Full Appendix of

Strategic Formation of Airline Alliances

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A Appendix: Definition of the relevant region R

A number of restrictions on the parameters d, θ and α have to be observed to ensure positive prices, quantities, marginal costs, margins and the compliance with non-arbitrage conditions are guaranteed in the three scenarios under consideration. Markets defined by a triple $\{d, \theta, \alpha\} \in R$ guarantee comparable results.

- Bounds on α. Positivity and non-arbitrage conditions in the three considered scenarios lead to several bounds in α. After comparing all these bounds and selecting the most stringent ones, we obtain $\alpha \in (\underline{\alpha}(d,\theta), \overline{\alpha}(d,\theta))$ with $\overline{\alpha}(d,\theta) = \min(B1, B2)$ and $\underline{\alpha}(d,\theta) = \max(B1, B3, 1)$ where

  $B1 \equiv \frac{4(4-6\theta+d(2\theta-3+d(5\theta-2)))}{d(2\theta-3)(7\theta-2)+2(2+\theta-6\theta^2)+2d^2(2+\theta(10\theta-9))}$,

  $B2 \equiv \frac{4\theta((d-1)d-3)-2(d^2-5)}{\theta(10-12\theta+(d-4)d(2\theta-1))}$ and $B3 \equiv \frac{4d(3\theta-2+d(2\theta-1))}{4d(2+d)-10+27\theta-4d(7+5d)\theta+6\theta^2(4d(1+d)-3)}$.[1]

[1] There are 20 bounds on α to take into account. Let us denote them by B$(\cdot)$, putting in the argument the equilibrium condition that gives rise to the bound. The precise expressions can be derived from the equilibrium values provided in the main text. *Pre-alliance*: B$(q^{na} > 0)$, B$(Q^{na} > 0)$ B$(1 - \theta(q^{na}+Q^{na}) > 0)$, and B$(p^{na} - 1 + \frac{\theta(q^{na}+Q^{na})}{2} > 0)$; *Single alliance*: B$(q_p^a > 0)$, B$(q_o^a > 0)$, B$(Q_p^a > 0)$, B$(Q_o^a > 0)$, B$(1 - \theta(q_o^a+Q_o^a) > 0)$, B$(1 - \theta(q_p^a+Q_p^a) > 0)$, B$(p_o^a - 1 + \frac{\theta(q_o^a+Q_o^a)}{2} > 0)$, B$(P_p^a - 2 + \theta(q_p^a+Q_p^a) > 0)$,

Specifically, $B1$ comes from ensuring positive equilibrium travel volume in the interline trip for outsiders in the single alliance situation; $B2$ from positive marginal cost for partners in the single alliance situation; $B3$ from the fulfillment of a non-arbitrage condition for partners in the single alliance situation.

Notice that $B1$ can be either a lower or an upper bound.

- An illustrative representation can be displayed in space (θ, d) - see Figure $A1$. To this end, we can compute the bounds on θ that come from the difference between $\overline{\alpha}(d,\theta)$ and $\underline{\alpha}(d,\theta)$, that is, the bounds ensuring the existence of a positive α such that we can find markets $\{d,\theta,\alpha\} \in R$. We obtain $\theta \in (0, \overline{\theta}(d))$ with

$$\overline{\theta}(d) = \begin{cases} L1 \text{ for } d < \frac{1}{2} \\ L3 \text{ for } d \in (\frac{1}{2}, 0.618] \\ L2 \text{ for } d > 0.618 \end{cases}$$

where $L1 \equiv \frac{2+3d+2d^2}{2(2+3d+d^2)}$, $L2 \equiv \frac{2d(2+d)-5}{2(d-3+2d^2)}$ and $L3 \equiv \frac{4+d}{6+4d}$. The case $d = \frac{1}{2}$ is a particular case: there is a discontinuity and α is bounded below by $B1 = B3 = 1$ and above by $B2 = \frac{38-52\theta}{47\theta-62\theta^2} > 1$.

