\right)^\alpha$$

$$\sigma(t) = \alpha A(t) \left(\frac{L(t)}{K(t)}\right)^{1-\alpha}$$

- Substituting these equations into the equilibrium condition $S_t = K(t+1)$ and solving for $K(t+1)$,

$$K(t+1) = \left(\frac{\frac{(1-\alpha)A(t)}{2} \frac{L_t(t)}{L(t)^\alpha}}{1 + \frac{1-\alpha}{2\alpha} \frac{L_t(t+1)}{L(t+1)}} \right) K(t)^\alpha .$$

- If A, L, and L_t all remain constant, the term within the parenthesis is a constant $a > 0$. The equation describing the dynamics of capital accumulation in equilibrium is then

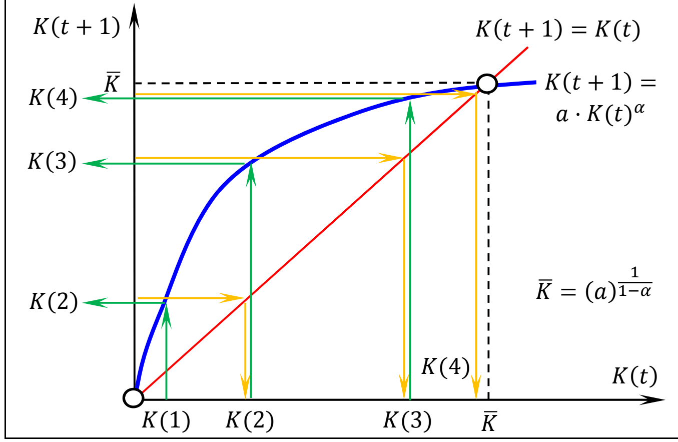
$$K(t+1) = a \cdot K(t)^\alpha .$$

- The steady state capital stock $\bar{K}$ is obtained when $K(t+1) = K(t) = \bar{K}$. That is, $\bar{K} = a \cdot \bar{K}^\alpha$. Accordingly,

$$\bar{K} = a^{1/(1-\alpha)} .$$

Population and technology constant

- The graph below represents $\bar{K}$ and equation $K(t+1) = a \cdot K(t)^\alpha$. No matter the initial stock $K(1) > 0$, the economy converges to $\bar{K}$.



Steady (stationary) state

- A steady state of the economy is characterized by the condition $K(t+1) = K(t)$.
- Once found a steady state value $\bar{K}$, then, assuming L and A constant, the value $\bar{Y}$ of output in the steady state can also be found: $\bar{Y} = A \cdot \bar{K}^\alpha \cdot L^{1-\alpha}$. Knowing this, both $\bar{\omega}$ and $\bar{\sigma}$ can be determined.
- From the equilibrium condition $S_t = K(t+1)$, it follows that $\bar{S} = \bar{K}$. Given this, as S_t is a function of $R(t)$, $\bar{R}$ can also be ascertained (in fact, in equilibrium, $\bar{R} = \bar{\sigma}$).

Population grows, technology constant

- With everything else the same, suppose $N(t+1) = N \cdot N(t)$, for some $N > 1$, and all generations have the same amount of labour.
- Let $L_0(0)$ be the labour endowment of the young at $t = 0$ and $L_0(1)$ the labour of the old at $t = 1$. Define $L(0) = L_0(0) + L_0(1)/N$.
- The total labour endowment (supply) of the young at t is

$$L_t(t) = N^t \cdot L_0(0)$$

and the labour endowment of the old at t is

$$L_{t-1}(t) = N^{t-1} \cdot L_0(1).$$

- Therefore, total labour supply at t is

$$L(t) = L_t(t) + L_{t-1}(t) = N^t \cdot L_0(0) + N^{t-1} \cdot L_0(1) = N^t \left(L_0(0) + \frac{L_0(1)}{N}\right) = N^t \cdot L(0).$$

- The savings function of each individual i at t is

$$s^i(t) = \frac{1}{2}\left(\omega(t) \cdot L_t^i(t) - \frac{\omega(t+1) \cdot L_t^i(t+1)}{R(t)}\right).$$