Figure $A1$ below represents $L1$, $L2$ and $L3$. We claim that, for any pair $\{d,\theta\}$ in the region delimited by $L1$, $L2$ and $L3$, there exist values of $\alpha \in (\underline{\alpha}(d,\theta), \overline{\alpha}(d,\theta))$ such that we can find markets $\{d,\theta,\alpha\} \in R$.

– Insert here Figure $A1$ –

More precisely,

- For $d < \frac{1}{2}$ and $\theta < L1$, there exist values of $\alpha \in (B1, B2)$ such that we can find markets $\{d,\theta,\alpha\} \in R$.

- For $d \in (\frac{1}{2}, 0.618]$ and $\theta \in [L1, L3)$, there exist values of $\alpha \in (B3, B1)$ such that we can find markets $\{d,\theta,\alpha\} \in R$.

B($p_p^a > 0$), B($p_o^a > 0$), B($P_p^a > 0$), B($2p_p^a - P_p^a > 0$) and B($P_p^a - p_p^a > 0$); *Double alliance*: B($q^{aa} > 0$) B($1-\theta(q^{aa}+Q^{aa}) > 0$), and B($P^{aa} - 2 + \theta(q^{aa}+Q^{aa}) > 0$). One can observe that B($p_p^a - 1 + \frac{\theta(q_p^a+Q_p^a)}{2} > 0$) and B($p^{aa} - 1 + \frac{\theta(q^{aa}+Q^{aa})}{2} > 0$) simply reduce to $\alpha > 1$. After comparing all these bounds and selecting the most stringent ones, we are left with $B1$, $B2$ and $B3$ where $B1 \equiv$B($Q_o^a > 0$), $B2 \equiv$B($1-\theta(q_p^a+Q_p^a) > 0$) and finally $B3 \equiv$B($2p_p^a - P_p^a > 0$).

- For $d > \frac{1}{2}$ and $\theta < \min(L1, L2)$, there exist values of $\alpha \in (B3, B2)$ such that we can find markets $\{d, \theta, \alpha\} \in R$.

In addition, we know that $\overline{\theta} < \frac{2}{3}$ from the second order conditions. This means that economies of traffic density cannot be too high. This makes sense because otherwise marginal costs would become negative. ■

B Appendix: Proofs

Proof of Proposition 1.

The difference $P_p^a - 2p^{na}$ yields an expression whose denominator is negative for $\{d, \theta, \alpha\} \in R$. The numerator is positive for $\alpha > \alpha^* \equiv \frac{4(d\theta-1)}{4d+9\theta+12d\theta^2-6-14d\theta-6\theta^2}$. We now compare α^* with the corresponding lower bounds in R. Thus, for $d < \frac{1}{2}$, the difference $B1-\ \alpha^*$ is positive and, for $d > \frac{1}{2}$, the difference $B3-\ \alpha^*$ is positive too. Therefore, $\alpha > \alpha^*$ is always verified in R. It is straightforward to check that $\alpha > \alpha^*$ also implies $Q_p^a > Q^{na}$, $p_o^a < p^{na}$, $Q_o^a < Q^{na}$ and $q_o^a > q^{na}$.

As for the fares and travel volumes for the partners' short markets, the difference $q_p^a - q^{na}$ yields an expression whose denominator is negative for $\{d, \theta, \alpha\} \in R$. The sign of the numerator depends on whether market size α is greater or smaller than $\frac{\phi_1(d,\theta)}{\phi_2(d,\theta)}$. The function $\phi_1(d, \theta)$ is positive for $\theta \in (\theta^-(d), \theta^+(d))$, where $\theta^+(d) > \frac{2}{3}$ and $\theta^-(d) = \frac{3-11d-d^2+4d^3+\sqrt{9-6d-5d^2-6d^3+17d^4}}{4d(2d^2-3)}$. The function $\phi_2(d, \theta)$ is positive for values of θ above $\tilde{\theta}(d)$, which is a decreasing function in d, it is discontinuous at $d = \frac{1}{2}$ and it lies above $\frac{2}{3}$ for $d > \frac{1}{2}$. When $\theta < \tilde{\theta}(d)$ the numerator in $q_p^a - q^{na}$ is positive; when $\theta > \tilde{\theta}(d)$ the numerator in $q_p^a - q^{na}$ is positive for $\alpha < \frac{\phi_1(d,\theta)}{\phi_2(d,\theta)}$. We have the following cases.

- Case $d < \frac{1}{2}$. For every $\{d, \theta, \alpha\} \in R$,

  i) for $\theta < \tilde{\theta}(d)$ the numerator in $q_p^a - q^{na}$ is positive and therefore $q_p^a - q^{na} < 0$.

  ii) for $\theta > \tilde{\theta}(d)$, $\frac{\phi_1(d,\theta)}{\phi_2(d,\theta)}$ is positive and greater than $B1$. If $\alpha < \frac{\phi_1(d,\theta)}{\phi_2(d,\theta)}$ the numerator in $q_p^a - q^{na}$ is positive and hence $q_p^a - q^{na} < 0$; if $\alpha > \frac{\phi_1(d,\theta)}{\phi_2(d,\theta)}$, then $q_p^a - q^{na} > 0$.

- Case $d = \frac{1}{2}$. For every $\{d, \theta, \alpha\} \in R$, the lower bounds on α are $B1 = B3 = 1$ and the upper bound is $B2 = \frac{38-52\theta}{47\theta-62\theta^2} > 1$. Since the numerator in $q_p^a - q^{na}$ is negative for every $\alpha < \frac{38-52\theta}{47\theta-62\theta^2}$, which is always the case, $q_p^a - q^{na}$ is positive.

- Case $d > \frac{1}{2}$. For every $\{d, \theta, \alpha\} \in R$,

  i) for $\theta < \theta^{-}(d)$ the numerator in $q_p^a - q^{na}$ is negative and therefore $q_p^a - q^{na} > 0$.

  ii) for $\theta \in (\theta^{-}(d), \theta^{+}(d))$, $\frac{\phi_1(d,\theta)}{\phi_2(d,\theta)}$ is positive but smaller than 1. Therefore, for $\alpha > \frac{\phi_1(d,\theta)}{\phi_2(d,\theta)}$, the numerator in $q_p^a - q^{na}$ is negative and $q_p^a - q^{na} > 0$.

The difference $p_p^a - p^{na}$ follows exactly the opposite pattern. ■

Proof of Proposition 2.

The difference $P^{aa} - 2p_o^a$ yields an expression whose denominator is negative for $\{d, \theta, \alpha\} \in R$. The numerator is positive for $\alpha > \alpha^*$, as previously defined, and it follows straightforward that $Q^{aa} > Q_o^a$, $P^{aa} < P_p^a$, $Q^{aa} < Q_p^a$, $p^{aa} > p_p^a$ and $q^{aa} < q_p^a$.

As for the fares and travel volumes for the partners' short markets, the difference $q^{aa} - q_o^a$ yields an expression whose denominator is negative for $\{d, \theta, \alpha\} \in R$. The sign of the numerator depends on whether market size α is greater or smaller than $\frac{\phi_1(d,\theta)}{\phi_2(d,\theta)}$. The function $\phi_1(d, \theta)$ is positive for $\theta \in (\theta^{-}(d), \theta^{+}(d))$, where $\theta^{+}(d) > \frac{2}{3}$ and $\theta^{-}(d) = \frac{6-23d+9d^3+\sqrt{36-84d+49d^2-52d^3+82d^4+d^6}}{4d(5d^2-6)}$. The function $\phi_2(d, \theta)$ is positive for values of θ above $\tilde{\theta}(d)$, which is a decreasing function in d, it is discontinuous at $d = \frac{1}{2}$ and it lies above $\frac{2}{3}$ for $d > \frac{1}{2}$. When $\theta < \tilde{\theta}(d)$ the numerator in $q^{aa} - q_o^a$ is positive; when $\theta > \tilde{\theta}(d)$ the numerator in $q^{aa} - q_o^a$ is positive for $\alpha < \frac{\phi_1(d,\theta)}{\phi_2(d,\theta)}$. We have the following cases.