- Aggregate savings at t are

$$S_t = N(t) \cdot s^i(t) = N^t \cdot N(0) \cdot s^i(t) = \frac{1}{2}\left(\omega(t) \cdot N^t \cdot L_0(0) - \frac{\omega(t+1) \cdot N^t \cdot L_0(1)}{R(t)}\right).$$

- The wage at t is

$$\omega(t) = \frac{\partial F}{\partial L(t)} = (1-\alpha) \cdot A(t) \cdot \left(\frac{K(t)}{N^t \cdot L(0)}\right)^{\alpha}.$$

- The price of capital at $t+1$ (which equals $R(t)$ in equilibrium) is

$$\sigma(t+1) = \frac{\partial F}{\partial K(t+1)} = \alpha \cdot A(t) \cdot \left(\frac{K(t+1)}{N^{t+1} \cdot L(0)}\right)^{\alpha-1}.$$

- Using these equations and the equilibrium condition $S_t = K(t+1)$, or simply recalling

$$K(t+1) = \left(\frac{\frac{(1-\alpha)A(t)}{2} \frac{L_t(t)}{L(t)^{\alpha}}}{1 + \frac{1-\alpha}{2\alpha} \frac{L_t(t+1)}{L(t+1)}}\right) K(t)^{\alpha}$$

which is the equation describing the equilibrium path of capital,

$$K(t+1) = \left(\frac{\frac{(1-\alpha)A(0)}{2} \frac{L_0(0)}{L(0)^{\alpha}}}{1 + \frac{1-\alpha}{2\alpha} \frac{L_0(1)}{N \cdot L(0)}}\right) \cdot N^{t(1-\alpha)} \cdot K(t)^{\alpha}.$$

- Denoting by B the term in parenthesis, $K(t+1) = B \cdot N^{t(1-\alpha)} \cdot K(t)^{\alpha}$.
- The gross growth rate of capital is

$$G_K(t+1) = \frac{K(t+1)}{K(t)} = \frac{B \cdot N^{t(1-\alpha)} \cdot K(t)^{\alpha}}{B \cdot N^{(t-1)(1-\alpha)} \cdot K(t-1)^{\alpha}} = \frac{1}{N^{\alpha-1}} \cdot G_K(t)^{\alpha} = N^{1-\alpha} \cdot G_K(t)^{\alpha}.$$

- Let G_K designate the limit of the gross growth rate of capital. As a result,

$$G_K = N^{1-\alpha} \cdot {G_K}^{\alpha}.$$

- Solving for G_K, ${G_K}^{1-\alpha} = N^{1-\alpha}$. In sum,

$$G_K = N.$$

- This says that, in the equilibrium steady state, capital accumulates at the same rate as the population grows: $K(t+1) = N \cdot K(t)$.
- The growth rate of the capital stock K and the growth rate of ouput Y eventually equal the growth rate of the population.

Population constant, technology grows

- Suppose now that technology improves at gross rate $G > 1$: $A(t+1) = G \cdot A(t)$. Since $Y = A \cdot K^{\alpha} \cdot L^{1-\alpha}$, technological growth is called neutral (changes in A affect the productivity of both capital and labour).
- Given $A(t) = G^t \cdot A(0)$ and constant population, the equilibrium path of capital is

$$K(t+1) = \left(\frac{\frac{(1-\alpha)A(0)}{2} \frac{L_0(0)}{L(0)^{\alpha}}}{1 + \frac{1-\alpha}{2\alpha} \frac{L_0(1)}{N \cdot L(0)}}\right) \cdot G^t \cdot K(t)^{\alpha}.$$

- Denoting by B the term in parenthesis, $K(t+1) = B \cdot G^t \cdot K(t)^{\alpha}$.
- The gross growth rate of capital is

$$G_K(t+1) = \frac{K(t+1)}{K(t)} = \frac{B \cdot G^t \cdot K(t)^{\alpha}}{B \cdot G^{t-1} \cdot K(t-1)^{\alpha}} = G \cdot G_K(t)^{\alpha}.$$

- If G_K is the limit of $G_K(t)$, $G_K = G \cdot {G_K}^{\alpha}$ and

$$G_K = G^{\frac{1}{1-\alpha}}.$$

- As $1/(1-\alpha) > 1$, $G_K > G$: the capital stock growth rate (which equals the ouput growth rate) is greater than the technology growth rate.