- Case $d < \frac{1}{2}$. For every $\{d, \theta, \alpha\} \in R$,

  i) for $\theta < \tilde{\theta}(d)$ the numerator in $q^{aa} - q_o^a$ is positive and therefore $q^{aa} - q_o^a < 0$.

  ii) for $\theta > \tilde{\theta}(d)$, $\frac{\phi_1(d,\theta)}{\phi_2(d,\theta)}$ is positive and greater than $B1$. If $\alpha < \frac{\phi_1(d,\theta)}{\phi_2(d,\theta)}$ the numerator in $q^{aa} - q_o^a$ is positive and hence $q^{aa} - q_o^a < 0$; if $\alpha > \frac{\phi_1(d,\theta)}{\phi_2(d,\theta)}$ then $q^{aa} - q_o^a > 0$.

- Case $d = \frac{1}{2}$. For every $\{d, \theta, \alpha\} \in R$, the lower bounds on α are $B1 = B3 = 1$ and the upper bound is $B2 = \frac{38-52\theta}{47\theta-62\theta^2} > 1$. Since the numerator in $q^{aa} - q_o^a$ is negative for every $\alpha < \frac{38-52\theta}{47\theta-62\theta^2}$, which is always the case, then $q^{aa} - q_o^a$ is positive.

- Case $d > \frac{1}{2}$. For every $\{d, \theta, \alpha\} \in R$,

  i) for $\theta < \theta^{-}(d)$ the numerator in $q^{aa} - q_o^a$ is negative and therefore $q^{aa} - q_o^a > 0$.

ii) for $\theta \in (\theta^-(d), \theta^+(d))$, $\frac{\phi_1(d,\theta)}{\phi_2(d,\theta)}$ is positive but smaller than 1. Therefore, for $\alpha > \frac{\phi_1(d,\theta)}{\phi_2(d,\theta)}$, the numerator in $q^{aa} - q^a_o$ is negative and $q^{aa} - q^a_o > 0$.

The difference $p^{aa} - p^a_o$ follows exactly the opposite pattern. ■

Proof of Lemma 1.

The denominator in $\Psi^a(d, \theta, \alpha) = \frac{\pi^a_p}{2} - \pi^{na}$ is positive for any $\{d, \theta, \alpha\} \in R$. The numerator can be written as $\alpha^2 K_1(d, \theta) + \alpha K_2(d, \theta) + K_3(d, \theta)$ where $K_1(d, \theta)$ may be either positive or negative. Solving $K_1(d, \theta) = 0$ for θ yields several solutions, from which only one is relevant in R. Denote this root by $\widetilde{\theta}(d)$ which is increasing in d. For any $\{d, \theta, \alpha\} \in R$, if $\theta > \widetilde{\theta}(d)$, the function $K_1(d, \theta)$ is positive and the numerator in $\Psi^a(d, \theta, \alpha)$ is a convex function in α. On the other hand, if $\theta < \widetilde{\theta}(d)$, the function $K_1(d, \theta)$ is negative and the numerator in $\Psi^a(d, \theta, \alpha)$ is a concave function in α. Solving the numerator in $\Psi^a(d, \theta, \alpha)$ for α results in $\alpha^-(d, \theta)$ and $\alpha^+(d, \theta)$. Thus, there are two constraints on α to be met to have a positive numerator in $\Psi^a(d, \theta, \alpha)$: $\alpha \notin (\alpha^-(d, \theta), \alpha^+(d, \theta))$ if $K_1(d, \theta)$ is positive; and $\alpha \in (\alpha^-(d, \theta), \alpha^+(d, \theta))$ if $K_1(d, \theta)$ is negative.

- If $K_1(d, \theta)$ is positive ($\theta > \widetilde{\theta}(d)$), the functions $\alpha^-(d, \theta)$ and $\alpha^+(d, \theta)$ are either non real or yield an interval outside region R. Hence if $\alpha \notin (\alpha^-(d, \theta), \alpha^+(d, \theta))$ then the numerator in $\Psi^a(d, \theta, \alpha)$ is positive and hence $\Psi^a(d, \theta, \alpha) > 0$.

  One can check that $d = 0.802$ when $\widetilde{\theta}(d) = 0$. Consequently, since $\widetilde{\theta}(d)$ is increasing in d, $d < 0.802$ is sufficient to ensure $\Psi^a(d, \theta, \alpha) > 0$.

- If $K_1(d, \theta)$ is negative ($\theta < \widetilde{\theta}(d)$), it is unclear whether α belongs to $(\alpha^-(d, \theta), \alpha^+(d, \theta))$. Nevertheless, one can check that $\Psi^a(d, \theta, \alpha)$ is decreasing in α for $d > 0.849$. Therefore, we study $\Psi^a(d, \theta, \alpha = \underline{\alpha} = B3)$ for $d > 0.849$. Solving $\Psi^a(d, \theta, \underline{\alpha}) = 0$, we obtain a function $\widehat{\theta}(d, \underline{\alpha})$ that is increasing in d as can be seen in Figure $A2$ below (since there is an upper bound for θ in region R , $\overline{\theta}(d) \equiv L2$ following the notation in Appendix 1, we include it in the figure):

– Insert here Figure $A2$ –

For $\theta > \widehat{\theta}(d, \underline{\alpha})$, $\Psi^a(d, \theta, \underline{\alpha}) > 0$ and then $\Psi^a(d, \theta, \alpha) > 0$ for any α in R. Since solving $\widehat{\theta}(d, \underline{\alpha}) = \overline{\theta}(d)$ yields $\theta = 0.08$, it is sufficient to require $\theta > 0.08$ to guarantee $\Psi^a(d, \theta, \alpha) > 0$ for any $\{d, \theta, \alpha\} \in R$.

The value $d = 0.856$ is obtained by a numerical method when $\Psi^a(d, \theta, \alpha = \overline{\alpha} = B2)$ since for $d > 0.849$ the function $\Psi^a(d, \theta, \alpha)$ is decreasing in α. Hence, for $d > 0.856$, $\Psi^a(d, \theta, \alpha = \overline{\alpha}) < 0$ and then $\Psi^a(d, \theta, \alpha) < 0$ for any $\{d, \theta, \alpha\} \in R$. ■

Proof of Lemma 2.

The first part of the proof is similar to Lemma 1. As for the sufficient conditions, for any $\{d, \theta, \alpha\} \in R$, one can check that $\Psi^{aa}(d, \theta, \alpha) = \frac{\pi^{aa}}{2} - \pi_o^a$ is increasing in α for low values of d in the interval $d \in (0.707, 0.870]$ and decreasing in α for high values of d in this interval. Solving $\Psi^{aa}(d, \theta, \alpha = \underline{\alpha}) = 0$ and $\Psi^{aa}(d, \theta, \alpha = \overline{\alpha}) = 0$ yields two functions, $\widehat{\theta}(d, \underline{\alpha})$ and $\widehat{\theta}(d, \overline{\alpha})$ that are increasing in d as can be seen in Figure $A3$ below.

– Insert here Figure $A3$ –

Therefore for low values of d in the interval, $\theta > \widehat{\theta}(d, \overline{\alpha})$ implies $\Psi^{aa}(d, \theta, \alpha = \overline{\alpha}) > 0$ and hence $\Psi^{aa}(d, \theta, \alpha) > 0$ for any α in R. Solving $\widehat{\theta}(d, \overline{\alpha}) = 0$ we obtain the value $d = 0.707$. Hence, for $d < 0.707$, $\theta > \widehat{\theta}(d, \overline{\alpha})$, we have that $\Psi^{aa}(d, \theta, \alpha = \overline{\alpha}) > 0$ and then $\Psi^{aa}(d, \theta, \alpha) > 0$.

In happens to be case that $\widehat{\theta}(d, \overline{\alpha}) = \widehat{\theta}(d, \underline{\alpha}) = \overline{\theta}(d)$ at $d = 0.828$ and $\theta = 0.195$ and $\Psi^{aa}(d, \theta, \alpha) = 0$ for any α in R. Therefore, for $\theta > 0.195$, both $\widehat{\theta}(d, \overline{\alpha})$ and $\widehat{\theta}(d, \underline{\alpha})$ are positive, then both $\Psi^{aa}(d, \theta, \alpha = \overline{\alpha})$ and $\Psi^{aa}(d, \theta, \alpha = \underline{\alpha})$ are also positive, and hence $\Psi^{aa}(d, \theta, \alpha) > 0$. Similarly, for $d > 0.828$ both $\widehat{\theta}(d, \overline{\alpha})$ and $\widehat{\theta}(d, \underline{\alpha})$ are negative, then both $\Psi^{aa}(d, \theta, \overline{\alpha})$ and $\Psi^{aa}(d, \theta, \underline{\alpha})$ are also negative, and hence $\Psi^{aa}(d, \theta, \alpha) < 0$. ■

Figures

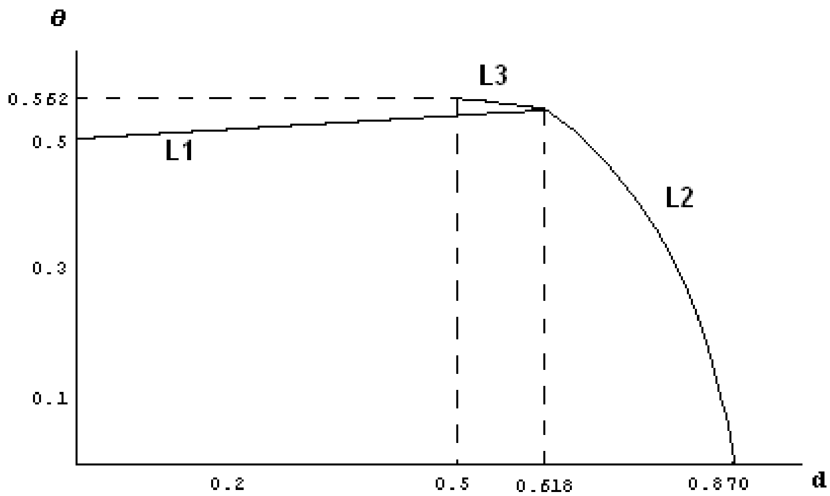

Figure $A1$: Bounds for d and θ in Region R

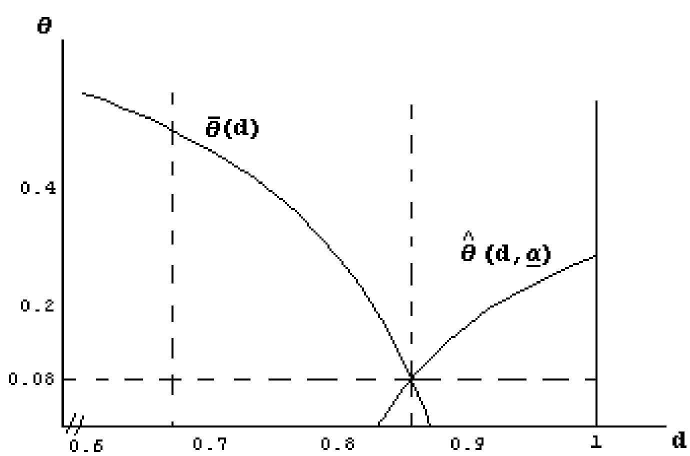

Figure $A2$: Proof of Lemma 1

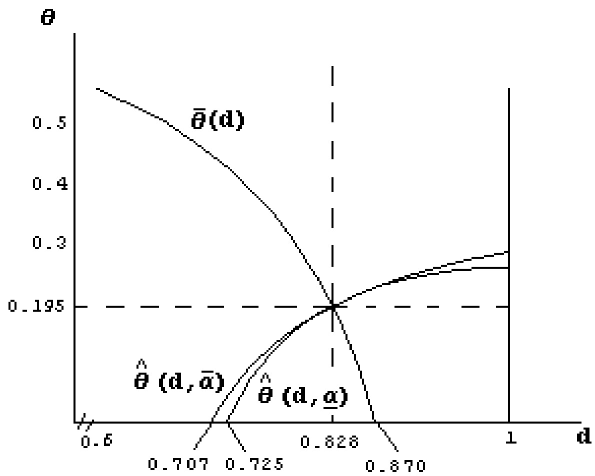

Figure $A3$: Proof of Lemma 2